$$(u^{k(l+r)-n}x')x = u^{k(l+r)}$$

Therefore $u^{k(l+r)-n}x'$ is a left inverse of x with identity $u^{k(l+r)}$.

From the discussion above we know that G is a group.

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A note on beta function, parseval identity, and a family of integrals in non-parametric regression

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A family of integrals over infinite intervals appears frequently in the statistical literature on non-parametric regression. This note presents two simple transform methods, based on beta function and Parseval identity, to explicitly evaluate these integrals. These methods will benefit researchers and practitioners working in non-parametric regression, whenever they encounter similar types of problems to solve.

1. Introduction

A family of integrals over infinite intervals,

$$\int_0^\infty \frac{1}{(1+t^{2k})^r}\,\mathrm{d}t \tag{1.1}$$

for parameters $k > 0$ and $r > (2k)^{-1}$, as well as other integrals that can be expressed as a linear combination of forms (1.1), has frequently appeared in the statistical literature on non-parametric regression. In the setting of curve estimation, these integrals arise from deriving the asymptotic expressions of non-parametric function estimators. Similarly, in the context of hypothesis testing, these integrals emerge from calculating the asymptotic mean and variance of test statistics based on non-parametric estimation. Henceforth, information on the true values of (1.1) plays an important role in making non-parametric inference, evaluating relative asymptotic efficiency, simplifying technical arguments, and will bring great convenience to practitioners. Examples can be found in, for instance, Watson and Leadbetter ([1], p. 486), Wahba ([2], p. 390), Davis ([3], p. 533), Craven and Wahba ([4], p. 391), Rice and Rosenblatt ([5], pp. 148–149), Cox ([6], p. 544 and p. 548), Cox ([7], p. 800), Eubank and Spiegelman ([8], p. 389), Messer ([9], p. 828), Eubank and LaRiccia ([10], pp. 5–6), Chen ([11], p. 68), Jayasuriya ([12], p. 1628), and Ramil-Novo and González-Manteiga ([13], p. 237). Although this reference list is not exhaustive, it does indicate the extent to which statisticians need a simple and quick method to obtain explicit results of (1.1). It seems that statisticians of the work above may not have been aware of an analytically transparent expression for (1.1), except in certain special cases. Silverman ([14], p. 589) provides the value of some integral, $\int_0^\infty (1+t^4)^{-1} \mathrm{d}t$, with reference to (2.141.4) of Gradshteyn and Ryzhik ([15]). Formulas for a particular type of integrals,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{t^s}{1+t^{2k}}\right)^2 \mathrm{d}t = \frac{2k-2s-1}{4k^2 \sin\{\pi(2s+1)/2k\}} \tag{1.2}$$

can be found in Messer and Goldstein ([16], p. 187); there, the derivational detail is not mentioned. The explicit expressions for (1.1), when $r = 1, 2, 3, 4$, are given in Fan, Zhang and Zhang ([17], p. 178), who used the computational power of Mathematica that may not work easily in other cases of r.

In recent years, the non-parametric regression technique has been developing rapidly. It is anticipated that more applications of (1.1) will be encountered and developed by statisticians in their future research work. Our goal in this paper is to present two simple transform methods which can easily and conveniently be applied by researchers working in the area of non-parametric regression. The first method, addressed in section 2, uses the change-of-variable and beta function, and is applicable to all examples above. The second method, illustrated in section 3, offers a statistical perspective on (1.1), i.e. applying Parseval identity to the probability density function of a gamma random variable. As will be seen, both methods do not need efforts put into searching tables in voluminous books, do not require background knowledge of contour integration (which is non-trivially complicated and requires integer-valued k and r) in complex analysis, and do not rely on machine intelligence and advanced level of computing skills in numerical integration.

