Advances and Applications in Statistics
© 2013 Pushpa Publishing House, Allahabad, India
Available online at http://pphmj.com/journals/adas.htm
Volume 36, Number 1, 2013, Pages 13-27

MOMENTS OF MATRIX VARIATE SKEW ELLIPTICALLY CONTOURED DISTRIBUTIONS

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Abstract

Matrix variate skew elliptically contoured distributions generalize several classes of important distributions. This paper defines and

Received: May 31, 2013; Accepted: June 24, 2013
2010 Mathematics Subject Classification: Primary 62H10; Secondary 62H05.
Keywords and phrases: matrix variate, skew elliptically contoured distribution, skew Pearson type VII distribution, skew normal distribution, stochastic representation, moment.

explores matrix variate skew elliptically contoured distributions. In particular, we discuss the first two moments of the matrix variate skew elliptically contoured distributions.

1. Introduction

The moments of matrix variate distributions are necessary both in theory and in applications, but often, their derivation is not straightforward and can constitute a work in and of itself. For example, Wen and Zhu derived stochastic representations and the first two moments, among other properties, of the multivariate skew Pearson type VII distribution and the skew *t*-distribution [1, 2]. In addition, Chen and Gupta extended several important properties of the multivariate skew normal distributions to the matrix variate case [3]. Also, Gupta et al. derived the first two moments of the multivariate skew normal distributions [4, 5, 6]. Furthermore, Akdemir and Gupta discussed more general matrix variate skew normal distributions, including moments [7].

In a previous work [8], we obtained the stochastic representations of the matrix variate skew elliptically contoured distributions (MSE). This paper extends that work by deriving the first two moments of the matrix variate skew elliptically contoured distributions (MSE).

In Section 2, we recall several definitions pertinent to the paper as well as the results from [8] for stochastic representations of matrix variate skew elliptically contoured distributions (MSE). In Section 3, we obtain moments of matrix variate skew elliptically contoured distributions.

2. Notation and Definitions

Matrix variate distributions have been studied by Gupta and Nagar, among others [9]. Below we include the results from [8] necessary for the derivation of moments. We refer the reader to [9] and [8] for additional definitions and theorems.

Let X be a $p \times m$ random matrix, which has a matrix variate normal distribution, i.e., $X \sim N_{p,m}(M, \Sigma \otimes \Psi)$, where $M \in \mathbb{R}^{p \times m}$ is a $p \times m$ mean matrix, Σ is a $p \times p$ positive definite matrix and Ψ is an $m \times m$ positive definite matrix. We use bold face upper and lower case letters to denote vectors, uppercase letters without bold face for matrices, and lowercase letters without bold face for elements of a vector or a matrix.

We list below the theorems and corollaries from [8] which are used in the following section.

Theorem 1 (Theorem 2 in [8]). *Assume*

$$V = \begin{pmatrix} \mathbf{V}_0 \\ V_1 \end{pmatrix} \sim E_{p+1,m}(0, \Sigma^* \otimes \Psi, \psi),$$

$$\Sigma^* = \begin{pmatrix} 1 & 0 \\ 0 & \Sigma \end{pmatrix}.$$

Σ *is* $p \times p$, Ψ *is* $m \times m$, $\mathbf{V}_0$ *is* $1 \times m$, V_1 *is* $p \times m$.

For $\delta_j \in (-1, 1)$, *let*

$$\begin{cases} \mathbf{X}_0 = |\mathbf{V}_0|, \mathbf{X}_j = \delta_j |\mathbf{V}_0| + (1-\delta_j^2)^{1/2} \mathbf{V}_j, \; j = 1, 2, \ldots, p, \\ |\mathbf{V}_0| = (|v_{01}|, |v_{02}|, \ldots, |v_{0m}|), \\ Z = (\mathbf{X}_0', X')', \; X = (\mathbf{X}_1', \ldots, \mathbf{X}_p')', \\ V^* = \begin{pmatrix} |\mathbf{V}_0| \\ V_1 \end{pmatrix}, \\ p_c = \int_{\mathbf{X}_0 < 0} |\Psi|^{-1/2} h(tr \mathbf{X}_0 \Psi^{-1} \mathbf{X}_0') d\mathbf{X}_0. \end{cases}$$

