Vernier scales and other early devices for precise measurement

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Vernier scales have been extensively used since the 17th century. They replaced the Nonius scale, a unpopular device due to difficulty in its fabrication and use, and they coexisted alongside other types of scales that increased measurement precision and accuracy in complementary ways. I suggest that the success of Vernier and diagonal scales is due not only to simplicity of fabrication, but also to their exploitation of visual hyperacuities.
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I. INTRODUCTION

Vernier scales on undergraduate teaching instruments such as calipers and spectrometers are being replaced by digital readouts, so students encounter Vernier scales less often. But students are often impressed by the results achievable by such a simple device, that such a simple idea escaped realization until the 17th century, and that, four centuries later, it has still not been completely superseded. Why do science and engineering still use a technology essentially unchanged since the 1630s?

The Vernier scale's success and its decline can be understood by examining how it relates to other devices with which it competes. This article surveys early technologies for precision length and angle measurement as a resource for teachers and students interested in the Vernier scale's origins, competitors, and long success.

II. MOTIVATIONS FOR PRECISE MEASUREMENT

Precision measurement has long been considered a hallmark of exact science, although it owes as much to commercial, industrial, government, and military concerns as to "pure" intellectual pursuits. Until the 18th century, nearly all precision measurements were concerned with length, weight, time, and angle. Lengths were typically standardized by lines that can still be found engraved on the exteriors of old European town halls and churches, weights by standard masses, and time by astronomical cycles. Angles were constructed by manually dividing circles, half-circles, and quarter circles to produce instruments such as circumferentors and quadrants for land surveying and astronomy. This division was done primarily by compass and rule, with dots marked at the arc and line intersections with a punch or drill (or burin, for lower-grade instruments). Protractors were used for plotting routes on navigational charts but generally not for measuring angles.

The standard angle measure system was inherited from ancient Mesopotamia.[1] The degree system appears to derive from astronomy, where 360 is approximately the number of steps that the sun takes around the ecliptic per year. 360 has many useful divisors, and fits readily into the sexagesimal number system in which ancient Mesopotamians counted. Although masons' and carpenters' squares have been used since antiquity for replicating right angles, more precise angles for surveying, cartography, navigation, artillery, and astronomical instruments were constructed anew.

Most efforts to standardize length, weight, volume, and currency can be understood as forms of social regulation because common standards facilitate trade and taxation, planning, property rights, and organization. Positional astronomy was especially significant in providing for cartography, timekeeping, and astrology, plus open-sea navigation from the 15th century onward. Hence, there was particular attention paid to measuring angles, and therefore instrumentation for astronomy generally represents the most advanced mechanical measuring technologies.

III. PRECISION FROM LARGENESS

The ancients and medievals had only one way to increase precision: They made their angle-measuring instruments bigger. Enlarging the instrument lengthens the arc segments for subdivision around the instrument's limb (its outer edge). The arc length increases in proportion to the radius, but the concomitant increases in size, weight, and bulk bring difficulties in manufacturing, transporting, mounting, and supporting the instrument, as well as in operating it. These difficulties were not uniquely medieval. Instrument makers remain concerned with the weight of instruments, causing them to sag or to misalign the walls and frames on which they were hung. Precision was enhanced by largeness, but ergonomics and accuracy can easily be undermined.

Medieval astronomical instruments could be tens of meters across. Well-known examples include Guō Shǒujìng's 13th century solar observation tower at Gàochéng in China and the excavated ruins of Ulugh Beg's 15th century observatory in Samarkand.[2,3] Even as modern astronomy gathered momentum from the 17th century, size remained a practical solution for instruments without large moving parts, as shown by Cassini's solar observatory built into the Basilica di San Petronio in Bologna in the 17th century to test for ellipticity of the Earth's orbit, and Jai Singh's 18th century system of observatories at Delhi, Jaipur, Mathura, and Varanasi.[4,5]

Most instruments were smaller, and seem to have been graduated to individual degrees. Ancient records, such as those in Ptolemy's *Almagest*, show that, even using portable instruments, astronomers measured angles to thirds, quarters, and fifths of a degree.[6] These fractions remained in use for a long time. At the end of the 16th century, for example, we find such fractions among Tycho's measurements, and surmise that the fractional parts were approximated by eye.

