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The 7 is written below the 3, and the remainder, $53 - 49 = 4$, above the 3. This remainder with the next digit in the dividend is 41; the largest multiple of $7 \leqslant 41$ is $35 = 7 \times 5$. The 5 is written below the 1, and the remainder, $41 - 35 = 6$, above the 1. The last remainder is 5. Usually the remainders, which have been written above the dividend, are carried mentally.

It is notable that this process involves guessing the largest multiple of the divisor that is less than a certain number, but only a few guesses are ever required and no practical difficulty is encountered.

Because $53,149 = 7,592 \times 7 + 5$, one may write (see 44) the ratio of 53,149 and 7 so that 7,592 is the partial quotient and 5 is the remainder.

(43)
```
    4 615
  7)53 149
    7 592
```

(44) $$\frac{53,149}{7} = 7,592\frac{5}{7}$$

(45)
```
621) 83 742 (134
     62 1
     21 64
     18 63
      3 012
      2 484
        528
```

(46) $$83,742 = 621 \times 134 + 528$$

(47)
```
621) 83 742 (134
     21 64
      3 012
        528
```

(48) $$\frac{83\ 742}{621} = 134\frac{528}{621}$$

(49)
```
621) 83 742 (134.8502
     21 64
      3 012
        528 0
         31 20
          1 500
            258
```

(50) $$\frac{83742.0000}{621} = 134.8502$$

(51) $$\frac{83742}{621} = \frac{837.42}{6.21} = 134.8502$$

(52)
```
       15 78.42 (39.72
        9
  69 | 6 78
     | 6 21
 787 | 57 42
     | 55 09
7942 |  2 33 00
     |  1 58 84
           74 16
```

If the divisor exceeds 10, long division is preferable (see 45). To divide 83,742 by 621, the 1 is written as the first digit of the quotient and 621×1 is subtracted from 837 since the largest multiple of $621 \leqslant 837$ is 621×1. Actually what is done is to subtract 621×100 from 83,742, leaving a remainder of 21,642. Since only the 2,164 will be used in the next step, the final 2 of the dividend need not be brought down yet. The largest multiple of $621 \leqslant 2,164$ is $621 \times 3 = 1,863$, and the remainder is 301. Now the final 2 must be brought down to make 3,012. The largest multiple of $621 \leqslant 3,012$ is $621 \times 4 = 2,484$, and the remainder is 528. Long division

Thus, the dividend (see 46) is the sum of a product and a remainder.

The preceding method may be considerably abbreviated by mentally subtracting the partial product as it is formed (see 47). Again, the resultant quotient may be expressed as the sum of an integer and a proper fraction (see 48). If it is desired to express the result as a decimal fraction, carried to a given number of decimal places, the above process is merely continued (see 49). Any two decimal numbers may be divided in this manner. The number of digits to the right of the decimal point in the quotient is equal to the number of such digits in the dividend, diminished by the number of such digits in the divisor, with care being taken to add to the dividend all the 0's that are brought down. Thus, in the present example (49) the proper fraction (see 50) is replaced by a decimal fraction. An alternative and exceptionally clear and simple method for determining the position of the decimal point in the quotient is (see 51) to divide both numerator and denominator by an appropriate power of 10; *i.e.*, expressing the divisor as a number lying between 0 and 10, which renders it obvious that the quotient lies between 100 and 1,000.

Divisibility rules. In both theory and practical application of arithmetic it is often important to factor a natural number; that is, to decompose it into numbers that, when multiplied together, will yield the given number as product.

The following tests for divisibility are therefore given. A composite number is divisible

- by 2 if it is even (*i.e.*, if it ends in 0, 2, 4, 6, or 8);
- by 3 if the sum of its digits is divisible by 3;
- by 4 if the number formed by its last two digits is divisible by 4;
- by 5 if it ends in 0 or 5;
- by 6 if it is even and the sum of its digits is divisible by 3;
- by 8 if the number formed by its last 3 digits is divisible by 8;
- by 9 if the sum of its digits is divisible by 9;
- by 10 if it ends with 0;
- by 11 if the difference between the sum of its digits in the odd places and that of the digits in the even places is either 0 or divisible by 11.

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Similar rules are easily devised with respect to other divisors by means of appropriate combinations of the rules given above. For example, if a number is to be divisible by 132, say, then it must satisfy the test for divisibility by 3, by 4, and by 11, respectively; *i.e.*, $3 \times 4 \times 11 = 132$.

Evolution. An algorithm for the determination of the square root of a decimal number, such as 1,578.42, is carried out as follows (see 52): Starting at the decimal point, the number is separated into periods of two digits each. The leftmost period is 15, and the largest square $\leqslant 15$ is $9 = 3^2$. The 3 is written at the right and the remainder $15 - 9 = 6$ is brought down. The next period 78 is brought down beside the 6, giving 678. The 3 is doubled and written at the left. By trial it is found that 9 is the largest digit, such that $69 \times 9 = 621 \leqslant 678$. The 9 is written after the 3 in the answer, and the difference, 57, is brought down, followed by the next period, 42. The partial answer 39 is doubled and written at the left under the 69. By trial it is found that $787 \times 7 = 5,509 \leqslant 5,742$ while $788 \times 8 > 5,742$. The next digit in the answer is therefore 7. The process is continued until the desired degree of accuracy is attained. In the present example (see 52) the last remainder, 7,416, exceeds half Method of square root of the last trial divisor, 7,942, so that 3 is a better approximation than 2 for the last digit. In fact, $39.73^2 = 1,578.4729$.