Then the random matrix X has a matrix variate skew elliptically contoured distribution. Specifically,

$$X \sim MSE_{p,m}(\Sigma_\delta^{-1}\delta, \Sigma_\delta, \Psi, \Omega_{11.2}\Psi, \psi),$$

where

$$\delta = (\delta_1, \delta_2, \ldots, \delta_p)', \; \Sigma_\delta = \delta\delta' + \Delta\Sigma\Delta, \; \Delta = diag\left((1-\delta_1^2)^{\frac{1}{2}}, \ldots, (1-\delta_p^2)^{\frac{1}{2}} \right),$$

and $\Omega_{11.2} = 1 - \delta'\Sigma_\delta^{-1}\delta.$

In the following corollaries, we use the notation and definitions found in Theorem 1.

Corollary 1. *Assume* $V = \begin{pmatrix} \mathbf{V}_0 \\ V_1 \end{pmatrix} \sim N_{p+1,m}(0, \Sigma^* \otimes \Psi)$. *Then the random matrix X has a matrix variate skew normal distribution. The pdf of the random matrix X can be written as*

$$f(X) = \left[\int_{\mathbf{X}_0<0} \det(\Psi)^{-\frac{1}{2}} (2\pi)^{-\frac{1}{2}m} \exp\left\{ -\frac{1}{2} \mathbf{X}_0 \Psi^{-1} \mathbf{X}_0' \right\} d\mathbf{X}_0 \right]^{-1}$$

$$\times \det(\delta\delta' + \Delta\Sigma\Delta)^{-\frac{1}{2}m} \det(\Psi)^{-\frac{1}{2}p} (2\pi)^{-\frac{1}{2}pm} \exp\left\{ -\frac{1}{2} tr(X'\Sigma_\delta^{-1} X \Psi^{-1}) \right\}$$

$$\times \int_{\mathbf{X}_0<0} \det(\Omega_{11.2})^{-\frac{1}{2}m} \det(\Psi)^{-\frac{1}{2}} (2\pi)^{-\frac{1}{2}m}$$

$$\times \exp\left\{ -\frac{1}{2} tr((\mathbf{X}_0 - \delta'\Sigma_\delta^{-1} X')' \Omega_{11.2}^{-1} (\mathbf{X}_0 - \delta'\Sigma_\delta^{-1} X) \Psi^{-1}) \right\} d\mathbf{X}_0$$

$$= \left[\int_{\mathbf{X}_0<0} \phi_m(X_0, 0, \Psi) d\mathbf{X}_0 \right]^{-1} \phi_{p,m}(X, 0, \Sigma_\delta \otimes \Psi)$$

$$\times \Phi_m(X'\Sigma_\delta^{-1}\delta, 0, \Omega_{11.2}\Psi)$$

which is denoted by $MSN_{p,m}(\Sigma_\delta^{-1}\delta, \Sigma_\delta, \Psi, \Omega_{11.2}\Psi)$.

Corollary 2. *Assume* $\begin{pmatrix} \mathbf{V}_0 \\ V_1 \end{pmatrix} \sim PVII_{p+1,m}(0, \Sigma^*, \Psi, q, r)$. *Then the random matrix X has a matrix variate skew Pearson type VII distribution. The pdf of the random matrix X can be written as*

$$f(X) = \left[\int_{\mathbf{X}_0 < 0} \frac{\Gamma(q)}{(\pi\lambda)^{m/2}\Gamma(q - m/2)|\,\Psi\,|^{1/2}} \left(1 + \frac{\mathbf{X}_0 \Psi^{-1} \mathbf{X}_0'}{\lambda}\right)^{-q} d\mathbf{X}_0 \right]^{-1}$$