IV. THE NONIUS

The first major advance in precision measurement scales was announced in the 1542 publication *De crepusculis*, a book on the duration of twilight as governed by the setting sun, by the mathematician Pedro Nunes (1492–1577).

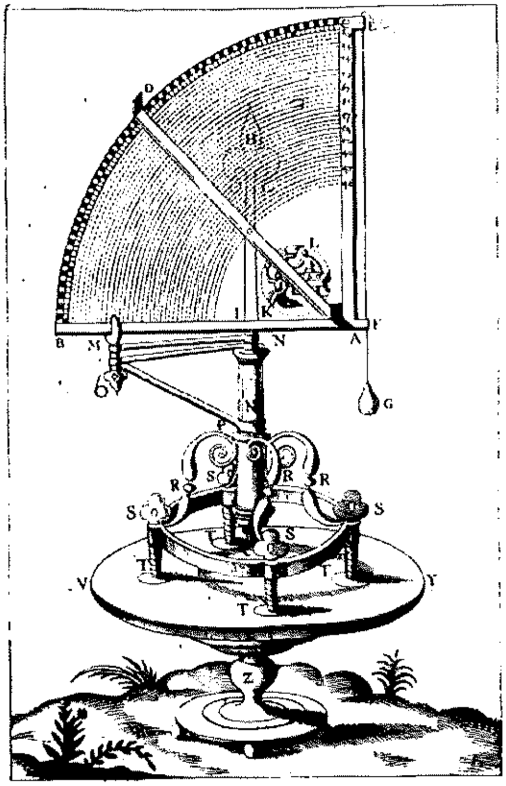

Fig. 1. A quadrant with a nonius, made and used by Tycho Brahe (Ref. 31).

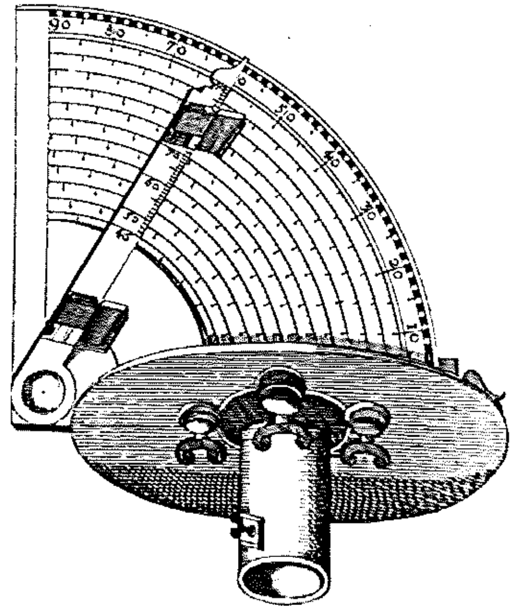

Fig. 2. A quadrant with nonius and alidade graduated for easier reading (Ref. 32).

Nunes, who worked when Portugal was most actively establishing colonies and trading posts, invented several instruments to assist navigation, and studied the paths followed by ships on a constant compass bearing: Loxodromes spiraling toward the poles.[7] His efforts to plot maps so that loxodromes appear as straight rhumb lines, while lines of latitude and longitude appear horizontally and vertically, led eventually to the Mercator projection.

Nunes's precision measuring device (named from the Latin form of his name, Petrus Nonius) fitted naturally into his navigational interests. At the time, angular measurements such as solar altitudes were made using cross-staves, astrolabes, and quadrants. These instruments typically measured to the nearest degree, or perhaps to a half or third of a degree. The nonius improved the quadrant, though in principle it could be adapted to any angle-measuring instrument with an alidade. The outer limb of the quadrant was divided into 90°, as usual. On an especially fine instrument, it might have been further divided into half-degrees, perhaps thirds or quarters. Nunes's innovation increased precision to the order of individual minutes.

