Binomial Identities and Hypergeometric Series

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1. Introduction. Combinatorial problems arise in many areas of mathematics and theoretical physics and their solutions often involve the evaluation of a sum of products of binomial coefficients. There are various methods for evaluating such sums and some procedures are discussed in Riordan [**13**] and Knuth [**11**]. Knuth has written that there are thousands of binomial identities; many excellent mathematicians have expended considerable amounts of ingenuity in proving these. In fact, contrary to the prevailing view, there is a very small number of essentially different binomial identities, and thus a great deal of mathematical ingenuity has been needlessly wasted. Many mathematicians have been unable to recognize that a given binomial identity is actually equivalent to one already known because the notation for a single binomial coefficient is very misleading when used to express sums of their products. This notation all too often serves to disguise several essentially identical sums and makes them appear very different. One reason for this is that binomial coefficients can be taken apart and then rearranged to take many different forms.

In this paper we shall explain how to write most single series of products of binomial coefficients in a canonical fashion, so that the real character of the series is easily discernible. This method was used by Euler and Gauss. These canonical series are known as hypergeometric series. The classical notation for hypergeometric series is easy to learn and use. This notation expresses explicitly certain important features of the sum; this allows for the kind of classification scheme which makes standardization possible. Here we consider some of the most important hypergeometric identities. These cover most of the single sums of products of binomial coefficients where the series can be summed. Thus, we shall show why the hypergeometric identities should be regarded as the standard forms. A number of examples taken from various sources will be given to illustrate how sums of binomial coefficients can be reduced to standard form.

The following section contains definitions and statements of four main hypergeometric identities. Since the proofs of these identities are unrelated to the applications, which are the prime concern of this paper, the proofs have been relegated to the fourth and last section. The third section contains the examples. The reader

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should also consult Andrews [**2**] where the view presented here is explicitly stated, and which contains examples complementing those presented here. For an interesting history of these identities, one may see Askey [**4**], [**5**].

2. Hypergeometric series. In general, a hypergeometric series (or to some, a generalized hypergeometric series) is a series

$$\sum C_n \tag{2.1}$$

such that C_{n+1}/C_n is a rational function of n. Note that this is a situation where the ratio test, the simplest of all convergence tests, applies. Also, if C_n is a product of binomial coefficients, then C_{n+1}/C_n is of this form. We can factor the rational function as

$$\frac{C_{n+1}}{C_n} = \frac{(n + a_1) \cdots (n + a_p)x}{(n + b_1) \cdots (n + b_q)(n + 1)} \tag{2.2}$$

and the series can then be written as a constant C_0 times

$$\sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!}, \tag{2.3}$$

where $(\alpha)_n$ are the shifted factorials defined by

$$(\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1), \quad (\alpha)_0 = 1. \tag{2.4}$$

The series (2.3) is usually denoted by

$${}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; x\right).$$

If $x = 1$, then we omit it. The case $p = q + 1$ arises very often and we shall consider only such cases. The parameters a_i and b_i are complex numbers, but for all our applications they will be real. Moreover, only the finite series is of interest to us here, but if the series is infinite, it converges for $|x| < 1$; if $x = 1$, then we require $(\Sigma b_i - \Sigma a_i) > 0$.

In the applications to binomial identities, q is very small, usually one or two, and the parameters $a_1, \dots, a_p$, $b_1, \dots, b_q$ satisfy certain relations. These relations play an important role in the classification of the series and contribute to the power of this method when applied to binomial sums. We shall say that a series is k-balanced if $x = 1$, if one of the a_i's is a negative integer, and if

$$k + \sum a_i = \sum b_i. \tag{2.5}$$

The case $k = 1$ is most important and then the series is called balanced or Saalschützian. (Note that, if a_i is a negative integer for some i, then the series (2.3) must be finite.) The series is called well-poised if

$$1 + a_1 = b_1 + a_2 = \cdots = b_{p-1} + a_p. \tag{2.6}$$

Recall that we have taken $q = p - 1$.

We now state the four hypergeometric identities which occur very often in practice. Numerous binomial identities can be reduced to these. From now on, we assume that n denotes a positive integer.