$$\times \frac{\Gamma(q)}{(\pi\lambda)^{pm/2}\Gamma(q - pm/2)|\,\Sigma_\delta\,|^{m/2}|\,\Psi\,|^{p/2}} \left(1 + \frac{tr(X'\Sigma_\delta^{-1} X \Psi^{-1})}{\lambda}\right)^{-q}$$

$$\times \int_{\mathbf{X}_0 < 0} \frac{\Gamma(q)}{(\pi\lambda)^{m/2}\Gamma(q - m/2)|\,\Omega_{11.2}\Psi\,|^{1/2}}$$

$$\times \left\{1 + \frac{[\mathbf{X}_0 - \delta'(\Sigma_\delta)^{-1} X](\Omega_{11.2}\Psi)^{-1}[\mathbf{X}_0 - \delta'(\Sigma_\delta)^{-1} X]'}{\lambda}\right\}^{-q} d\mathbf{X}_0.$$

This matrix variate skew Pearson type VII distribution is denoted by

$$MSPVII_{p,m}(\Sigma_\delta^{-1}\delta,\ \Sigma_\delta,\ \Psi,\ \Omega_{11.2}\Psi,\ q,\ \lambda).$$

In the next theorem, we use the same notation and definitions as in Theorem 1.

Theorem 2 (Theorem 3 in [8]). *Assume*

$$V = \begin{pmatrix} \mathbf{V}_0 \\ V_1 \end{pmatrix} \sim E_{p+1,m}(0,\ \Sigma^* \otimes \Psi,\ \psi).$$

Also assume that $\Sigma = AA'$, *and* $\Psi = BB'$. *For* $\delta_j \in (-1, 1)$, *let*

$$\begin{pmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_p \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_p \end{pmatrix} |\,\mathbf{V}_0\,| + \Delta \begin{pmatrix} \mathbf{V}_1 \\ \vdots \\ \mathbf{V}_p \end{pmatrix}, \text{ or } X = \delta|\,\mathbf{V}_0\,| + \Delta V_1.$$

Then the random matrix X has the following stochastic representation:

$$X = R(\delta|\,\mathbf{U}_1\,| + \Delta A U_2)B', \tag{1}$$

where R *is a nonnegative random variable,* $\mathbf{U}_1$ *is* $1 \times m$ *and uniformly distributed on sphere* S_m, U_2 *is* $p \times m$ *and* $vec(U_2')$ *is uniformly distributed* *on sphere* S_{pm}. *In addition*, R, $\mathbf{U}_1$, *and* U_2 *are independent*. *Furthermore*, $\psi(x) = \int_0^\infty \Omega_{pm}(r^2 x) dF(r)$, $x \geq 0$, *where* $\Omega_{pm}(\mathbf{t}'\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^{pm}$ *denotes the characteristic function of* $vec(U_2')$, *and* $F(r)$ *denotes the distribution function of* r.

The next corollary follows immediately. Moreover, its statement uses the same notation as in Theorem 2 and also assumes that $\Sigma = AA'$, $\Psi = BB'$.

Corollary 3. *Assume* $V = \begin{pmatrix} \mathbf{V}_0 \\ V_1 \end{pmatrix} \sim N_{p+1,m}(0, \Sigma^* \otimes \Psi)$. *For* $\delta_j \in (-1, 1)$, *let*

$$\begin{pmatrix} \mathbf{Z}_1 \\ \vdots \\ \mathbf{Z}_p \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_p \end{pmatrix} | \mathbf{V}_0 | + \Delta \begin{pmatrix} \mathbf{V}_1 \\ \vdots \\ \mathbf{V}_p \end{pmatrix}, \text{ or } Z = \delta | \mathbf{V}_0 | + \Delta V_1.$$

Then the random matrix Z is distributed as matrix variate skew normal and has the following stochastic representation:

$$Z = R_0(\delta | \mathbf{U}_1 | + \Delta A U_2) B', \tag{2}$$

where $\mathbf{U}_1$, U_2, A, B *are the same as in Theorem* 2.