Just inside the usual arc, Nunes added another arc that divided the quarter-circle into 89 equal parts. Inside this 89-part arc was another quarter-circle of 88 equal parts, and so on to the innermost one with 45 equal parts. The user had only to see which of these marks the alidade aligned with. For alignment with the ath mark of the arc divided into n parts, the angle is $90° \times a/n$. Calculating the angle involved some arithmetic which, in the era of manual computation, was made easier, more reliable, and faster by a pre-calculated table.

In spite of its theoretical simplicity, the nonius was never widely used. It was cumbersome to read, as Tycho found with the two nonius-equipped quadrants that he made around 1580, one of which is shown in Fig. 1.[8] The forest of dots on Tycho's instruments would have been difficult to tell apart, but some noniuses, such as those shown in Fig. 2, have every fifth or tenth dot enlarged, showing that users were expected to count the dots. This particular instrument also has a graduated alidade scale to save the user from having to count which arc the dot is on, and only 12 supplementary arcs, far fewer than Nunes's 44. Although the book containing this plate does not describe the instrument, the instrument matches one now at the Istituto e Museo di Storia della Scienza in Florence, attributed to the London manufacturer James Kynvyn ca. 1600.[9]

In addition to being difficult to read, noniuses were difficult to engrave. Dividing a quarter-circle into 90 equal parts required a high degree of skill to accurately construct the easier angles via regular polygons and bisection (from the pentagon, 72°, 36°, 18°, and 9°; from the equilateral triangle 60°, 30°, 15°, and 7.5°; from the square, 45°), plus further ingenuity for the non-Euclidean trisections and trial and error divisions for individual degrees and halves or thirds thereof. We know from surviving instruments, notably astrolabes, that instrument makers sometimes hastened this task by projecting a divided arc through the center, and by duplicating divided intervals using a template, because such methods replicate discrepancies by reflectional and translational symmetry.[10,11] Dividing the other arcs was in most cases much more difficult due to the absence of well-known constructions for them. Even when exact procedures were known, anything beyond the simplest ones was inaccurate due to errors compounding from the many steps of construction. Nunes's design involves constructing or otherwise locating more than 3000 points.

Few noniuses were built, but the idea was not ignored.

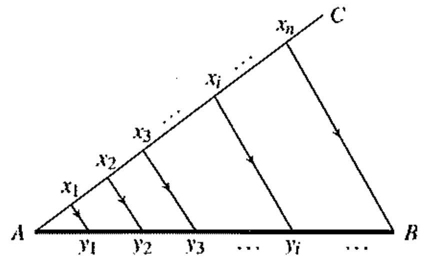

Fig. 3. Dividing a line AB into n equal parts.

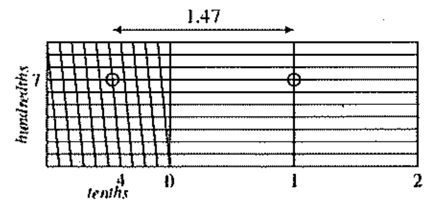

Fig. 4. Ruler with a diagonal scale to measure hundredths of a unit. The circles indicate the points used for a length of 1.47 units.

Holy Roman Emperor Rudolph II's pro-chancellor Jacob Kurz (Jacobus Curtius, 1554–1594) developed an improved version of a quadrant for Tycho Brahe who moved to the imperial court in Prague after losing his welcome in Denmark. The outermost arc had 90 conventional degrees. Inside it were 59 further arcs, each divided into 60 parts. The first of these arcs subtended 61°, the second 62°, and so on for 60 graduated arcs in total, the innermost one subtending 119°. The outer scale, the conventional one, reads whole degrees. The inner scales all read minutes: If the alidade coincides with a mark on the arc having 60 intervals spread over $(60+d)$ degrees, the fractional part of the angle is d minutes.[12–14] Curtius's device was much easier to read than Nunes's, but it involved locating and engraving over 5000 dots—many more than Nunes's laborious version required, and all involving the awkward task of dividing an arc into 60 equal parts.