2. The change-of-variable approach: beta function

To illustrate the first approach, we will introduce an auxiliary function, denoted by

$$\mathcal{I}(k, r, \ell) = \int_0^{\infty}\left(\frac{1}{1+t^{2k}}\right)^r \left(\frac{t^{2k}}{1+t^{2k}}\right)^{\ell} \mathrm{d}t, \tag{2.1}$$

for valid parameters k, r, and ℓ. Consideration of (2.1) offers two advantages. First, when $\ell = 0$, (2.1) reduces to (1.1). Second, the right side of (2.1) can easily be obtained by making a change-of-variable transform. Namely, in (2.1), setting $1/(1+t^{2k}) = x$, i.e. $t = (1/x - 1)^{1/(2k)}$, leads to $\mathrm{d}t = 1/(2k)(1/x-1)^{1/(2k)-1}(-1/x^2)\mathrm{d}x$. Consequently,

$$\int_0^\infty \left(\frac{1}{1+t^{2k}}\right)^r \left(\frac{t^{2k}}{1+t^{2k}}\right)^\ell \mathrm{d}t = \int_0^1 x^r(1-x)^\ell \frac{1}{2k}\left(\frac{1-x}{x}\right)^{1/(2k)-1}\frac{1}{x^2}\,\mathrm{d}x$$

$$= \frac{1}{2k}\int_0^1 x^{r-1/(2k)-1}(1-x)^{\ell+1/(2k)-1}\mathrm{d}x \tag{2.2}$$

Notice that (2.2) is the integral representation of a beta function. Thus for $k > 0$, $r > (2k)^{-1}$, and $\ell > -(2k)^{-1}$, we obtain from (2.1) and (2.2) that

$$\mathcal{I}(k,r,\ell) = \frac{1}{2k}B\Big(r-\frac{1}{2k},\ell+\frac{1}{2k}\Big) = \frac{1}{2k}\Gamma\Big(r-\frac{1}{2k}\Big)\Gamma\Big(\ell+\frac{1}{2k}\Big)\Big/\Gamma(r+\ell) \tag{2.3}$$

where $B(\cdot,\cdot)$ and $\Gamma(\cdot)$ denote the beta function and the gamma function, respectively.

All the integrals in the special cases, considered by previous authors, can easily be deduced from the method for obtaining (2.3). To see this point, some illustrations are given below.

(1 For
$\ell = 0$, $r = 1$, and $k > 1/2$, result (2.3) indicates that

$$\int_0^\infty \frac{\mathrm{d}t}{1+t^{2k}} = \mathcal{I}(k,1,0) = \frac{1}{2k}\Gamma\Big(1-\frac{1}{2k}\Big)\Gamma\Big(\frac{1}{2k}\Big) = \frac{\pi/(2k)}{\sin\{\pi/(2k)\}} \tag{2.4}$$

in which validity of the last-step equality uses the reflection formula (cf. Abramowitz and Stegun, [18], p. 256),

$$\Gamma(1-z)\Gamma(z) = \pi/\sin(\pi z) \tag{2.5}$$

which holds for any $0 < z < 1$.

(2
More generally, for $\ell = 0$ and $r > (2k)^{-1}$, where $k > 0$, an immediate application of (2.3) to (1.1) leads to

$$\int_0^\infty \frac{1}{(1+t^{2k})^r}\,\mathrm{d}t = \mathcal{I}(k,r,0) = \frac{1}{2k}\frac{\Gamma(r-(1/2k))\Gamma(1/2k)}{\Gamma(r)} \tag{2.6}$$

Furthermore, if r, satisfying $r > (2k)^{-1}$, is an integer (≥ 2), then (2.6), combined with (2.5), becomes

$$(2k)^{-1}\left\{\prod_{j=1}^{r-1}\Big(j-\frac{1}{2k}\Big)\right\}\Gamma\Big(1-\frac{1}{2k}\Big)\Gamma\Big(\frac{1}{2k}\Big)\Big/\prod_{j=1}^{r-1} j = \prod_{j=1}^{r-1}\Big(1-\frac{1}{2kj}\Big)\frac{\pi/(2k)}{\sin\{\pi/(2k)\}}$$

$$= \frac{\Pi_{j=1}^{r-1}(2kj-1)}{(2k)^{r-1}(r-1)!}\,\frac{\pi/(2k)}{\sin\{\pi/(2k)\}} \tag{2.7}$$

Clearly, the explicit expressions (2.7) and (2.4) are very easy to use, because only the sine function needs to be evaluated.