The Chu-Vandermonde identity:

$$ {}_2F_1\left(\begin{matrix}-n, & -b\\ & c\end{matrix}\right) = \frac{(c+b)_n}{(c)_n}. \tag{2.7}$$

The Pfaff-Saalschütz identity:

$$ {}_3F_2\left(\begin{matrix}-n, & -a, & -b\\ & c, & d\end{matrix}\right) = \frac{(c+a)_n(c+b)_n}{(c)_n(c+a+b)_n}, \tag{2.8}$$

where $d = 1 - a - b - n - c$, that is, the series is balanced.

The Sheppard-Andersen identity:

$$ {}_3F_2\left(\begin{matrix}-n, & -a, & -b\\ & c, & 2-n-a-b-c\end{matrix}\right)$$
$$= \frac{(c+b-1)_n(c+a)_n}{(c+a+b-1)_n(c)_n}\left[1 - \frac{a}{(c+b-1)(a+c+n-1)}\right]. \tag{2.9}$$

Note that the series (2.9) is 2-balanced. The first two identities are quite old; Chu published the first one in 1303. The second identity was obtained by Pfaff [**12**] in 1797, then forgotten and rediscovered by Saalschütz [**14**] in 1890. Sheppard [**15**] published the identity for a k-balanced ${}_3F_2$ in 1912 and the particular case $k = 2$ was rediscovered by Andersen [**1**] in some work in probability theory.

The final identity is for a well-poised ${}_3F_2$ and is due to Dixon [**7**]. This identity is best expressed in terms of the gamma function $\Gamma(s)$. This is defined for $s > 0$ by the integral

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\,dx, \tag{2.10}$$

and extended to other real values except $s = 0, -1, -2, \dots,$ by

$$\Gamma(s+1) = s\Gamma(s). \tag{2.11}$$

Some important properties of this function are

$$\Gamma(s+1) = s! \quad \text{if } s \text{ is an integer} \geqslant 0, \tag{2.12}$$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin s\pi}, \tag{2.13}$$

and

$$\lim_{s\to\infty} \frac{\Gamma(s+1)e^s}{\sqrt{2\pi s}\,s^s} = 1 \quad \text{(Stirling's formula)}. \tag{2.14}$$

Dixon's identity is:

$$
{}_3F_2\left(\begin{matrix} a, & -b, & -c \\ & a+b+1, & a+c+1 \end{matrix}\right)
$$
$$
= \frac{\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a+b)\Gamma(1+a+c)\Gamma\left(1+\frac{a}{2}+b+c\right)}{\Gamma(1+a)\Gamma\left(1+\frac{a}{2}+b\right)\Gamma\left(1+\frac{a}{2}+c\right)\Gamma(1+a+b+c)}. \tag{2.15}
$$

Note that this series can be infinite; then we require $a + 2b + 2c + 2 > 0$ for convergence.

We shall need the following elementary identities to reduce sums of products of binomial coefficients to hypergeometric series:

$$
(f)_{n-k} = \frac{(-1)^k (f)_n}{(-f-n+1)_k}, \tag{2.16}
$$

and if $f = 1$ we get

$$
(n-k)! = \frac{(-1)^k n!}{(-n)_k}. \tag{2.17}
$$

$$
(a-n)_k = (-1)^k(-a+n-k+1)_k, \tag{2.18}
$$

and

$$
(a)_{2k} = 2^{2k}\left(\frac{a}{2}\right)_k\left(\frac{a+1}{2}\right)_k. \tag{2.19}
$$

These identities are very easily proven. For example, the right side of (2.16) is

$$
\frac{f(f+1)\cdots(f+n-1)}{(f+n-1)\cdots(f+n-k)} = f(f+1)\cdots(f+n-k-1)
$$
$$
= (f)_{n-k}. \tag{2.20}
$$

The relations (2.18) and (2.19) are proven in a similar way.

3. Examples. We now give a few examples to show how sums of products of binomial coefficients can be reduced to hypergeometric series. Moreover, we illustrate that though such sums come in many disguises, they reduce to very few different kinds of hypergeometric series. The reader can best be convinced of this by taking a number of binomial identities from a textbook or the problem section of the MONTHLY and verifying them by the method explained here.