Another refinement was introduced by Christoph Clavius (1538–1612), now remembered mostly for his part in the reform of the Gregorian calendar. Clavius is speculated to have learned about the nonius while studying in Lisbon, where it was possible for him to be taught by Nunes himself. Clavius was later posted at the Collegio Romano, where he designed a quadrant with 39 supplementary graduated arcs subtending angles from 91° to 128°. In some respects, Clavius's arcs were like Curtius's, but Clavius divided each of his supplementary arcs into 128 parts so they could be constructed simply by repeated bisection. Clavius's aim was to make construction easier and more accurate, but his design still entailed much work.[14,15]

The key point to draw from the nonius and its refinements is that astronomers were willing to invest a great deal in precision measurement, but the skill and effort needed to make and use these devices discouraged their widespread use.

V. TRANSVERSALS

The problem of dividing lengths is much easier than dividing angles. As early modern instrumentation manuals explained,[16] any length can be divided arbitrarily into many equal parts using the well-known and simple construction shown in Fig. 3. Line AC is drawn branching off the given line AB, and n equidistant points x_i are stepped off using dividers. Line x_nB is joined, then the lines x_iy_i are constructed parallel to x_nB, giving n equal parts of AB. The parallel line construction recommended by the texts is, like the previous explanation, from Euclid's *Elements*.[17]

Dividing lines in this way is well suited to practical applications because the small number of steps limits construction inaccuracies. The precision is limited mainly by the fineness of the scriber, and subdividing the scale into smaller parts makes it less legible as more lines are crammed into the available space. Both of these problems were ameliorated by diagonal scales, which came into use as early as the 14th century.

A diagonal scale is shown in Fig. 4. The unit lengths are supplemented by an extra interval added to one end. This extra interval is divided into tenths, using a similar triangles construction. We can now measure in tenths of units, transferring distances on or off the ruler using a pair of dividers.

Ten more horizontals are then spaced evenly above the baseline, and the diagonal lines, called transversals, drawn across each tenth unit. The intersections of these transversals with the horizontals gave hundredths. To take a length of 1.47 units, for example, we look for the vertical corresponding to 1, count back from the zero to find the transversal for 0.4, and count up from the baseline to find where the transversal intersects the horizontal for 0.07. The divider tips go between the two points thus identified.

Confining the fractional parts into a single block to the left of the zero reduced the labor, and hence the instrument's cost, by eliminating the need to engrave transversals in every unit block to the right.

Transversals were applied to angle scales by drawing concentric circles instead of parallel lines. The resulting division was only approximate, but for small angular intervals, it could be good enough. Tycho included diagonal scales on several large instruments, placing a dot at each intersection point along the transversals. One of his explanatory diagrams shows 2° of arc divided into 10 min intervals, with transversal dots giving the individual minutes (see Fig. 5). Tycho also showed diagonal scales giving intervals of 10 arcsec on his famous wall-mounted quadrant of 2 m radius, and claimed that he could measure to 5 arcsec—obviously by bisecting the smallest intervals by eye—without difficulty. Tycho's published description is largely responsible for the

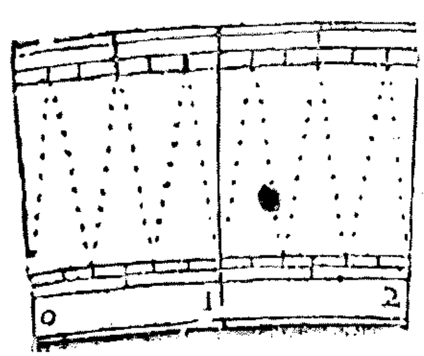

Fig. 5. Part of a dotted diagonal scale shown in Tycho's *Mechanica* (Ref. 33). The dots divide these two degrees of arc into individual minutes.

widespread adoption of diagonal scales, though they had been in scattered use at least since the 14th century.[18] Most diagonal scales are drawn with lines, though Tycho's dot design was used elsewhere. For example, the Museum Boerhaave has a large quadrant with dotted transversals made ca. 1610 by the cartographer W.J. Blaeu who studied with Tycho at Uraniborg. This instrument was commissioned by Willebrord Snel who likewise had visited Tycho at Uraniborg.