Based on the definitions and the stochastic representation obtained in [8], the moments of matrix variate skew elliptically contoured distribution can be derived, as we now show.

3. Moments of Matrix Variate Skew Elliptically Contoured Distribution

The moments of matrix variate skew elliptical distributions can be found using the moments of the matrix variate skew normal distribution since the latter is easy to find. Suppose that the random matrix X is distributed as a general matrix variate skew elliptically contoured distribution (MCD) such as in Theorem 2 and also suppose that the random matrix Z is distributed as a matrix variate skew normal as in Corollary 3. Then the moments of the skew

MCD *X* are associated with the moments of skew normal *Z*. Specifically, we have the following relationship between them:

$$\mathbb{E}\left(\prod_{i=1}^{p}\prod_{j=1}^{m} X_{ij}^{s_{ij}}\right) = \frac{\mathbb{E}\left(R^{\sum_{i=1}^{p}\sum_{j=1}^{m} s_{ij}}\right)}{\mathbb{E}\left(R_0^{\sum_{i=1}^{p}\sum_{j=1}^{m} s_{ij}}\right)} \mathbb{E}\left(\prod_{i=1}^{p}\prod_{j=1}^{m} Z_{ij}^{s_{ij}}\right), \tag{3}$$

where X_{ij} is the $(i,\ j)$ element of random matrix *X*, Z_{ij} is the $(i,\ j)$ coefficient of the random matrix *Z*, and R_0^2 is distributed as χ^2 with degrees of freedom *pm*. Equation (3) leads to the following results:

$$\mathbb{E}(X) = \frac{\mathbb{E}(R)}{\mathbb{E}(R_0)} \times \mathbb{E}(Z), \quad \mathbb{E}(XX') = \frac{\mathbb{E}(R^2)}{\mathbb{E}(R_0^2)} \times \mathbb{E}(ZZ'), \tag{4}$$

$$\mathbb{E}(vec(X') \otimes vec(X')') = \frac{\mathbb{E}(R^2)}{\mathbb{E}(R_0^2)} \times \mathbb{E}(vec(Z') \otimes vec(Z')'). \tag{5}$$

It also leads to our next theorem.

Theorem 3. *Suppose that the random matrix X is distributed as a matrix variate skew Pearson type VII defined in Corollary* 2, *i.e.*, $X \sim MSPVII_{p,m}(\Sigma_\delta^{-1}\delta,\ \delta\delta' + \Delta\Sigma\Delta,\ \Psi,\ \Omega_{11.2}\Psi,\ q,\ \lambda)$. *Suppose also that the random matrix Z is distributed as a matrix variate skew normal defined in Corollary* 1, *i.e.*, $Z \sim MSN_{p,m}(\Sigma_\delta^{-1}\delta,\ \Sigma_\delta,\ \Psi,\ \Omega_{11.2}\Psi)$. *Then we have*

$$\mathbb{E}(X) = \frac{\Gamma\left(q - \frac{(p+1)m+1}{2}\right)}{\Gamma\left(q - \frac{(p+1)m}{2}\right)} \sqrt{\frac{\lambda}{2}} \times \mathbb{E}(Z), \tag{6}$$

$$\mathbb{E}(XX') = \frac{\lambda}{2q - (p+1)m - 2} \times \mathbb{E}(ZZ'), \tag{7}$$

$$\mathbb{E}[vec(X') \otimes vec(X')'] = \frac{\lambda}{2q - (p+1)m - 2} \times \mathbb{E}[vec(Z') \otimes vec(Z')']. \tag{8}$$