EXAMPLE 1. The following is problem E 3065 which appeared in this MONTHLY, December 1984. One has to evaluate in closed form the sum

$$S=\sum_{j=0}^{n}(-1)^{j}\frac{\binom{k}{j}\binom{k-1-j}{n-j}}{j+1}. \tag{3.1}$$

We shall reduce it to a hypergeometric series which will show that it is merely the Chu-Vandermonde series in disguise. The first step is to rewrite (3.1) in factorials to get

$$S=\sum_{j=0}^{n}(-1)^{j}\frac{k!}{j!(k-j)!}\frac{(k-1-j)!}{(n-j)!(k-1-n)!}\frac{1}{j+1}. \tag{3.2}$$

Next we take terms which do not depend on j outside the summation and the rest of the terms we write as shifted factorials $(\alpha)_j$. To do the latter we use (2.17) and arrive at

$$S=\frac{(k-1)!}{(k-1-n)!n!}\sum_{j=0}^{n}\frac{(-k)_j}{j!}\frac{(-n)_j}{(-k+1)_j}\frac{1}{j+1}$$

$$=\binom{k-1}{n}\sum_{j=0}^{n}\frac{(-k)_j(-n)_j}{(1)_{j+1}(-k+1)_j}. \tag{3.3}$$

The last series would be a standard form of a hypergeometric series except for the term $(1)_{j+1}$, so we effect the following changes:

$$S=\binom{k-1}{n}\frac{(-k)}{(n+1)(k+1)}\sum_{j=0}^{n}\frac{(-k-1)_{j+1}(-n-1)_{j+1}}{(1)_{j+1}(-k)_{j+1}}. \tag{3.4}$$

(Note that $(-k)_j=(-k-1)_{j+1}/-(k+1)$ and a similar relation is true for the other terms.)

If we set $j+1=l$ the sum may be written as

$$\sum_{l=1}^{n+1}\frac{(-k-1)_l(-n-1)_l}{(1)_l(-k)_l} \tag{3.5}$$

and if we add one to this sum we get the hypergeometric series ${}_2F_1\left(\begin{matrix}-n-1, & -k-1\\ & -k\end{matrix}\right)$.

(Recall that a hypergeometric series begins with one.) Thus,

$$S = \binom{k-1}{n}\frac{k}{(n+1)(k+1)}\left[1 - {}_2F_1\left(\begin{matrix}-n-1, & -k-1\\ & -k\end{matrix}\right)\right]$$

$$= \binom{k-1}{n}\frac{k}{(n+1)(k+1)}\left[1 - \frac{(1)_{n+1}}{(-k)_{n+1}}\right], \quad \text{by (2.7)}. \tag{3.6}$$

The last expression is easily simplified to

$$\frac{1}{k+1}\left[\binom{k}{n+1} + (-1)^n\right].$$

The following are three more cases of the Chu-Vandermonde identity as the reader can easily verify:

$$\sum_{k=0}^{n}\binom{r+k}{k} = \binom{r+n+1}{n}, \quad \text{where } r \text{ is real}, \quad n = \text{integer} \geqslant 0; \tag{3.7}$$

$$\sum_k \binom{r}{k}\binom{s}{n+k} = \binom{r+s}{r+n}, \quad n = \text{integer}, \quad r = \text{integer} \geqslant 0; \tag{3.8}$$

$$\sum_k (-1)^k\binom{r}{k}\binom{s+k}{n} = (-1)^r\binom{s}{n-r}, \quad n, r \text{ as in (3.8)}. \tag{3.9}$$

We now see that the verification of many binomial identities can be reduced to a routine calculation by using hypergeometric series.