Many rulers with diagonal scales survive from the Renaissance through to the 18th century (when they came to be supplanted by caliper jaws and Vernier scales), and diagonal scales continued to appear on maps throughout the 20th century. Their primary advantage is that, due to the large spacings, they are easy to construct accurately and to read. They could be constructed also for intervals other than tenths and hundredths; twelfths and powers of two were (and still are) commonly used in pre-decimal unit systems.

VI. THE VERNIER SCALE

Due to its origin, the Vernier scale was originally called a nonius and apparently considered a refined version of it. The name "Vernier scale" came into use in French and then in English in the 18th century, after the astronomer Jérôme Lalande, citing an historical study by Esprit Pezenas at the Marseille Observatory, wrote that "Nonius is not its inventor. ... The true inventor of our device was Pierre Vernier who publicized it in a small book printed at Brussels in 1631. ... I believe, then, that it is just to restore to the true inventor his rights, that is to call a Vernier, rather than Nonius, this method of division that he worked on."[19,20] The name "nonius" persists, however, in some languages including German and Danish.

In contrast to Nunes, Curtius, and Clavius who had added numerous subsidiary scales, Pierre Vernier (1580–1637) added just one. He also made his subsidiary scale mobile so that it would contribute equally over the entire length or arc. Vernier's scale was also much easier to read, due to having far fewer marks for the user to count. Vernier made his point by demonstrating an instrument that measured angles to the nearest 20 arcsec. This precision was of the same order of magnitude that Tycho claimed for his huge wall-mounted quadrant, but Vernier's instrument had a radius of only 2 ft, compared with Tycho's 2 m, and it was much easier to read than a nonius. The large radii of medieval instruments had finally been overcome.

Although Vernier scales can still be found in many teaching laboratories, most students spend very little time with them. Therefore it seems worthwhile to give a brief explanation of their use. Vernier-equipped instruments have both a fixed scale in standard units, and a subsidiary Vernier scale in units of a slightly different size. Most commonly, Vernier scale units are scaled to fit across a space 90 times wider than they measure, greatly magnifying the detail which the user must see. Because length-measuring instruments are more readily fitted with digital readouts than angle-measuring instruments, the only Vernier scales in undergraduate laboratories are usually on spectrometers. Some of these read in base 10 and some in base 60.

Let us consider the Vernier scale in Fig. 6 for measuring millimeters. The main scale is divided into millimeters, and the subsidiary scale is 9 mm wide and divided into 10 equal parts. The Vernier scale is fixed to one side of the measuring instrument, say a caliper jaw, and the main scale is fixed to

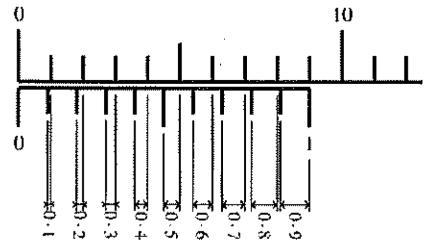

Fig. 6. Close-up of a scale in millimeters with a Vernier scale below it. The dimensions show the arithmetically increasing offsets between corresponding lines.

the other. When the caliper jaws are separated by an integer number of millimeters, the Vernier zero aligns exactly with a millimeter mark on the main scale. The other Vernier lines will misalign such that one line will be offset by 0.1 mm from the next main scale line, the two by 0.2 mm, the three by 0.3 mm, and so on. Sliding the Vernier scale 0.1 mm to the right (by opening the jaws 0.1 mm) therefore brings one line into alignment, and misaligns the zero by 0.1 mm. Each time the Vernier scale advances 0.1 mm, another of its lines comes into alignment, and all other lines are offset. By identifying the aligned line, we can read the Vernier scale's position, and hence the caliper jaw separation, to a precision of 0.1 mm.

More generally, if the main scale divides the unit L into n parts, and the Vernier scale has n parts spread over an interval $L \times (n \pm 1)/n$, an alignment will occur each time the Vernier scale advances by a distance L/n^2. For base 10, $n=10$, so the Vernier scale measures hundredths of the main unit. All the user has to do is look for the mark on the Vernier scale aligned with any mark on the main scale. There are usually 10 or 30 marks, far fewer than on a nonius. Smaller Vernier scales may require a magnifier. With practice, it becomes possible to discern half-increments by comparing the offsets of adjacent lines, and even thirds, quarters, and fifths.

