VANISHING OF HIGHER COHOMOLOGY GROUPS

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1. Injective objects (finalé)

We use our powerful theorem to give a simple proof of the fact that the category of sheaves of abelian groups has enough injectives (every object is isomorphic to a sub-object of an injective object).

Theorem 1.1. *Let $\mathcal{O}_X$ be a sheaf of rings on a topological space X. Then the additive category $\mathfrak{Mod}(\mathcal{O}_X)$ of $\mathcal{O}_X$-modules on X is an abelian category satisfying the axiom (AB5) and has a generator.*

Proof :(Sketch) Let $(\mathcal{F}_i)$ be a family of sheaves. The direct sum S of these sheaves are constructed by taking the sheaf associated to the presheaf that associates to each open set U the direct sum of the $\mathcal{F}_i$. It is clear that kernels and cokernels exist. Since direct limits exist, we can verify (AB5) is also satisfied in this category.

We now show that $\mathfrak{Mod}(\mathcal{O}_X)$ has a generator. Let $U \subseteq X$ be an open subset of X. We denote by $\mathcal{O}_U$ the sheaf obtained by extension by zero outside U. The morphisms $\mathcal{O}_U \to \mathcal{F}(U)$ are identified with the elements $\mathcal{F}(U)$. Thus the family $\mathcal{O}_U$ forms a family of generators for $\mathfrak{Mod}(\mathcal{O}_X)$. □

Corollary 1.2. *Every $\mathcal{O}_X$-module is isomorphic to a sub-$\mathcal{O}_X$-module of an injective $\mathcal{O}_X$-module.*

Corollary 1.3. *Let X be a topological space, then every sheaf of abelian groups $\mathcal{F}$ on X is isomorphic to a subsheaf of an injective sheaf.*

Proof : If we take $\mathcal{O}_X$ be the constant sheaf of rings $\mathbb{Z}$ then the category of $\mathfrak{Mod}(\mathcal{O}_X)$ is the same as the category of sheaves of abelian groups on X. □

2. Injective resolutions and Derived functors

Let $\mathbb{C}$ be an abelian category with enough injectives. An *injective resolution* of an object $A \in \mathbb{C}$ is a complex $I^\cdot$ defined in degrees $i \geq 0$ such that $\epsilon : A \to I^0$, each object I^i is injective $i \geq 0$ and the following sequence

$$0 \to A \xrightarrow{\epsilon} I^0 \to I^1 \to \cdots$$

is exact.

Proposition 2.1. *Let $\mathbb{C}$ be an abelian category with enough injectives. Then every object $A \in \mathbb{C}$ has an injective resolution.*

Proof :(Sketch) If $B \in \mathbb{C}$ is any object we will use $\mathcal{I}(B)$ to denote any injective object which has B as a sub-object. We let $I^0 = \mathcal{I}(A)$ and ϵ the canonical injection. Now take $I^1 = \mathcal{I}(\text{coker } \epsilon)$ and proceed to get an injective resolution. □

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Let $\mathbb{A}, \mathbb{B}$ be two abelian categories and $F : \mathbb{A} \to \mathbb{B}$ be a covariant functor. F is said to be *additive* if for any two objects A, B the map $\hom(A, B) \to \hom(FA, FB)$ is a homomorphism of abelian groups. F is said to be *left exact* if it is additive and for every short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

the sequence

$$0 \to FA' \to FA \to FA''$$

is exact.

Let $\mathbb{A}$ be an abelian category with enough injectives and let $F : \mathbb{A} \to \mathbb{B}$ be a covariant left exact functor, where $\mathbb{B}$ is abelian catgory. We define the *right derived functors* R^iF for $i \geq 0$ as follows. Choose an injective resolution $I^{\cdot}$ for each object $A \in \mathbb{C}$. Define $R^iF(A) = h^i(F(I^{\cdot}))$, where h^i denotes the i-th cohomology object of a complex.

Proposition 2.2. *All the nice properties of any self-respecting cohomology theory that ought to hold, do indeed hold for the functors* R^iF.

Proposition 2.3. *If* I *is any injective object then* $R^iF(I) = 0$ *for* $i > 0$.

Proof : Choose the injective resolution $0 \to I \to I \to 0 \to 0 \cdots$ to conclude the result. □

Definition 2.4. With the notation above, an object $A \in \mathbb{A}$ is *acyclic* for F if $R^iF(A) = 0$ for all $i > 0$.

The injective objects are acyclic. Acyclic objects allow us to compute the right derived functors of an object.

Proposition 2.5. *With the notation of this section. If we have an exact sequence*

$$0 \to A \to J^0 \to J^1 \to \cdots$$

where each J^i *is acyclic for* F, $i \geq 0$. *Then there is a natural isomorphism* $R^iF(A) \cong h^i(F(J^{\cdot}))$.

The claim is trivial for an injective resolution.