EXAMPLE 2. The next example is the most difficult of the worked problems on binomial identities in Knuth [**11**]. The problem is to evaluate

$$S = \sum_{k\geqslant 0}\binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1}. \tag{3.10}$$

This is only the Pfaff-Saalschütz identity in another form. In showing this we also illustrate the use of the duplication formula (2.19). We write S as

$$\sum_{k\geqslant 0}\frac{(n+k)!(2k)!(-1)^k}{(m+2k)!(n-m-k)!k!k!(k+1)}$$

$$= \frac{n!}{m!(n-m)!}\sum_{k\geqslant 0}\frac{(n+1)_k(-n+m)_k(1)_{2k}}{(m+1)_{2k}(1)_k(1)_{k+1}}, \quad \text{by (2.17)}. \tag{3.11}$$

We now rewrite $(1)_{2k}, (m+1)_{2k}$ using (2.19) to get

$$S = \binom{n}{m} \sum_{k \geqslant 0} \frac{(n+1)_k(-n+m)_k(1)_k\left(\frac{1}{2}\right)_k}{\left(\frac{m+1}{2}\right)_k\left(\frac{m}{2}+1\right)_k(1)_k(1)_{k+1}}$$

$$= \binom{n}{m} \frac{\frac{m}{2}\left(\frac{m-1}{2}\right)}{(-1-n+m)n\left(-\frac{1}{2}\right)} \sum_{k \geqslant 0} \frac{(-1-n+m)_{k+1}(n)_{k+1}\left(-\frac{1}{2}\right)_{k+1}}{\left(\frac{m-1}{2}\right)_{k+1}\left(\frac{m}{2}\right)_{k+1}(1)_{k+1}}$$

$$= \binom{n}{m} \frac{m(m-1)}{2n(n+1-m)}\left[{}_3F_2\left(\begin{matrix}-1-n+m,\ n,\ -\frac{1}{2}\\ \frac{m-1}{2},\ \frac{m}{2}\end{matrix}\right) - 1\right]$$

$$= \frac{(n-1)!}{2(m-2)!(n-m+1)!}\left[\frac{\left(\frac{m-1}{2}-n\right)_{n+1-m}\left(\frac{m}{2}\right)_{n+1-m}}{\left(\frac{m-1}{2}\right)_{n+1-m}\left(\frac{m}{2}-n\right)_{n+1-m}} - 1\right], \quad \text{by (2.8)}$$

$$= \frac{(n-1)!}{2(m-2)!(n-m+1)!}\left[\frac{\left(\frac{m+1}{2}\right)_{n+1-m}\left(\frac{m}{2}\right)_{n+1-m}}{\left(\frac{m-1}{2}\right)_{n+1-m}\left(\frac{m}{2}\right)_{n+1-m}} - 1\right], \quad \text{by (2.18)}.$$

$$= \frac{(n-1)!}{2(m-2)!(n-m+1)!}\left[\frac{n+\frac{1}{2}-\frac{m}{2}}{\frac{m-1}{2}} - 1\right]$$

$$= \binom{n-1}{m-1}. \tag{3.12}$$

The reader may like to verify that $\sum_{k \geqslant 0}\binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n}$, though somewhat different in appearance from (3.10), also reduces to the Pfaff-Saalschütz series. This sum is also from Knuth [**11**].

I. J. Good [**9**] encountered the following sum in some work in probability theory:

$$\sum_{\nu=0}^{s}(-1)^{\nu}\binom{\beta}{\nu}\binom{\beta+s-\nu}{\beta}\frac{\alpha}{\alpha+s-\nu}. \tag{3.13}$$

He remarks that it is difficult to see directly that this sum is positive when $\alpha > \beta$. However, (3.13) is equal to

$$\frac{(\beta+s)!}{s!\beta!}\frac{\alpha}{\alpha+s}{}_3F_2\left(\begin{matrix}-s,\ -\beta,\ -\alpha-s\\ -\beta-s,\ -\alpha-s+1\end{matrix}\right), \tag{3.14}$$

which is again the Pfaff-Saalschütz series (2.8) and the value of the sum (3.13) is

$$\frac{(\alpha-\beta)_s}{(\alpha+1)_s},\quad \text{which is positive when } \alpha > \beta. \tag{3.15}$$

Now consider the identity,

$$\sum_{j=0}^{k}\binom{k}{j}^2\binom{n+2k-j}{2k} = \binom{n+k}{k}^2, \tag{3.16}$$

where k and n are nonnegative integers. Takács **[16]** has written a brief history of the various proofs of this identity which have been given in the past fifty years; some of these are quite complicated. However, L. Carlitz pointed out that this identity too is a particular case of the Pfaff-Saalschütz summation (2.8). The reader will not find it difficult to verify this.

