SQUAREFREE INTEGERS WITHOUT LARGE PRIME FACTORS IN SHORT INTERVALS

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ABSTRACT. We show that for every $\epsilon > 0$ there are squarefree integers that are free of prime factors $> \exp(\log^{\delta} X)$ in the interval $[X \cdots X + X^{\frac{1}{2}+\epsilon}]$ for all large enough X and δ sufficiently close to 1. The approach used is a variant of the methods used by Balog [Ba87] and by Harman [Har91] in their study of smooth integers in short intervals.

1. INTRODUCTION

The number of integers below x having no prime factors greater than y is denoted by $\Psi(x, y)$. Let $\Psi((x \cdots z], y) = \Psi(z, y) - \Psi(x, y)$. The behaviour of $\Psi((x \cdots x + x^{\epsilon}], y)$ is still largely a mystery for small ϵ. Friedlander and Lagarias considered this problem in [FL87] and showed that intervals of size $x^{1-2\alpha\left(1-2^{-\lfloor\frac{1}{\alpha}\rfloor}\right)}$ contain x^{α}-smooth integers. This result was later improved by Balog ([Ba87]) who showed the existence of x^{δ}-smooth integers in any interval of size $x^{\frac{1}{2}+\epsilon}$. Later Harman [Har91] went further by showing the existence of $\exp\left((\log X)^{\frac{2}{3}+\epsilon}\right)$-smooth integers in the same interval. Here we consider the question of the existence of integers which are both squarefree and $\exp\left(\log^{\delta} X\right)$-smooth in intervals of size $x^{\frac{1}{2}+\epsilon}$. We show that indeed there are such integers in these intervals using analytic arguments if δ is sufficiently close to 1. The approach is from Balog [Ba87] who considered a weighted sum of the smooth integers in the interval. This approach in turn was inspired by the work of Heath-Brown and Iwaniec [HI79]. We also employ the extra-averaging idea of Harman [Har91] in estimating this weighted sum. The following is known about the distribution of squarefree numbers without any smoothness restriction. We know that intervals of size $x^{\frac{1}{5}} \log x$ contain squarefree integers [FT92]. Recently Granville [Gr98] has shown the existence of squarefree integers in interval of size x^{ϵ} for every $\epsilon > 0$, if one assumes the ABC-conjecture. Our results indicate that the methods used to prove the existence of smooth numbers in short intervals are not sensitive to the contribution by numbers that are *not* squarefree. If we wish to reduce the smoothness bound significantly then we need to develop methods that take the contribution of the non-squarefree such integers into consideration.

Date: 31 Aug 2001.

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2000 *Mathematics Subject Classification:* 11N25.

2. Preliminary Results

Let

$$\begin{aligned} 0 < \epsilon < \frac{1}{2},\ 0 < \delta < 1 \\ Y = \frac{1}{2}X^{\frac{1}{2}+\epsilon}, \\ u = \left\lfloor \log^{1-\delta} X \right\rfloor + 1, \\ \mathcal{I}(x) = [x \cdots x + Y]. \end{aligned}$$

Define L by

$$L^u = X^{1-\gamma}$$

where $0 < \gamma < \frac{1}{u}$ (we will impose further restrictions on γ later).
Let

$$H(x) = \sum_{\substack{n \in \mathcal{I}(x) \\ n = mp_1p_2\cdots p_u \\ p_i \in (L\cdots eL], 1 \le i \le u.}} \mu(m)^2 \log p_1 \log p_2 \cdots \log p_u.$$

Note that by the choice of the parameters n is $\exp(\log X^\delta)$-smooth since $\gamma < \frac{1}{u}$ but the sum is over numbers some of which are not squarefree.
We will prove a lower bound for the integral:

$$\int_X^{X+Y} H(x)dx.$$

Let $P(s) = \sum_{L<p\le eL} \frac{\log p}{p^s}$.