In particular, for integers $r = 2, 3, 4$, (2.7) gives the identities,

$$\int_0^\infty \frac{1}{(1+t^{2k})^2}\,\mathrm{d}t = \frac{2k-1}{2k}\frac{\pi/(2k)}{\sin\{\pi/(2k)\}}$$

$$\int_0^\infty \frac{1}{(1+t^{2k})^3}\,\mathrm{d}t = \frac{(4k-1)(2k-1)}{8k^2}\frac{\pi/(2k)}{\sin\{\pi/(2k)\}}$$

$$\int_0^\infty \frac{1}{(1+t^{2k})^4}\,\mathrm{d}t = \frac{(6k-1)(4k-1)(2k-1)}{48k^3}\frac{\pi/(2k)}{\sin\{\pi/(2k)\}}$$

These results, along with (2.4), are exactly the same as given in ([17], p. 178).

(3 We

now use (2.3) to illustrate (1.2) given in [16]. Note that the integrand function in (1.2) can be written as,

$$\frac{t^{2s}}{(1+t^{2k})^2} = \frac{(t^{2k})^{s/k}}{(1+t^{2k})^2} = \left(\frac{1}{1+t^{2k}}\right)^{2-(s/k)}\left(\frac{t^{2k}}{1+t^{2k}}\right)^{s/k}$$

Thus the left side of (1.2) becomes

$$\begin{aligned}\frac{1}{\pi}\mathcal{I}(k, 2-s/k, s/k) &= \frac{1}{2k\pi}\Gamma\left(2-\frac{2s+1}{2k}\right)\Gamma\left(\frac{2s+1}{2k}\right)\Big/\Gamma(2)\\ &= \frac{1}{2k\pi}\left(1-\frac{2s+1}{2k}\right)\Gamma\left(1-\frac{2s+1}{2k}\right)\Gamma\left(\frac{2s+1}{2k}\right)\\ &= \frac{1}{2k\pi}\left(1-\frac{2s+1}{2k}\right)\frac{\pi}{\sin\{\pi(2s+1)/(2k)\}}\end{aligned}$$

which coincides with the right side of (1.2).

3. The Fourier-transform approach: Parseval identity

Fourier analysis is one of the most useful tools in statistical inference; see [19] for details and the references therein. In this section, we explore the applicability of Parseval identity, a well-known result in Fourier analysis, to evaluating (1.1). Given a function f, satisfying $\int_{-\infty}^{\infty}|f(x)|\mathrm{d}x < \infty$, its Fourier transform, $\phi(t) = \int_{-\infty}^{\infty} f(x)e^{itx}\mathrm{d}x$, can be well defined for every real t. Conversely, the knowledge of ϕ also enables f to be recovered, according to the inverse Fourier transform, $f(x) = (2\pi)^{-1}\int_{-\infty}^{\infty}\phi(t)e^{-itx}\mathrm{d}t$. Parseval identity establishes the connection between the L_2-norm of f and the L_2-norm of ϕ via the expression,

$$\int_{-\infty}^{\infty}|f(x)|^2\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\phi(t)|^2\mathrm{d}t \tag{3.1}$$

In statistics, if f is the probability density function of a random variable, then the Fourier transform ϕ of f, is also called the characteristic function of that random variable. This observation motivates us to study (1.1) via Parseval identity.