Proof. Based on (2.5.17) on p. 59 of [10], the pdf of the random variable R can be found using the function h defined in (6) of [8]:

$$g(r) = \frac{2\pi^{\frac{1}{2}(p+1)m}}{\Gamma\left(\frac{1}{2}(p+1)m\right)} r^{(p+1)m-1} h(r^2)$$

$$= \frac{2\pi^{\frac{1}{2}(p+1)m}}{\Gamma\left(\frac{1}{2}(p+1)m\right)} r^{(p+1)m-1} \frac{\Gamma(q)}{(\pi\lambda)^{\frac{(p+1)m}{2}} \Gamma\left(q - \frac{(p+1)m}{2}\right)} \left(1 + \frac{r^2}{\lambda}\right)^{-q}$$

$$= \frac{2}{\Gamma\left(\frac{1}{2}(p+1)m\right)} \frac{\Gamma(q) r^{(p+1)m-1}}{\lambda^{\frac{(p+1)m}{2}} \Gamma\left(q - \frac{(p+1)m}{2}\right)} \left(1 + \frac{r^2}{\lambda}\right)^{-q}.$$

Thus,

$$\mathbb{E}(R) = \int_0^{+\infty} r g(r) dr$$

$$= \frac{2\Gamma(q)}{\lambda^{\frac{(p+1)m}{2}} \Gamma\left(\frac{(p+1)m}{2}\right) \Gamma\left(q - \frac{(p+1)m}{2}\right)} \int_0^{+\infty} r^{(p+1)m} \left(1 + \frac{r^2}{\lambda}\right)^{-q} dr$$

$$= \frac{\Gamma\left(q - \frac{(p+1)m+1}{2}\right) \Gamma\left(\frac{(p+1)m+1}{2}\right)}{\Gamma\left(q - \frac{(p+1)m}{2}\right) \Gamma\left(\frac{(p+1)m}{2}\right)} \lambda^{\frac{1}{2}}.$$

Next, we find the expectation of R_0. By (2.5.17) on p. 59 of [10], we have

$$g(r_0) = \frac{2\pi^{\frac{1}{2}(p+1)m}}{\Gamma\left(\frac{1}{2}(p+1)m\right)} r_0^{(p+1)m-1} h(r_0^2)$$

$$= \frac{2\pi^{\frac{1}{2}(p+1)m}}{\Gamma\left(\frac{1}{2}(p+1)m\right)} r_0^{(p+1)m-1} \left(\frac{1}{(2\pi)^{\frac{(p+1)m}{2}}} \right) e^{-\frac{1}{2}r_0^2}.$$

Let $s = r_0^2$. Then

$$g(s) = \left(\Gamma\left(\frac{m}{2}\right) 2^{\frac{1}{2}(p+1)m} \right)^{-1} s^{\frac{(p+1)m}{2}-1} e^{-\frac{1}{2}s}.$$

Therefore, S or $R_0^2 \sim \chi^2((p+1)m)$ and

$$\mathbb{E}(R_0) = \mathbb{E}(\sqrt{S})$$

$$= \int_0^{+\infty} \left(\Gamma\left(\frac{1}{2}(p+1)m\right) 2^{\frac{1}{2}(p+1)m} \right)^{-1} s^{\frac{(p+1)m}{2}-1+\frac{1}{2}} e^{-\frac{1}{2}s} ds$$

$$= \frac{\Gamma\left(\frac{(p+1)m+1}{2}\right)}{\Gamma\left(\frac{(p+1)m}{2}\right)} \sqrt{2},$$

$$\frac{\mathbb{E}(R)}{\mathbb{E}(R_0)} = \frac{\Gamma\left(q - \frac{(p+1)m+1}{2}\right)}{\Gamma\left(q - \frac{(p+1)m}{2}\right)} \sqrt{\frac{\lambda}{2}}. \tag{9}$$