There is nothing to prevent the Vernier scale in Fig. 6 from being spread over a wider interval, and Vernier recognized that small instruments needed larger Vernier scales for legibility. There are disadvantages such as running out of room for the fixed main scale to slide along (because it must be adjacent to the main scale for reading), but also the advantage of greater spacing between its lines. The interval must be chosen so that only one Vernier line aligns with a main scale line at a time. This criterion can be satisfied by choosing co-prime divisors (no common factors other than 1) though few such possibilities are used. The main advantages of using such a Vernier scale are legibility, and permitting other unit systems such as sexagesimal angle measure, and twelfths, sixteenths, and so on in English units.

VII. SCREW DIALS

Vernier scales were not the only possibility, nor always the favored approach. The closely spaced lines are difficult to distinguish in the dark, for example, and very fine lines can require magnification even in the light. And reading them is relatively slow. An alternative, which came into use around 1670, was the screw dial that Robert Hooked added to the alidade of a quadrant.[21] Hooke, who competed with Johannes Hevelius for the best methods for precision astronomical measurements, developed screw movements to completely avoid the problem of graduating the arc along the quadrant's edge. In spite of the high degree of skill employed in constructing those scales, there were errors due to the accumulation of unavoidable inaccuracies at each step in each construction. Hooke's design involved a screw on the alidade meshing with gear teeth along the quadrant's circumference so that, as the astronomer turned the screw, the alidade—with a telescopic sight attached—moved around the quadrant. By counting the number of turns, it was possible to deduce the angle through which the telescope had moved. All the astronomer needed to know in advance was the number of gear teeth around the quadrant's edge and the number of turns needed to traverse them. To read the number of turns, Hooke added a large brass disk to the end of the screw, and marked it off into a dial plate. On some instruments the dial plate turned past a fixed pointer, and on others, the plate remained still, while a geared pointer spun over the graduated scale.

The screw dial's success bolstered Hooke's arguments against Hevelius's refusal to use telescopic sights, but neither the screw nor gears could be cut well enough to meet expectations. Techniques greatly improved, however, and a version was put to important use by James Bradley in the 1720s.[22] The telescope with which Bradley serendipitously discovered the aberration of starlight (he was looking for the long-sought annual parallax to prove that the Earth orbited the Sun, and not vice versa) hung from a fixed hinge at the top of a chimney. The side of the tube, near its base, rested against the end of a screw running through a fixed block. By turning the screw, Bradley could tilt the telescope very gradually. The knob on the end of the screw was studded with small buttons to mark the angles through which it turned, making them much more visible, and even tactile, than the fine lines of a Vernier scale. Bradley's telescope, minus its weight, survives in the Royal Observatory, Greenwich.[23]

VIII. THE 96-PART QUADRANT

Although the Vernier and screw dials provided excellent subdivision, they did not address the difficulty of dividing an arc into 90° and parts of degrees. The problems are that there is no Euclidean construction for a single degree, and the combination of exact constructions and approximations used to obtain single degrees involves a progressive accumulation of error at every step. However, there is an exact construction for dividing a right angle into 96 equal parts: First construct the 30° and 60° angles, then repeatedly bisect into intervals of (15/16)°. Such units are easily converted into degrees by multiplication.

This construction approach was invented by George Graham in 1725.[24] It completely eliminates the need for trisections and trial and error approximations, and there was no reason not to add a Vernier scale. Combining the two technologies reflects how they addressed different problems: One increased precision; the other, accuracy.

Construction error could not be completely eliminated, due to the impossibility of perfect compass control. Quadrants divided into 96 parts included a conventional 90° arc as well, and each arc was equipped with its own Vernier scale. We can see this combination of 96-part and 90-part arcs on a quadrant that Graham made for Greenwich in 1725.[25]

IX. CURVED TRANSVERSALS FOR ANGLES

As mentioned, the precision offered by Vernier scales came at the cost of slower reading. When less precision and greater speed were required, diagonal scales prevailed. They were improved throughout the 18th century for the lower precision instruments used in surveying.

