SOME HIGHLY TRIVIAL THEOREMS ABOUT PERFECT NUMBERS

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A **perfect number** is a positive integer n such that $\sigma(n) = 2n$ i.e., $\sum_{d \mid n} d = 2n$, in other words the sum of the proper divisors of n is exactly n. For example $6 = 3+2+1$ and $28 = 14+7+4+2+1$ are perfect. As of now we know that even perfect numbers are Mersenne primes in disguise and we believe there are no odd perfect numbers. Needless to say we do not know if there are infinitely many perfect numbers. A number is said to be **squarefree** if it is not divisible by any square. Thus n is squarefree iff it factors as a product of distinct primes. A number n is said to be **k-smooth** if all the prime factors of n are $\leq k$.

Theorem 0.1. *6 is the only squarefree perfect number.*

Proof : Suppose $n > 6$ is perfect and squarefree, then write $n = \prod_{1 \leq i \leq k} p_i$ where p_i are prime and $p_i \neq p_j$ if $i \neq j$. Assume that the primes have been ordered such that $p_1 > p_2 > \cdots > p_k$. Since $n \neq 6$ we can assume $p_1 > 3$. Now the sum of the divisors of n is

$$\prod_{1 \leq i \leq k} (p_i + 1)$$

and this quantity is $2n$ since n is perfect. Thus we must have

$$\frac{(p_1+1)(p_2+1)\cdots(p_k+1)}{p_1 p_2 \cdots p_k} = 2,$$

but this means that p_1 in the denominator must divide one of the $p_i + 1$ (as p_1 is prime.) Clearly p_1 does not divide $p_1 + 1$ and $p_i + 1 < p_1$ for $i \neq 1$ since $p_1 > 3$, thus we have a contradiction. □

In fact, the proof of the previous theorem begs us to generalize further and let us indulge it.

Theorem 0.2. *If n is an odd perfect number, then there is exactly one prime that divides it to an odd power.*

Proof : Let $n = \prod_i p_i^{k_i}$ be the prime factorization where the p_i are odd primes. We have

$$\sigma(n) = \prod_i \left(p_i^{k_i} + p_i^{k_i - 1} + \cdots + 1 \right)$$

and if n were perfect, we must have $\sigma(n) = 2n$. Now if k_i is odd then $p_i^{k_i} + \cdots + 1$ is even, since n is perfect this happens exactly once. □

Given the above result, the next one is not surprising (as if anything else was surprising):

Theorem 0.3. *Fix $\delta > 0$. If n is large enough and perfect, then n is $n^{\frac{1}{2}+\delta}$-smooth.*

Proof : Let n be a perfect number and assume that a prime $p > n^{\epsilon}$ divides n. Then reducing the sum of the divisors relation $\mod p$ we get

$$\sum_{d \mid n,\ (d,p)=1} d \equiv 0 \mod p.$$

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Thus we have the bounds

$$n^{\epsilon} \leq \sum_{d \mid n,(d,p)=1} d \leq n^{1-\epsilon+\delta}$$

for any δ, where we have used the fact the number of divisors of a number m is $o(m^{\delta})$ for any $\delta > 0$. The theorem then follows from this inequality for any $\epsilon > \frac{1}{2}$. $\square$

Notes: A stronger version of the first theorem was known to Euler. The third theorem is tight, since even perfect numbers look like $2^{p-1}(2^p - 1)$ with p and $2^p - 1$ both prime.

Acknowledgements: I thank insomnia and a hard bed for the above results. I should have learnt to count sheep.

Here is my song for the asking
Ask me and I will play...
This is my tune for the taking
Take it don't turn away...
- Simon & Garfunkel.