On the other hand,

$$\mathbb{E}(R^2) = \int_0^{+\infty} r^2 g(r) = \frac{(p+1)m}{2q-(p+1)m-2}\lambda, \text{ and } \mathbb{E}(R_0^2) = (p+1)m,$$

$$\frac{\mathbb{E}(R^2)}{\mathbb{E}(R_0^2)} = \frac{\lambda}{2q-(p+1)m-2}. \tag{10}$$

Hence, equations (6), (7) and (8) are proved using (9) and (10). □

In order to use Theorem 3 to find the first two moments of any matrix variate skew elliptically contoured distribution, we have to first find the first two moments of matrix variate skew normal distributions. Gupta et al. derived the first two moments of multivariate skew normal distributions [4, 5, 6]. To generalize these results to matrix variate skew normal distribution, we utilize moment generating functions. In particular, Arellano-Valle and Azzalini derived the moment generating function (mgf) of a multivariate unified skew normal distribution [11]. Harrar and Gupta presented the mgf for general matrix variate skew normal distributions [12]. The following lemma is from Harrar and Gupta [12].

Lemma 1. *Assume* $X \sim MSN_{p,m}(\mathbf{b}, \Sigma, \Psi, \Omega)$. *Then the moment generating function of X can be written as*

$$M_X(T) = c \times etr\left\{\frac{1}{2}\Sigma T\Psi T'\right\}\Phi_m(\Psi T'\Sigma\mathbf{b},\ \Omega + \mathbf{b}'\Sigma\mathbf{b}\Psi), \tag{11}$$

where $c = [\Phi_m(0;\ \Omega + \mathbf{b}'\Sigma\mathbf{b}\Psi)]^{-1}$.

In order to simplify notation in the sequel, we let $\Xi = \Omega + \mathbf{b}'\Sigma\mathbf{b}\Psi$ and $\boldsymbol{\theta} = \Psi T'\Sigma\mathbf{b}$. Note that $\boldsymbol{\theta} = vec(\Psi T'\Sigma\mathbf{b}) = [(\mathbf{b}'\Sigma) \otimes \Psi] \times vec(T')$. Furthermore, we let $\mathbf{t} = vec(T')$, $D = (\mathbf{b}'\Sigma) \otimes \Psi$.

Theorem 4. *Suppose* $X \sim MSN_{p,m}(\mathbf{b}, \Sigma, \Psi, \Omega)$. *Then the expectation of X can be written as*

$$\mathbb{E}(X) = [\Phi_m(0;\ \Xi)]^{-1} \times \sum_{i=1}^{m}\sum_{j=1}^{p} G_{ij}^{mp}(0;\ D,\ \Xi) \times H_{ij}, \tag{12}$$

where the matrix $H_{ij}(m \times p)$ *has unit element at the* $(i,\ j)th$ *place and zero elsewhere and that*

$$G_{ij}^{mp}(0;\ D,\ \Xi) = \frac{\partial}{\partial t_{ij}}\Phi_m(D\mathbf{t};\ \Xi)\,|_{T=0}$$

$$= \frac{1}{\sqrt{2\pi}|\,\Xi\,|^{1/2}}\sum_{k=1}^{m} D_{k,\,m(i-1)+j}|\,(\Xi_{(k)}^{-1})^{-1}\,|^{1/2}\,\Phi_{m-1}[0;\ (\Xi_{(k)}^{-1})^{-1}],$$

where $D_{k,\, m(i-1)+j}$ *is the element* $(k,\, m(i-1)+j)$ *of matrix D,* $\Xi^{-1}_{(k)}$ *is the matrix constructed by eliminating the kth row and the kth column of* Ξ^{-1}, *with the convention that* $\Xi^{-1}_{(k)} = (\Xi^{-1})_{(k)}$.

