ABELIAN CATEGORIES - II

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1. Sub-objects in a category

Let $\mathbf{C}$ be a category and let $A \in \mathbf{C}$ be an object. We define a pre-order $\leq$ on monomorphisms $u : B \to A$ and $u' : B' \to A$ by defining $u \leq u'$ if u factors as follows:

$$\begin{tikzcd} B \arrow[r, "u"] \arrow[dr, "v"'] & A \\ & B' . \arrow[u, "u'"'] \end{tikzcd}$$

We call u and u' *equivalent* if $u \leq u'$ and $u' \leq u$. If u and u' are equivalent then B and B' are isomorphic. The set of equivalence classes of monomorphisms into A will be called the *sub-objects* of A (Grothendieck calls them *sous-trucs*). A sub-object of A is thus an object B together with a monomorphism $u : B \to A$ and this map is called the *canonical injection*. The pre-order in this case becomes an ordering, i.e. if $B \leq C$ and $C \leq B$ then $C = B$.

2. Generators

Let $\mathbf{C}$ be a category and let $(U_i)_{i \in I}$ be a family of objects of $\mathbf{C}$. We say that this is a *family of generators* of $\mathbf{C}$ if for each object $A \in \mathbf{C}$ and all sub-objects $B \neq A$ one can find an index $i \in I$ and a morphism $u : U_i \to A$ which does not arise from B. This implies that every sub-object of A is completely determined by those maps $U_i \to A$ which arise from the maps $U_i \to B$. We say that an object U of $\mathbf{C}$ is a *generator* if the family $\{U\}$ of subobjects of U is a generator.

For some of our investigations we will need the following axiom to be satisfied by our abelian categories.

(AB3) For all families $(A_i)_{i \in I}$ of objects in $\mathbf{C}$ the direct sum exists.

Proposition 2.1. *Let $\mathbf{C}$ be an abelian category satisfying (AB3). Let $(U_i)_{i \in I}$ be a family of objects of $\mathbf{C}$, and let $U = \oplus_{i \in I} U_i$ be their direct sum. Then the following conditions are equivalent:*

1. *(U_i) is a family of generators of $\mathbf{C}$.*
2. *U is a generator of $\mathbf{C}$.*
3. *Each $A \in \mathbf{C}$ is isomorphic to a quotient of a direct sum $U^{(I)}$ of objects each one being U.*

Proof : The equivalence of (1) and (2) follows immediately from the definitions. For (2) $\Rightarrow$ (3) we let $I = \hom(U, A)$. Now we describe a map $\bigoplus_I U \to A$. Since $\bigoplus_I U$ is a direct sum and using the fact that $\Phi : \prod_{i \in I} \hom(U_i, A) \to \hom(\bigoplus_I U_i, A)$ given by

$$\begin{tikzcd} A_i \arrow[d] \arrow[dr] & \\ \bigoplus_I U_i \arrow[r] & A \end{tikzcd}$$

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is an isomorphism. So to give a map $\bigoplus_I U \to A$ we can describe the map on its components. The map on the u-th component, where $u : U \to A$ is simply the map u itself. Let the map we constructed be $\varphi : \bigoplus_I U \to A$. We claim $\mathrm{Im}\varphi = A$. Suppose this is false, then $\mathrm{Im}\varphi = B$ a sub-object of A. But U is a generator so there is a map $u \in \hom(U, A)$ such that $u(U)$ does not arise from B. Thus image of φ cannot be contained in any proper subobject of A, consequently A is a quotient of $U^{(I)}$.

For (3) $\Rightarrow$ (2), we have that A is a quotient of $U^{(I)}$, so for any proper sub-object B of A, there is an i such that the canonical image in A of the i-th factor $U^{(I)}$ does not inject into B. Thus we have a morphism $U \to A$ which does not arise from B and so U is a generator. $\square$

3. Injective objects

An object M in an abelian category $\mathbf{C}$ is called *injective* if the functor $A \mapsto \hom(A, M)$ is exact. This means that all morphisms $u : B \to M$ for B a sub-object of A extends to a morphism $u' : A \to M$. In other words there is a morphism u' making the following diagram commute:

$$\begin{tikzcd} B \arrow[r] \arrow[d, "u"] & A \arrow[dl, "u'"] \\ M & \end{tikzcd}$$

where $B \to A$ is the canonical injection.
We conclude with the following characterization of injective modules.

Theorem 3.1. *Let L be a left A-module. Then L is injective iff for all left ideals I of A and all homorphisms $f : I \to L$ there is an $x \in L$ such that $f(\lambda) = \lambda x$ for all $\lambda \in I$.*

Proof : Suppose L is injective, then in particular the maps $f : I \to L$ will lift to maps $f : A \to L$. Thus $f(a) = af(1) \in L$.

Conversely, let X be a module, X' a sub-module and let $f : X' \to L$ be a morphism. Consider the pairs (Y, g) where Y is a sub-module of X containing X' and g an extension of f to the sub-module Y. By Zorn's lemma there is a maximal element call this (Y, g). Suppose $X \neq Y$ then there is an $x \in X$ that is not in Y. Let I be the ideal of elements $\lambda \in A$ such that $\lambda x \in Y$ and a map $h : I \to A$ given by $h(\lambda) = g(\lambda x)$. By hypothesis there is an element $u \in L$ such that $\lambda x \in Y$ implies that $g(\lambda x) = \lambda u$. Thus we can enlarge g to the module generated by Y and x by defining $g(\lambda x) = \lambda u$. This contradicts the maximality of Y. Thus $X = Y$ and we are done. $\square$