The above process is based upon the relation $(a + b)^2 = a^2 + (2a + b)b$. In each step the part of the square root already obtained is a; the part remaining to be found is b. In the example (see 52), $(a + b)^2 = 1,578.42$, $a = 30$. Then $2ab + b^2 = 678.42$. In order to determine the largest integer n in b, one must find the largest integer n (namely 9), such that $(2a + n)n \leqslant 678.42$. The trial divisor is $2a + n = 69$. In the next step $a = 39$, $2ab + b^2 = 57.42$, etc.

Cube and higher roots. The cube root of a number may be calculated by a similar algorithm based upon the relation (see 53) that expresses the difference of two cubes as a product involving a quadratic as a factor. Thus, it is possible to find the approximate cube root of 279,463 by proceeding as in the example (see 54). This last remainder is too large but much closer than the remainder resulting from using 7 as the last digit.

(53) $$(a+b)^3 - a^3 = (3a^2 + 3ab + b^2)b$$

(54)

	279 463.000 (65.38–
	216
10 800	63 463
900	
25	
11 725	58 625
1 267 500	4 838 000
5 850	
9	
1 273 359	3 820 077
127 922 700	1 017 923 000
156 720	
64	
128 079 484	1 024 635 872

A fourth root is easily obtained as the square root of the square root. Fifth and higher roots can be obtained by an algorithm similar to those just given, based upon the expansion of $(a + b)^n$ by the binomial theorem. The method is cumbersome and is seldom used, for the method of logarithms is easy and rapid.

LOGARITHMS

Napier's contribution

Another marvel of human inventiveness in mathematics is the logarithm, introduced with virtually no precedent in 1614 by the Scottish mathematician John Napier (*q.v.*). As a concept, its hallmarks are simplicity, economy, and utility.

Formally, if $N = b^p$, then $\log_b N = p$, which is read as "the logarithm of the number N to the base b is p." Thus, $\log_{10} 100 = 2$ because $100 = 10^2$, and $\log_2 8 = 3$ because $8 = 2^3$. The simplicity is self-evident. Next, it is noted that if $N = 100,000,000 = 10^8$, then $\log_{10} N = 8$, and the impressive economy effected in the representation of the number N is observed, cut down as it is from no fewer than nine digits in conventional notation to but one in terms of the corresponding logarithm. These characteristics alone are striking. The full measure of the value of this concept, however, becomes apparent when it is realized that, apart from its simplicity and economy, it reduces all multiplications and divisions to mere additions and subtractions, and all involution (raising to a power) and evolution (determination of square root) to mere multiplications, respectively. The importance of this is that the number of operations generally involved in multiplying (or dividing) two numbers always exceeds the number of operations involved in adding (or subtracting) them. Furthermore, this discrepancy in labour rapidly increases with the magnitudes of the particular numbers involved.

As for illustrative examples, if $N = 10^a$, and $M = 10^b$, then, using the basic results of arithmetic (see above *Fundamental definitions and laws* and *Theory of divisors*), it will be found that $N \cdot M = 10^a \cdot 10^b = 10^{a+b}$, and hence, $\log (N \cdot M) = a + b$, a simple sum. (In the preceding and below, the convention is used that if no base is specified, log refers to base 10.) Similarly, then, $N/M = 10^{a-b}$, so that $\log (N/M) = a - b$, a simple difference.

Next, in considering involution and evolution, in which, for the sake of clarity, it is assumed that n is a real number, if $P = N^n$, then $\log P = \log (N^n) = \log ((10^a)^n) = \log (10^{an}) = an = n \cdot \log N$, and it is seen that involution has been transformed into a mere product. (N has been used here as defined in the preceding paragraph.) Similarly, if $R = N^{1/n}$, then $\log R = \log ((10^a)^{1/n}) = \log (10^{a/n}) = a/n = (1/n) \cdot \log N$, and the tedious process of determining the value of the nth root of N has been reduced to a mere multiplication (or, depending on one's point of view, to a simple division) at most. It is to be noted that in each of these examples the quantities a, b, n need not be positive (or negative) integers: instead, in accord with above *Fundamental definitions and laws* and *Theory of divisors*, they may be fractions, proper or improper, positive or negative. Indeed, they may also be irrational numbers, or logarithms themselves; for example, $\log N^{\log M} = \log N^b = b \cdot \log N = ab$.

The expression $\log 1 = 0$ holds for any base b because $b^0 = 1$ for all (nonzero) values of b. Similarly, for any base b $(\neq 0)$, $\log_b b = 1$ because $b^1 = b$.

Sometimes it is necessary to transform logarithms from one base a to another, say b. To see how this transformation can be accomplished, N may now be taken to be a number such that $\log_a N = x$, so that $a^x = N$. It follows that $\log_b N = \log_b (a^x) = x \cdot \log_b a = (\log_b a) \cdot \log_a N$. A case of particular interest is that in which $a = 10$, $b = e$, the symbol e denoting the base of the so-called natural or Naperian logarithms. From the result above, it follows at once that $\log_e N = (\log_e 10) \times (\log_{10} N)$, but $\log_e 10$ is a well-known constant $(2.30258 \cdots)$. The preceding equation thus enables one to compute $\log_e N$ if $\log_{10} N$ is known, and conversely. (C.C.MacD.)

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