Proof. Since $\frac{\partial tr(\Sigma T \Psi T')}{\partial T} = 2\Sigma T \Psi,$ the derivative with respect to T of the moment generating function (11) can be written in the form

$$\frac{\partial M_X(T)}{\partial T} = c \times etr\left\{\frac{1}{2}\Sigma T \Psi T'\right\}(\Sigma T \Psi) \times \Phi_m(\boldsymbol{\theta},\, \Xi) + c \times etr\left\{\frac{1}{2}\Sigma T \Psi T'\right\} \times \frac{\partial}{\partial T}\Phi_m(\boldsymbol{\theta},\, \Xi). \tag{13}$$

After a lengthy calculation (the reader is referred to similar derivations in [4, 5, 6] for details), we have

$$\mathbb{E}(X) = \left.\frac{\partial M_X(T)}{\partial T}\right|_{T=0} = [\Phi_m(0;\, \Xi)]^{-1} \times \left.\frac{\partial}{\partial T}\Phi_m(\boldsymbol{\theta},\, \Xi)\right|_{T=0},$$

where

$$\left.\frac{\partial}{\partial T}\Phi_m(\boldsymbol{\theta},\, \Xi)\right|_{T=0} = \left.\frac{\partial}{\partial T}\Phi_m(D\mathbf{t};\, \Xi)\right|_{T=0}$$

and

$$\left.\frac{\partial}{\partial T}\Phi_m(D\mathbf{t};\, \Xi)\right|_{T=0} = \sum_{i=1}^{m}\sum_{j=1}^{p} G_{ij}^{mp}(0;\, D,\, \Xi) H_{ij}.$$

Therefore, the theorem is proved. □

Finally, we have the following theorem, whose proof – like that of Theorem 4 – requires a long elementary calculation of a second derivative which is similar in details to derivations in [4, 5, 6].

Theorem 5. *Suppose* $X \sim MSN_{p,m}(\mathbf{b}, \Sigma, \Psi, \Omega)$. *Then the expectation of* $vec(X') \otimes vec(X')'$ *can be written as*

$$\mathbb{E}(vec(X') \otimes vec(X')')$$

$$= \Sigma \otimes \Psi + [\Phi_m(0; \Xi)]^{-1} \times \sum_{i'=1}^{pm} \sum_{j'=1}^{pm} G_{i'j'}^{mp}(0; D, \Xi) \times H_{i'j'},$$

where the matrix $H_{i'j'}(pm \times pm)$ *has unit element at the* $(i', j')th$ *place and zero elsewhere,*

$$i' = m(i-1) + j,\ j' = m(a-1) + b;\ i,\ a = 1, ..., m;\ j, b = 1, ..., p;$$

and

$$G_{i'j'}^{pm}(0, D, \Xi)$$

$$= \frac{\partial}{\partial t_{ij} \partial t_{ab}} \Phi_m(D\mathbf{t}, \Xi)\Big|_{\mathbf{t}=0}$$

$$= \frac{1}{2\pi |\Xi|^{1/2}} \sum_{k=1}^{m} \sum_{l \neq k}^{m} D_{l, m(i-1)+j} D_{k, m(a-1)+b} |\Xi_{(k,l)}^{-1}| \Phi_{m-2}(0; [\Xi_{(k,l)}^{-1}]^{-1})$$

$$+ \sum_{k=1}^{m} \sum_{\substack{i=1 \\ i \neq k}}^{m} \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} D_{l, m(i-1)+j} D_{k, m(a-1)+b} \Xi^{ki} x_i$$

$$\times \phi_{m-1}[x_1, ..., x_{k-1}, x_{k+1}, ..., x_m; (\Xi_{(k)}^{-1})^{-1}] dx_1 \cdots dx_{k-1} dx_{k+1} \cdots dx_m,$$

where Ξ^{kk} *is the element* (k, k) *of* Ξ^{-1} *and* $\Xi_{(i,j)}^{-1}$ *is the matrix constructed by eliminating the ith and jth rows and the ith and jth columns of* Ξ^{-1}, *with* $\Xi_{(i,j)}^{-1} = (\Xi^{-1})_{(i,j)}$.