3. Vanishing of higher cohomology groups

Definition 3.1. Let X be a topological space $\mathfrak{Ab}(X)$ the category of sheaves of abelian groups on X and $\mathfrak{Ab}$ the category of abelian groups. Let $\Gamma(X, \cdot)$ be the global section functor $\mathfrak{Ab}(X) \to \mathfrak{Ab}$. We define the *cohomology functors* $H^i(X, \cdot)$ to be the right derived functors of the global section functor.

Our aim is to prove the following theorem of Grothendieck (our treatment here follows Hartshorne, the proof in Grothendieck is shorter but needs more background).

Theorem 3.2 (Grothendieck)**.** *Let* X *be a noetherian topological space of dimension* n. *Then for all* $i > n$ *and all sheaves of abelian groups* $\mathcal{F}$ *on* X, *we have* $H^i(X, \mathcal{F}) = 0$.

Definition 3.3. A sheaf $\mathcal{F}$ on a topological space is said to be *flasque* if for every inclusion of open sets $V \subseteq U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

Lemma 3.4. *If* X *is a topological space and* $\mathcal{O}_X$ *a sheaf of rings on* X, *then any injective* $\mathcal{O}_X$*-module is flasque.*

Proof : For any open subset $U \subseteq X$, consider the sheaf $\mathcal{O}_U$ obtained by extending the sheaf $\mathcal{O}_X|_U$ by zero outside U. Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Suppose $V \subseteq U$ are open sets. Then we have the inclusion $0 \to \mathcal{O}_V \to \mathcal{O}_U$ of sheaves of $\mathcal{O}_X$-modules. Since $\mathcal{I}$ is injective every map from the sub-object $\mathcal{O}_V \to \mathcal{I}$ extends to $\mathcal{O}_U$ so we get a surjection

$$\hom(\mathcal{O}_U, \mathcal{I}) \to \hom(\mathcal{O}_V, \mathcal{I}) \to 0.$$

But as we saw earlier, the family $\mathcal{O}_U$ is a generator and in fact $\hom(\mathcal{O}_U, \mathcal{I}) = \mathcal{I}(U)$ and $\hom(\mathcal{O}_V, \mathcal{I}) = \mathcal{I}(V)$, so $\mathcal{I}$ is flasque. □

Proposition 3.5. *If $\mathcal{F}$ is a flasque sheaf on a topological space X, then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

Proof : Embed $\mathcal{F}$ into an injective object $\mathcal{I}$ and let $\mathcal{G}$ be the quotient so that we have the exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0.$$

$\mathcal{F}$ is flasque by assumption and $\mathcal{I}$ is flasque by Lemma 3.4 and so the exact sequence implies that $\mathcal{G}$ is flasque. Taking global sections and using the fact that these are flasque sheaves we obtain an exact sequence

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{I}) \to \Gamma(X, \mathcal{G}) \to 0.$$

Now taking the cohomology functors one obtains the long exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & H^0(X, \mathcal{F}) \arrow[r] & H^0(X, \mathcal{I}) \arrow[r] & H^0(X, \mathcal{G}) \arrow[dll] \\ & H^1(X, \mathcal{F}) \arrow[r] & H^1(X, \mathcal{I}) \arrow[r] & \cdots \end{tikzcd}$$

But $\mathcal{I}$ is injective so $H^i(X, \mathcal{I}) = 0$ for $i > 0$ and using the fact that $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$ and the above exact sequence of global sections, we find that $H^1(X, \mathcal{F}) = 0$ and also $H^i(X, \mathcal{F}) \cong H^{i-1}(X, \mathcal{G})$ for $i \geq 2$. Using the fact that $\mathcal{G}$ is also flasque and induction on i we conclude that $H^i(X, \mathcal{F}) = 0$ for $i > 0$. □

We require the following three lemmas which we state without proof.

Lemma 3.6. *On a noetherian topological space, a direct limit of flasque sheaves is flasque.*

Lemma 3.7. *Let X be a noetherian topological space, and let $(\mathcal{F}_\alpha)$ be a direct system of abelian sheaves. Then there are natural isomorphisms for each $i \geq 0$*

$$\varinjlim H^i(X, \mathcal{F}_\alpha) \cong H^i(X, \varinjlim \mathcal{F}_\alpha).$$

Lemma 3.8. *Let Y be a closed subset of X, let $\mathcal{F}$ be a sheaf of abelian groups on Y, let $j : Y \to X$ be the inclusion. Then $H^i(Y, \mathcal{F}) = H^i(X, j_*\mathcal{F})$ where $j_*\mathcal{F}$ is the extension of $\mathcal{F}$ by zero outside Y.*

Proof :(of Theorem 3.2). If U is an open subset of X, by $\mathcal{F}_U$ we mean the sheaf $\mathcal{F}|_U$ extended by zero outside the open set U. If Y is a closed subset of X, by $\mathcal{F}_Y$ we mean the direct image sheaf $j_*(\mathcal{F}|_Y)$ via the inclusion $j : Y \to X$.
The proof will be a series of reductions which we do in stages.

