ABELIAN CATEGORIES - IV

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1. Injective Objects (contd.)

Here we sketch the proof of the following fundamental theorem:

Theorem 1.1 (Grothendieck). *Let $\mathbb{C}$ be an abelian category satisfying the axiom (AB5) and assume that $\mathbb{C}$ has a generator. Then for every object $A \in \mathbb{C}$, there is a monomorphism $f(A) : A \to M$, where M is an injective object.*

Proof :(Sketch) Let $A \in \mathbb{C}$ and U be a generator of $\mathbb{C}$. Let $I(A)$ be the set of maps $u_i : V_i \to A$ from sub-objects V_i of U. Now consider the map $\varphi : \oplus V_i \to A \times U^{(I(A))}$ ($U^{(I(A))} = \oplus_{i \in I(A)} U$) given by its action on the i-th component by $-u_i : V_i \to A$ and the canonical injection of V_i in the i-th factor of the direct sum $U^{(I(A))}$. Let $M_1(A)$ be coker φ. Consider the map $f(A) : A \to M_1(A)$ given by the composition $A \to A \times U^{(I(A))} \to M_1(A)$ where the first map is $(1_A, 0, \cdots, 0)$ and the second is the canonical projection onto the quotient. The map $f(A)$ is a monomorphism, this can be seen for example by considering $\ker f(A)$ which is by definition a sub-object of every $\mathrm{Im} u_i$ and since the V_i are a family of generators has to be trivial. The constructed object $M_1(A)$ has the property that every map $u_i : V_i \to A$ can be "extended" to a map $U \to M_1(A)$ by

$$U \longrightarrow A \times U^{(I(A))} \longrightarrow M_1(A)$$

where the first map U to the i-th factor of $U^{(I(A))}$. If every map from $V_i \to M_1(A)$ extended to a map from $U \to M_1(A)$ we would be done, but this cannot be guaranteed at this stage. Next, we "saturate" $M_1(A)$ to get an object that has this property as follows. We define an object by transfinite induction. Take $M_0(A) = A$ and $M_1(A)$ as defined. Assume that the definition has been completed for all ordinals $< i$, and if i is of the form $j+1$ we set $M_i(A) = M_1(M_j(A))$ and the morphism $M_j(A) \to M_{j+1}(A)$ will be $f(M_j(A))$. If i is a limit ordinal, we set $M_i(A) = \lim_{j<i} M_j(A)$ and we take the morphisms $M_j(A) \to M_i(A)$ for $j < i$ to be the canonical morphisms (one can show this limit always exist in an abelian category satisfying (AB5)).

Let k be the smallest ordinal whose power is strictly larger than the power of the ordinal representing the set of sub-objects of U. Define $M(A) = M_k(A)$. We argue that $M(A)$ is injective by showing it satisfies the hypothesis of Lemma 1 in (Ab. Cat. III.) Let V be a sub-object of U and $\nu : V \to M(A)$, we will show that $\nu(V)$ is contained in $M_i(A)$ for some $i < k$ and this will prove the theorem. Since $M_k = \sup M_i$ we can take $V = \sup_{i<k} \nu^{-1}(M_i)$. Since the power of the ordinal representing the set of sub-objects of U is less than the power of k and since the set of ordinals $< k$ having k as a limit is a power of k, it follows that $\nu^{-1}(M_i)$ is constant for an $i_0 < k$, thus $V = \nu^{-1}(M_{i_0})$ and this proves the theorem. □

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