Proof. Based on (11), we have

$$\frac{\partial M_X(T)}{\partial vec(T')} = c \times etr\left\{\frac{1}{2}\Sigma T \Psi T'\right\} \frac{\partial tr\left(\frac{1}{2}\Sigma T \Psi T'\right)}{\partial vec(T')} \Phi_m(\boldsymbol{\theta}, \Xi)$$

$$= c \times etr\left\{\frac{1}{2}\Sigma T \Psi T'\right\} \frac{\partial}{\partial vec(T')} \Phi_m(\boldsymbol{\theta}, \Xi).$$

Thus,

$$\frac{\partial M_X(T)}{\partial vec(T')\partial vec(T')'}$$

$$= c \times etr\left\{\frac{1}{2}\Sigma T \Psi T'\right\} \frac{\partial tr\left(\frac{1}{2}\Sigma T \Psi T'\right)}{\partial vec(T')} \otimes \frac{\partial tr\left(\frac{1}{2}\Sigma T \Psi T'\right)}{\partial vec(T')'} \Phi_m(\boldsymbol{\theta}, \Xi)$$

$$+ c \times etr\left\{\frac{1}{2}\Sigma T \Psi T'\right\} \frac{\partial tr\left(\frac{1}{2}\Sigma T \Psi T'\right)}{\partial vec(T')\partial vec(T')'} \Phi_m(\boldsymbol{\theta}, \Xi)$$

$$+ c \times etr\left\{\frac{1}{2}\Sigma T \Psi T'\right\} \frac{\partial \Phi_m(\boldsymbol{\theta}, \Xi)}{\partial vec(T')} \otimes \frac{\partial tr\left(\frac{1}{2}\Sigma T \Psi T'\right)}{\partial vec(T')'}$$

$$+ c \times etr\left\{\frac{1}{2}\Sigma T \Psi T'\right\} \frac{\partial^2 \Phi_m(\boldsymbol{\theta}, \Xi)}{\partial vec(T')\partial vec(T')'}$$

$$= c \times etr\left\{\frac{1}{2}\Sigma T \Psi T'\right\} \left\{ vec(\Psi T'\Sigma) \otimes (vec(\Psi T'\Sigma))' \Phi_m(\boldsymbol{\theta}, \Xi) \right.$$

$$+ (\Sigma \otimes \Psi)\Phi_m(\boldsymbol{\theta}, \Xi) + \frac{\partial \Phi_m(\boldsymbol{\theta}, \Xi)}{\partial vec(T')} \otimes (vec(\Psi T'\Sigma))'$$

$$\left. + \frac{\partial^2 \Phi_m(\boldsymbol{\theta}, \Xi)}{\partial vec(T')\partial vec(T')'} \right\}.$$

Hence,

$$\mathbb{E}[vec(X') \otimes vec(X')'] = \left. \frac{\partial M_X(T)}{\partial vec(T') \partial vec(T')'} \right|_{T=0}$$

$$= \Sigma \otimes \Psi + c \times \left. \frac{\partial^2 \Phi_m(\boldsymbol{\theta}, \Xi)}{\partial vec(T') \partial vec(T')'} \right|_{T=0}. \qquad \square$$

4. Concluding Remarks

The set of skew elliptically contoured distributions contains the elliptically contoured distributions. In [8], we defined and explored the matrix variate skew elliptically contoured distribution, obtaining in the process two stochastic representations. In this work, we have extended that effort by obtaining the first two moments of a subfamily of this large family of distributions.

In particular, the moment results obtained in this paper are useful in both theory and application. Future directions include expanding upon the utility and importance of these results.

Acknowledgments

The authors graciously acknowledge the financial and collaborative support of the Institute for Quantitative Biology at East Tennessee State University and Zhang's research is supported by the US NSF grant DMS-1106586 and Wisconsin Alumni Research Foundation. The authors would also like to thank the referees for many useful suggestions and comments which improved the presentation in this paper.

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