3.1. *The case when* $k=1$

To provide a simple illustration of how Parseval identity can be used to evaluate (1.1), we first consider the integral, $\int_0^\infty (1+t^2)^{-r}\mathrm{d}t$, where $r > 1/2$, corresponding to (1.1) with $k=1$. To this end, we will use the probability density function, of a gamma random variable with parameter r, expressed as

$$f(x) = \begin{cases} x^{r-1}e^{-x}/\Gamma(r), & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases} \tag{3.2}$$

The characteristic function of this random variable, via some straightforward calculus, is given by

$$\phi(t) = (1 - it)^{-r}, \qquad t \in (-\infty, \infty) \tag{3.3}$$

see also Evans, Hastings, and Peacock ([20], p. 55) and many other standard texts on mathematical statistics. We observe that

$$|\phi(t)|^2 = (1 + t^2)^{-r}$$

The Parseval identity (3.1), applied to f in (3.2) and ϕ in (3.3), yields

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{(1+t^2)^r} = 2\pi \int_0^\infty \left\{x^{r-1}e^{-x}/\Gamma(r)\right\}^2 \mathrm{d}x$$

$$= \frac{2\pi}{2^{2r-1}} \int_0^\infty \frac{y^{2r-2}e^{-y}}{\{\Gamma(r)\}^2}\mathrm{d}y = \frac{2\pi\Gamma(2r-1)}{2^{2r-1}\{\Gamma(r)\}^2}$$

and thus

$$\int_0^\infty \frac{1}{(1+t^2)^r}\mathrm{d}t = \mathcal{I}(1, r, 0) = \frac{\pi\Gamma(2r-1)}{2^{2r-1}\{\Gamma(r)\}^2} \tag{3.4}$$

Does $\mathcal{I}(1, r, 0)$ obtained from (3.4), via Parseval identity, coincide with the counterpart resulted from (2.6), via Beta function? In other words, could we equate $\sqrt{\pi}\,\Gamma(2r-1)$ with $2^{2(r-1)}\Gamma(r-1/2)\Gamma(r)$? Readers can answer this question by directly applying the Legendre duplication formula (cf. [18], p. 256),

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2) \qquad z > 0$$

and setting $z = r - 1/2$.

3.2. *The case when* $k > 1$

For $k > 1$, we could not find $\phi(t)$, among the commonly used characteristic functions, that satisfies $|\phi(t)|^2 = (1 + t^{2k})^{-r}$. As a result, the approach illustrated in the previous case of $k=1$ needs to be modified.

We first extend Parseval identity (3.1) as follows.

Lemma 1. For $j=1,2$, assume that $\int_{-\infty}^{\infty} |f_j(x)|\mathrm{d}x < \infty$ and that $\phi_j(t)$ is the Fourier transform of $f_j(x)$. Then

$$\int_{-\infty}^{\infty} f_1(x)\overline{f_2(x)}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_1(t)\,\overline{\phi_2(t)}\,\mathrm{d}t \tag{3.5}$$

where the overline denotes complex conjugate.

Proof. Since the Fourier transform of f_1+f_2 is $\phi_1+\phi_2$, (3.1) gives $\int|f_1+f_2|^2=\int(|f_1|^2+2f_1\overline{f_2}+|f_2|^2)=(2\pi)^{-1}\int|\phi_1+\phi_2|^2=(2\pi)^{-1}\int(|\phi_1|^2+2\phi_1\overline{\phi_2}+|\phi_2|^2)$, which combined with Parseval identities, $\int|f_j|^2=(2\pi)^{-1}\int|\phi_j|^2$, $j=1,2$, leads to the desired (3.5). □

We now apply lemma 1 to (1.1). For $k>1$ and $r>(2k)^{-1}$, we substitute $t^{2k}=s^2$. This yields

$$\int_0^\infty \frac{1}{(1+t^{2k})^r}\,\mathrm{d}t=\int_0^\infty \frac{1}{(1+s^2)^r}\frac{1}{k}s^{1/k-1}\mathrm{d}s=\frac{1}{2k}\int_{-\infty}^\infty \frac{1}{(1+s^2)^r}\frac{1}{|s|^{1-1/k}}\,\mathrm{d}s \tag{3.6}$$