(1) *Reduction to irreducible* X. If X is reducible, let Y be one of its irreducible components. Then we have $0 \to \mathcal{F}_U \to \mathcal{F} \to \mathcal{F}_Y \to 0$. We will show that $H^i(X, \mathcal{F}_Y) = H^i(Y, \mathcal{F}|_Y) = 0$ for $i > n$. Further, since U has one fewer irreducible component by induction on the number of irreducible components we have that $H^i(X, \mathcal{F}_U) = 0$. Thus the exact sequence implies that $H^i(X, \mathcal{F}) = 0$ for $i > n$. Thus it suffices to show the theorem for X irreducible.

(2) *Base case* $\dim X = 0$. If X is irreducible of dimension 0. Then the only open subsets of X are X and the empty set. Thus the global section functor $\Gamma(X, \cdot) : \mathfrak{Ab}(X) \to \mathfrak{Ab}$ is an equivalence of categories. In particular, $\Gamma(X, \cdot)$ is an exact functor and so $H^i(X, \mathcal{F}) = 0$ for $i > 0$ and all $\mathcal{F}$.

(3) *Reduction to sheaves generated by a single section.* Let X be irreducible of dimension n. Let $B = \cup_{U \subseteq X} \mathcal{F}(U)$ and let A be the set of finite subsets of B. For each $\alpha \in A$, let $\mathcal{F}_\alpha$ be the subsheaf of $\mathcal{F}$ generated by the sections in α. Now A is a directed set and $\mathcal{F} = \varinjlim \mathcal{F}_\alpha$. By lemma 3.7 it suffices to show that cohomology with coefficients in $\mathcal{F}_\alpha$ vanishes. If $\alpha' \subseteq \alpha$ then we have

$$0 \to \mathcal{F}_{\alpha'} \to \mathcal{F}_\alpha \to \mathcal{G} \to 0.$$

Now $\mathcal{G}$ is a sheaf generated by $\sharp(\alpha - \alpha')$ sections. Thus we reduce to the case that $\mathcal{F}_\alpha$ is a sheaf generated by a single section over the open set U. In that case $\mathcal{F}$ is a quotient of the sheaf $\mathbb{Z}_U$ (i.e. the constant sheaf $\mathbb{Z}$ extended by zero outside U). Thus we get

$$0 \to \mathcal{R} \to \mathbb{Z}_U \to \mathcal{F} \to 0$$

by setting $\mathcal{R}$ to be the kernel. Again by long exact sequences it suffices to show vanishing for the sheaves $\mathcal{R}$ and $\mathbb{Z}_U$, where now $\mathcal{R}$ is a subsheaf of $\mathbb{Z}_U$.

(4) *Reduction to the sheaf* $\mathbb{Z}_U$. Let U be an open subset of X and let $\mathcal{R}$ be a subsheaf of $\mathbb{Z}_U$. For each $x \in U$, the stalk $\mathcal{R}_x$ is a subgroup of $\mathbb{Z}$. If $\mathcal{R}$ is the zero sheaf then we can skip to the last stage. If not, let d be the least positive integer which occurs in any of the groups $\mathcal{R}_x$. Then there is a non-empty open subset $V \subseteq U$ such that $\mathcal{R}|_V \cong d \cdot \mathbb{Z}|_V$. Thus $\mathcal{R}_V \cong \mathbb{Z}_V$ and we have an exact sequence

$$0 \to \mathbb{Z}_V \to \mathcal{R} \to \mathcal{R}/\mathbb{Z}_V \to 0.$$

Now the sheaf $\mathcal{R}/\mathbb{Z}_V$ is supported on the closed subset $(U - V)^-$ of X which has dimension $< n$ since X is irreducible. So by lemma 3.8 and induction hypothesis we have $H^i(X, \mathcal{R}/\mathbb{Z}_V) = 0$ for $i \geq n$. So by long exact sequence of cohomology, we reduce to the case of showing vanishing only for $\mathbb{Z}_V$.

(5) *The case* $\mathbb{Z}_U$. We need to show that for any $U \subseteq X$, $H^i(X, \mathbb{Z}_U) = 0$ for $i > n$. Let $Y = X - U$. Then we have an exact sequence

$$0 \to \mathbb{Z}_U \to \mathbb{Z} \to \mathbb{Z}_Y \to 0.$$

Now $\dim Y < \dim X$ as X is irreducible, so by lemma 3.8 and induction hypothesis we have $H^i(X, \mathbb{Z}_Y) = 0$ for $i \geq n$. Now $\mathbb{Z}$ being a constant sheaf on an irreducible space is flasque. Thus by proposition 3.5, $H^i(X, \mathbb{Z}) = 0$. Thus from the long exact sequence of cohomology we conclude that $H^i(X, \mathbb{Z}_U) = 0$ for each $i > n$.

□