**CS 809**
**Fall 2018**
**Homework #3**

**Due in class Friday, Oct. 12, 2018.**

**Rules for Homework.** See Homework 1.

3. This problem is concerned with an analog of the Feller "card dealing" algorithm, for choosing a random equivalence relation of $\{1, \ldots, n\}$.

   Let $B_n$ denote the number of different equivalence relations on $\{1, \ldots, n\}$. $B_0 = 1$ by convention, and you can verify that the next few values are given by

$$\begin{array}{rcccc} n = & 0 & 1 & 2 & 3 \\ B_n = & 1 & 1 & 2 & 5 \end{array} .$$

   a) For $k = 1, \ldots, n$, find the number of equivalence relations on $\{1, \ldots, n\}$ having 1 in a class of size $k$. Your formula can involve values of $B_j$ for $j < n$.
   b) Note that 1 must go into some class. Using a), give a recursive formula for $B_n$. Don't forget your initial condition(s).
   c) Use this to design a recursive algorithm for choosing a random equivalence relation.
   d) Let $E_n$ be the expected number of recursive calls in your solution to c). (This will also be the mean number of equivalence classes in a randomly chosen equivalence relation on $\{1, \ldots, n\}$.) Write down a recurrence relation for $E_n$, and show that $B_{n+1}/B_n - 1$ is the solution to it.
   e) (*) For a random permutation on $\{1, \ldots, n\}$, the size of the part (cycle) containing 1 is uniformly distributed. This is not true for random equivalence relations, as you can see from small examples (even $n = 3$ is big enough). What *can* you say about its distribution?