The sums above reduced to a balanced ${}_3F_2$. The next example from Riordan [**13**, p. 87] reduces to a 2-balanced ${}_3F_2$. The problem is to evaluate

$$\sum_{k=0}^{n}\frac{2n}{n+k}\binom{n+k}{2k}\binom{2k}{k}(k+p)^{-1}(-1)^k. \tag{3.17}$$

It is easily seen that (3.17) is

$$\frac{2}{p}{}_3F_2\left(\begin{matrix}-n,\ n,\ p\\ 1,\ p+1\end{matrix}\right),\quad \text{which can be evaluated by (2.9).} \tag{3.18}$$

EXAMPLE 3. We now consider examples which reduce to Dixon's well-poised ${}_3F_2$ given by (2.15). Problem 62 on page 73 of Knuth **[11]** is to show that

$$\sum_{k=-l}^{l}(-1)^k\binom{2l}{l+k}\binom{2m}{m+k}\binom{2n}{n+k} = \frac{(l+m+n)!(2l)!(2m)!(2n)!}{(l+m)!(m+n)!(n+l)!l!m!n!}, \tag{3.19}$$

where we are assuming that $l = \min(l, m, n)$. To write (3.19) as a hypergeometric series, set $j = k + l$ in (3.19) to get

$$(-1)^l\sum_{j=0}^{2l}(-1)^j\binom{2l}{j}\binom{2m}{m-l+j}\binom{2n}{n-l+j}. \tag{3.20}$$

By the methods already explained, this reduces to the series

$$\frac{(-1)^l (2m)!(2n)!}{(m-l)!(m+l)!(n-l)!(n+l)!}\,{}_3F_2\left(\begin{matrix}-2l, & -m-l, & -n-l\\ & m-l+1, & n-l+1\end{matrix}\right), \tag{3.21}$$

which is Dixon's well-poised ${}_3F_2$. However, Dixon's result cannot be applied directly since we get the term $\Gamma(1-l)/\Gamma(1-2l)$ on the right, which is undefined. To take care of this difficulty we consider the following case of Dixon's formula:

$$\begin{aligned}&{}_3F_2\left(\begin{matrix}-2l-2\varepsilon, & -m-l-\varepsilon, & -n-l-\varepsilon\\ & m-l-\varepsilon+1, & n-l-\varepsilon+1\end{matrix}\right)\\ &\quad=\frac{\Gamma(1-l-\varepsilon)\Gamma(1+m-l-\varepsilon)\Gamma(1+n-l-\varepsilon)\Gamma(1+m+n+l+\varepsilon)}{\Gamma(1-2l-2\varepsilon)\Gamma(1+m)\Gamma(1+n)\Gamma(1+m+n)}.\end{aligned} \tag{3.22}$$

We now apply the formula (2.13) to the right side of (3.22) to get

$$\frac{\sin\pi(2l+2\varepsilon)}{\sin\pi(l+\varepsilon)}\cdot\frac{\Gamma(2l+2\varepsilon)}{\Gamma(l+\varepsilon)}\cdot\frac{\Gamma(1+m-l-\varepsilon)\Gamma(1+n-l-\varepsilon)\Gamma(1+m+n+l+\varepsilon)}{\Gamma(1+m)\Gamma(1+n)\Gamma(1+m+n)}.$$

We let $\varepsilon\to 0$ and obtain (3.19). Riordan [**13**, p. 89] has the following series:

$$\sum_{k=1}^{m} 2k\binom{2p}{k+p}\binom{2n}{k+n}, \tag{3.23}$$

and this too reduces to Dixon's sum.

4. Proofs of the hypergeometric identities. In this section we give inductive proofs of the identities stated in Section 2. There are many other proofs and the reader can consult Askey [**3**] or Bailey [**6**] for these. Askey's analytic proofs best indicate why these identities hold.