Lemma 2.1. *Let* $s = \sigma + it$. *Then for* $X \ge t \ge T_0 = \exp\{(\log X)^\theta\}$, $0 < \theta < 1$ *and* $1 - \frac{1}{\log^\lambda X} \le \sigma \le 1$. *If* λ *is sufficiently close to* 1 *then there is a* μ, $0 < \mu < 1$ *such that*

$$|P(s)| \ll \exp\{-(\log X)^\mu\}.$$

Proof : We use the effective Perron's formula to estimate partial sums of $\sum_{1\le n} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}$. Now $-\frac{\zeta'(s)}{\zeta(s)} = \sum_{1\le n} \frac{\log p}{p^s} + f(s)$, where $f(s)$ converges for $\sigma > \frac{1}{2}$. In particular since

$$\sum_{L\le p^m \le eL, m>1} \frac{\log p}{p^{ms}} \ll \frac{\log^2 L}{L^{\frac{1}{2}}}$$

so we can ignore the contribution from the higher prime powers in this interval.
From [TH86] (p. 60-63) we have

$$\sum_{n<eL} \frac{\Lambda(n)}{n^s} = \frac{-1}{2\pi i}\int_{c-iT}^{c+iT} \frac{\zeta'(s+w)}{\zeta(s+w)}\frac{x^w}{w}dw + O\left(\frac{L^c}{Tc}\right) + O\left(\frac{\log^2 L}{T}\right).$$

In this case we take $c = \frac{h}{\log^\delta X}$, where $h > 1$. Shifting the contour of integration to: $\{\sigma \pm iT \mid -f < \sigma < c\} \cup \{-f + it \mid |t| \le T\}$ where f is picked such that no other poles apart from the one at $w = 0$ and at $w = 1 - s$ are introduced.

We get

$$\sum_{L<p\le eL}\frac{\log p}{p^s}=\frac{(eL)^{1-s}-L^{1-s}}{1-s}+O\left(\frac{L^c}{Tc}\right)+\left(\frac{\log^2 L}{T}\right)+O\left(\frac{L^c\log^9 T}{T}\right)+O\left(\frac{\log^{10} T}{L^f}\right)$$

and now the lemma follows if $T=\exp\{\log^{\frac{1}{10}} X\}$ and $\lambda>\frac{9}{10}$. □

We will also make use of the following theorem from the theory of Mean-values of Dirichlet polynomials see [Mon71] Chapters 6 and 9.

Theorem 2.2. *Let* b_n *be any sequence of complex numbers, and* $S(s)=\sum_{1\le n\le N}\frac{b_n}{n^s}$. *Then*

$$\int_0^T\left|\sum_{1\le n\le N}\frac{b_n}{n^{it}}\right|^2 dt=\left(T+\theta\frac{4\pi}{\sqrt{3}}N\right)\left(\sum_{1\le n\le N}|b_n|^2\right)$$

where $|\theta|\le 1$.

3. Proof of the theorem

Theorem 3.1. *Let* X, Y, u, δ *and* $H(x)$ *be the quantities defined earlier. If* δ *is sufficiently close to* 1 *then*

$$\int_X^{X+Y} H(x)dx \gg b^{-u}Y^2$$

for some constant $b>0$.

Proof : Using Perron's formula we have

$$H(x)=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{\zeta(s)}{\zeta(2s)}P^u(s)\left\{\frac{(x+Y)^s-x^s}{s}\right\}ds.$$

Thus we have

$$\int_X^{X+Y} H(x)dx=\frac{1}{2\pi i}\int_{2-i\infty}^{2+\infty}\frac{\zeta(s)}{\zeta(2s)}P^u(s)A(s)ds$$

where

$$A(s)=\frac{(X+2Y)^{s+1}-2(X+Y)^{s+1}+X^{s+1}}{s(s+1)}$$

by a justifiable interchange of the integrals. We note that

$$A(s)\ll\min\{Y^2X^{\sigma-1},X^{1+\sigma}|t|^{-2}\}$$

this is display (11) in [Har91].
We now shift the contour of integration to $\mathcal{C}=\mathcal{C}_1\cup\mathcal{C}_2\cup\mathcal{C}_3$. Let $T_0=\exp{(\log X)^\theta}$, $a=\frac{d}{\log^\lambda X}$, $d>0$, where $\frac{9}{10}<\lambda<1$, d is a constant and

$$\mathcal{C}_1=\{1+it \mid |t|\ge T_0\}$$
$$\mathcal{C}_2=\left\{\sigma+it\;\middle|\;|t|=T_0, 1-a\le\sigma\le 1\right\}$$
$$\mathcal{C}_3=\left\{1-a+it\;\middle|\;|t|\le T_0\right\}.$$