Define $\phi_1(t)=1/(1+t^2)^r$, and $\phi_2(t)=1/|t|^{1-1/k}$. Thus (3.6) indicates

$$\int_0^\infty \frac{1}{(1+t^{2k})^r}\,\mathrm{d}t=\frac{1}{2k}\int_{-\infty}^\infty \phi_1(t)\,\overline{\phi_2(t)}\,\mathrm{d}t \tag{3.7}$$

The inverse Fourier transform of $\phi_1(t)$ can be obtained, from (17.34.4) of Gradshteyn and Ryzhik ([15], p. 1151) and a scaling factor $1/\sqrt{2\pi}$, by

$$f_1(x)=\frac{1}{\sqrt{\pi}\Gamma(r)2^{r-1/2}}|x|^{r-1/2}K_{r-1/2}(|x|)\qquad x\in(-\infty,\infty)$$

where $K_v(z)$ is the 'Bessel function of imaginary argument', whereas the inverse Fourier transform of $\phi_2(t)$ is given, via (17.23.5) of [15, p. 1147] and a scaling factor $1/\sqrt{2\pi}$, by

$$f_2(x)=\frac{\Gamma(1/k)\sin\{(1-1/k)\pi/2\}}{\pi|x|^{1/k}}=\frac{1}{2\sin\{\pi/(2k)\}\Gamma(1-1/k)|x|^{1/k}}\qquad x\in(-\infty,\infty)$$

Direct application of lemma 1 to the right side of (3.7) leads to

$$\begin{aligned}\int_0^\infty \frac{1}{(1+t^{2k})^r}\,\mathrm{d}t&=\frac{\pi}{k}\int_{-\infty}^\infty f_1(x)\overline{f_2(x)}\,\mathrm{d}x\\ &=\frac{2\pi}{k}\frac{1}{2\sin\{\pi/(2k)\}\Gamma(1-1/k)\sqrt{\pi}\Gamma(r)2^{r-1/2}}\int_0^\infty x^{r-1/k-1/2}K_{r-1/2}(x)\,\mathrm{d}x\end{aligned}$$

in which

$$\int_0^\infty x^{r-1/k-1/2}K_{r-1/2}(x)\mathrm{d}x=2^{r-1/k-1/2-1}\Gamma\left(r-\frac{1}{2k}\right)\Gamma\left(\frac{1}{2}-\frac{1}{2k}\right)$$

(cf. [15], (6.56.16), p. 684). Thus

$$\begin{aligned}\int_0^\infty \frac{1}{(1+t^{2k})^r}\,\mathrm{d}t&=\frac{\sqrt{\pi}}{k}\frac{1}{\sin\{\pi/(2k)\}\Gamma(1-(1/k))\Gamma(r)}2^{-1/k-1}\Gamma\left(r-\frac{1}{2k}\right)\Gamma\left(\frac{1}{2}-\frac{1}{2k}\right)\\ &=\frac{\pi}{k}\frac{1}{\sin\{\pi/(2k)\}2^{-1/k}\Gamma((1/2)-(1/2k))\Gamma(1-(1/2k))\Gamma(r)}\\ &\quad\times 2^{-1/k-1}\Gamma\left(r-\frac{1}{2k}\right)\Gamma\left(\frac{1}{2}-\frac{1}{2k}\right)\\ &=\frac{1}{2k}\Gamma\left(\frac{1}{2k}\right)\Gamma\left(r-\frac{1}{2k}\right)\Big/\Gamma(r)\end{aligned}$$

This result agrees with (2.6).

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Variations in the solution of linear first-order differential equations

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A special project which can be given to students of ordinary differential equations is described in detail. Students create new differential equations by changing the dependent variable in the familiar linear first-order equation $(\mathrm{d}v/\mathrm{d}x) + p(x)v = q(x)$ by means of a substitution $v = f(y)$. The student then creates a table of the new equations and describes how they are solved. Applications are also given.

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CORRECTION

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The authors of the paper would like to correct the following errors.

1. On page 309, [14] in **References** should be

   Silverman, B. W. (1984). A fast and efficient cross-validation method for smoothing parameter choice in spline regression. *J. Amer. Statist. Assoc.*, **79**, 584–589.