The obvious problem in applying a diagonal scale to an angle-measuring instrument is that a straight diagonal does not divide the sectors uniformly if the parallels are evenly spaced. Tycho considered the discrepancy small enough to tolerate, but with the advent of the Enlightenment, such compromises were deemed problematic. The French instrument maker Guillaume Ferrier had early on applied curved transversals to an instrument for Descartes, and the method was taken up some decades later by Philippe de la Hire (1640–1718), and again by Nicholas Bion (1652–1733) through whose *Traité de la construction et des principaux usages des instrumens de mathematique* it is best known.[26] The curve chosen is a circular arc through three points: The two opposite corners of the diagonal scale, plus the center of the instrument. The concentric parallels along the scale are engraved at uniformly separated radii. Bion's description shows a main arc divided into 5° increments, and the transversals give the individual degrees.[27]

The resulting scale is much quicker to read than a Vernier because it is larger, and thus does not require magnification, because the fiduciary points are widely separated. It also does not require deliberation over which of two nearly aligning lines align best. Unlike the Vernier scale, Bion's curved transversal appears to have fallen completely into disuse, replaced by scales cut using the symmetry method of the Duc de Chaulnes, and later by the development of dividing engines.[28] These advances made it possible to mark the units individually rather than by subdividing a larger interval. Due to the relatively low precision required, no Vernier is needed.

X. WHY HAS THE VERNIER LASTED SO LONG?

The mathematical ingenuity of the Vernier scale and simplicity of its construction have long been appreciated but these features alone cannot explain practicality. I suggest that the Vernier scale's longevity is due also to legibility, and hence our innate visual abilities, and that this factor is also important in the Vernier scale's present decline.

Reading the earliest scales involved guessing the fraction of the smallest marked interval, as can be seen in Ptolemy's or Tycho's measurements. We are accustomed to base 10 and the metric system and commonly estimate fractions by eye in tenths, but users of measuring instruments from antiquity through about the 17th century estimated predominantly in halves, thirds, quarters, and fifths. This tendency toward simple fractions can be explained partially by visual hyperacuity. Hyperacuity is the ability to discern details beyond what is expected from ocular and retinal geometry through post-processing deep in the retina, in the optic nerve, and in the brain. The hyperacuity associated with visually dividing an interval into a small number of equal parts lies behind tests for "bisection acuity," the ability to judge the middle of a line segment.

The hyperacuity associated with detecting slight misalignments of two end-to-end line segments is fittingly called "Vernier acuity." Hyperacuity for misalignments is what makes the nonius superior to guessing the fraction, and what makes the Vernier scale better still. Adults can typically detect Vernier line misalignments subtending as little as a few seconds of arc at the eye, and the skill can be improved by practice.[29,30]

Let us consider what hyperacuity means for the scales we have described. Hyperacuity is thwarted by overcrowding, which makes adjacent lines difficult to distinguish, and by low contrast. Overcrowding is reduced by spreading out the marks, whether along the transversal of a diagonal scale, or on the expanded length of a Vernier scale. Contrast was typically increased by engraving on an inlaid strip of silver or ivory, rather than on the brass, iron, steel, or wood used for instrument frames.

Progress in scale design is often thought of in mathematical and mechanical terms: The development of new constructions, symmetry tests, dividing engines, and other ways to reduce fabrication error. But technique is wasted if the scale cannot be read. Historically, successful designs are distinguished as much by their exploiting of visual hyperacuities as by their mathematical ingenuity. In addition to being accurate and precise, the most widely used precision scales over the past four centuries—essentially the whole modern era—were the more legible ones. In conjunction with advances in manufacturing, legibility and speed of reading help to explain why the nonius failed, why transversals and the Vernier scale succeeded, and why digital readouts are now replacing the Vernier.

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[30]J. Saarinen and D. M. Levy, "Perceptual learning in vernier acuity: What is learned?" Vision Res. **35** (4), 519–527 (1995).
[31]Reference 8, "Quadrans minor orichalcicus inauratus."
[32]Robert Dudley, *Dell' arcano del mare* (Florence, 1646–1647), Book V, Plate 44.
[33]Reference 8, "Supplementum de subdivisione."