The only pole encountered in the shifting is at $s = 1$. Thus by the theorem of residues:

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} = \frac{Y^2}{\zeta(2)}P^u(1) + \frac{1}{2\pi i}\left\{\int_{\mathcal{C}_1} + \int_{\mathcal{C}_2} + \int_{\mathcal{C}_3}\right\}.$$

Since $P(1)$ is a constant, we have that $\frac{Y^2P^u(1)}{\zeta(2)} \gg Y^2b^{-u}$ for some positive constant b. The theorem will follow if we can show that the integral over the contour is asymptotically smaller than Y^2b^{-u}.

Contour $\mathcal{C}_1 : 1+it, |t| > X$: We use the following estimates:

$$\begin{aligned} A(1+it) &\ll \frac{X^2}{|t|^2} \\ P(s) &\ll 1, \\ \zeta(1+it) &\ll \log|t| \\ \frac{1}{\zeta(2+it)} &\ll 1. \end{aligned}$$

Thus

$$\begin{aligned} \int_{1+it\ :\ |t|\geq X} &\ll X^2\int_{t\geq X}\frac{\log t}{t^2}dt \\ &\ll \frac{X^2\log X}{X} \text{ by partial integration} \\ &\ll Y^{2-2\epsilon}\log X. \end{aligned}$$

Contour $\mathcal{C}_1 : 1+it, \frac{X}{Y} < |t| \leq X$: Assume that $u = 2k+1$.

$$\begin{aligned} \int_{1+it\ :\ \frac{X}{Y}<|t|\leq X} &\ll X^2\log X\int\frac{P(1+it)^{u-1}}{t^2}P(1+it)dt \\ &\ll X^2\log X\int\frac{P(1+it)^{2k}}{t^2}\exp(-(\log X)^{\mu})dt \text{ by Lemma (2.1),} \\ &\ll X^2\log X\exp(-(\log X)^{\mu})\left[\frac{1}{T^2}\int_0^T P(1+it)^{2k}dt\right]_{\frac{X}{Y}}^{X} \text{ by partial integration} \\ &\ll Y^2\log X\exp(-(\log X)^{\mu})\left[\left(\frac{X}{Y}+L^k\right)\frac{\log^{2k}X}{L^k}\right] \end{aligned}$$

by Theorem (2.2) applied to the function $P(1+it)^k$

$$\ll Y^2\log^{1+2k}X\exp(-(\log X)^{\mu})\left(X^{-\epsilon+\frac{\gamma}{2}+\frac{1}{2u}}\right).$$

Now $\log^u X = \exp(u\log\log X)$, $\frac{1}{u} = o(1)$ and if γ is small then this term is asymptotically smaller (provided δ is close to 1) than the leading term.

Suppose $u = 2k$ then we split the product $P(s)^u$ into two parts one which has a square term and another with $P(s)^{2(k-1)}$ and proceed as above.

Contour $\mathcal{C}_1 : 1 + it, \exp(\log X)^\theta < |t| < \frac{X}{Y}$: In this case we use the upper bound $A(1+it) \ll Y^2$ and proceed as in the previous region of the contour.
Thus

$$\int_{\exp(\log X)^\theta < |t| < \frac{X}{Y}} \ll Y^2 \log X \exp\big(-(\log X)^\mu\big) \left[(T + L^k) \frac{\log^{2k} X}{L^k} \right]_{\exp(\log X)^\theta}^{\frac{X}{Y}}$$
$$\ll Y^2 \log^{1+2k} X \exp\big(-(\log X)^\mu\big).$$

Contour $\mathcal{C}_2 : \sigma + iT_0, 1 - a \le \sigma \le 1$: In this region we use $\zeta(\sigma + it) = O(\log t)$ and $\frac{1}{\zeta(1+it)} = O(\log t)$. Also $A(s) \ll Y^2 X^{\sigma-1}$ and $P(s) \ll \exp\{-\log^\mu X\}$, because if X is large enough we can use Lemma (2.1).