We begin by proving the Pfaff-Saalschütz theorem. This proof is due to Dougall [**8**]. We first observe the symmetry in a, b and n by rewriting (2.8) as

$${}_3F_2\left(\begin{matrix}-a, & -b, & -n\\ & c, & d\end{matrix}\right)=\frac{\Gamma(a+c+n)\Gamma(b+c+n)\Gamma(a+b+c)\Gamma(c)}{\Gamma(c+a)\Gamma(b+c)\Gamma(c+n)\Gamma(a+b+c+n)}, \tag{4.1}$$

where we have used property (2.11) of the gamma function. Now since $d=1-a-b-n-c$, to prove (2.8) we must show that

$$(c)_n(c+a+b)_n\sum_{j=0}^{n}\frac{(-n)_j(-a)_j(-b)_j}{j!(c)_j(1-a-b-n-c)_j}=(c+a)_n(c+b)_n. \tag{4.2}$$

Note that it follows from (2.16) that both sides of (4.2) are polynomials in b of degree n. Therefore, it is sufficient to prove that they are equal for $n+1$ distinct values of b. Clearly the result is true for $n=0$. Assume the result true for $n=0,1,\ldots,k-1$. Now set $n=k$. By symmetry in b and n, it follows from the inductive hypothesis that (4.2) is true for $b = 0, 1, \dots, k-1$. If we can find one more value of b for which the relation is true, we would be done. Observe that (2.16) implies

$$\frac{(c+a+b)_n}{(1-a-b-n-c)_j} = (-1)^j (c+a+b)_{n-j}. \tag{4.3}$$

Thus, when $b = -a-c$, both sides of (4.2) are equal to $(c+a)_k(-a)_k$ and the result is proved.

We can derive Chu-Vandermonde from (4.1). Let $a = m$, an integer, let $n \to \infty$ and use Stirling's formula (2.14) together with (2.11) to get

$${}_2F_1\left(\begin{matrix} -m, & -b \\ & c \end{matrix}\right) = \frac{(c+b)_m}{(c)_m}. \tag{4.4}$$

An inductive argument can be used to prove the Sheppard-Andersen identity as well. The identity of Dixon lies a little deeper. Dougall showed that a much more general identity could be proved by the method used to prove (4.1). The reader may reconstruct the proof himself or consult Dougall's paper. The argument is also reproduced in Bailey **[6]** and Hardy **[10]**. The identity gives the sum of a very well-poised 2-balanced ${}_7F_6$.

$${}_7F_6\left(\begin{matrix} a, \ 1+\frac{1}{2}a, & -b, \ -c, \ -d, \ -e, \ -n \\ \frac{1}{2}a, & 1+a+b, \ 1+a+c, \ 1+a+d, \ 1+a+e, \ 1+a+n \end{matrix}\right)$$

$$= \frac{(1+a)_n(1+a+b+c)_n(1+a+b+d)_n(1+a+c+d)_n}{(1+a+b)_n(1+a+c)_n(1+a+d)_n(1+a+b+c+d)_n}, \tag{4.5}$$

where $1+2a+b+c+d+e+n = 0$ and n is a positive integer. The last relation simply means that the series is 2-balanced. The adjective "very" in very well-poised refers to the factor

$$\frac{\left(\frac{a}{2}+1\right)_k}{\left(\frac{a}{2}\right)_k} = \frac{a+2k}{a}$$

in the series. To derive Dixon's identity we merely set $d = -\frac{1}{2}a$ in (4.5), let $n \to \infty$, apply Stirling's formula (2.14), and use (2.11).

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UNSOLVED PROBLEMS

EDITED BY RICHARD GUY

In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics.*

A Conjecture Related to Chi-Bar-Squared Distributions

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Consider a polyhedral, convex, closed cone C in $\mathbb{R}^n$. We write $\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n$ and $\|x\| = \langle x, x\rangle^{1/2}$. Let $P:\mathbb{R}^n \to C$ be the projection mapping onto C, which is defined as assigning to each $x \in \mathbb{R}^n$ the closest point in C, that is

$$\|x - P(x)\| = \min_{y\in C}\|x - y\|.$$

Let $X = (X_1,\ldots, X_n)$ be a multivariate random variable having the standard