$$\int_{1-a+iT_0}^{1+iT_0} \ll Y^2 \log^2 X \ \exp\{-u \log^\mu X\}.$$

Contour $\mathcal{C}_3 : 1 - a + it, |t| \le T_0$:
Here we use the $A(s) \ll Y^2 X^{\sigma-1}$, and $P(s) \ll L^a$, and the same bounds on the zeta function as in $\mathcal{C}_2$. Thus we get

$$\int_{\mathcal{C}_3} \ll \frac{Y^2}{X^a} L^{au} \log^2 X.$$

Now $L^{au} = X^{a(1-\gamma)}$, so we get

$$\int_{\mathcal{C}_3} \ll \frac{Y^2 \log X}{X^{-a\gamma}}$$
$$\ll Y^2 \exp\big(-\log^{\delta-\lambda} X + \log\log X\big) \text{ if we chose } \gamma \text{ close to } \frac{1}{u}.$$

Now this term is also asymptotically smaller than the leading term.

Thus the integral over the contour is indeed asymptotically smaller than the leading term of the integral and the theorem follows. □

Corollary 3.2. *There is a squarefree* $\exp(\log X^\delta)$*-smooth integer in the interval* $[X \cdots X + X^{\frac{1}{2}+\epsilon}]$ *if* $\delta < 1$ *is sufficiently close to* 1.

Proof : Let $Y = X^{\frac{1}{2}+\epsilon}$. By theorem 3.1 we know that there is an interval $I(x)$ with $X \le x \le X + Y$ such that $H(x) \gg Yb^{-u}$. The maximum weight given to any integer in this interval is $O(\log^u X) \times O(u!) = O(\log^{2u} X)$ since $u!$ is a bound on the number of times any integer involved in the sum $H(x)$ is counted. We immediately infer that the number of integers of the form $mp_1 \cdots p_u$, where $p_i \in (L \cdots eL]$ and $m \le X^\gamma$ is squarefree is $\gg \frac{Yb^{-u}}{\log^{2u} X} = Y\exp(-(\log b)u - 2u \log\log X) = Y\exp(-\log^{1-\delta} X(2\log\log X + \log b))$. Now the number of integers in the interval

$[X \cdots X+Y]$ that are divisible by a *square* of a prime $p \in (L \cdots eL]$ is at most

$$\sum_{p \in (L \cdots eL]} \left\lfloor \frac{Y}{p^2} \right\rfloor \ll \frac{YL}{L^2} + O(L)$$
$$\ll Y \exp\left(-\frac{(1-\gamma)}{u} \log X\right)$$
$$\ll Y \exp(-(1-\gamma)\log^{\delta} X).$$

Thus if δ is close to 1 then the number of integers involved in the sum $H(x)$ that are also squarefree is $\Omega\left(\frac{Yb^{-u}}{\log^{2u} X}\right)$. $\square$

Acknowledgements: I would like to thank professors Eric Bach, A. Balog and S. W. Graham for very useful discussions. Special thanks to professor Balog and the Referee for pointing out several improvements in a previous version of this paper.

References

[Ba87] Balog, A.; *On the distribution of integers having no large prime factor*, Astérisque **147-148**, 27-31, (1987).

[FT92] Filaseta, M.; Trifonov, O; *On gaps between squarefree numbers. II.*, J. London Math. Soc. (2), **45**, no. 2, 215-221, (1992).

[FL87] Friedlander, J., B.; Lagarias, J. C., *On the distribution in short intervals of integers having no large prime factor*, J. Number Theory, **25**, 249 - 273, (1987).

[Gr98] Granville, A.; ABC *allows us to count squarefrees.*, International Math. Res. Notices, no. 19, 991-1009, (1998).

[Har91] Harman, G.; *Short intervals containing numbers without large prime factors.*, Math. Proc. Camb. Phil. Soc., **109**, 1 - 5, (1991).

[HI79] Heath-Brown, D., R.; Iwaniec, H.; *On the difference between consecutive primes*, Inventiones Math., **55**, 49-69, (1979).

[Kar93] Karatsuba, A., A.; *Basic Analytic Number Theory*, Springer-Verlag, 1993.

[Mon71] Montgomery, H., L.; *Topics in Multiplicative Number Theory*, Springer-Verlag, (1971).

[TH86] Titchmarsh, E., C.; *The Theory of the Riemann Zeta-function*, Second Edition (revised by Heath-Brown, D. R.), Oxford at the Clarendon Press, (1986).