Wheels on Wheels on Wheels —Surprising Symmetry

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While designing a computer laboratory exercise for my calculus students, I happened to sketch the curve defined by this vector equation:

$$(x, y) = (\cos(t), \sin(t)) + \frac{1}{2}(\cos(7t), \sin(7t)) + \frac{1}{3}(\sin(17t), \cos(17t)).$$

I was thinking of the curve traced by a particle on a wheel mounted on a wheel mounted on a wheel, each turning at a different rate. The first term represents the largest wheel, of radius 1, turning counter-clockwise at one radian per second. The second term represents a smaller wheel centered at the edge of the first, turning 7 times as fast. The third term is for the smallest wheel centered at the edge of the second, turning 17 times as fast as the first, clockwise and out of phase. See FIGURE 1. As you can notice from FIGURE 2, this curve displays a 6-fold symmetry, a fact that one would probably not guess by looking at the formulas.

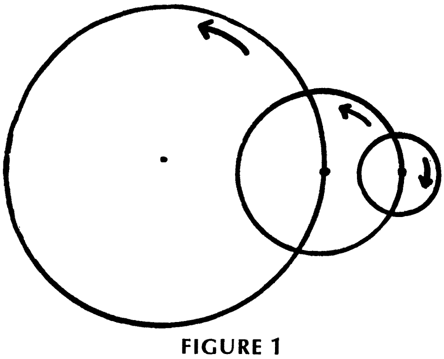


FIGURE 1

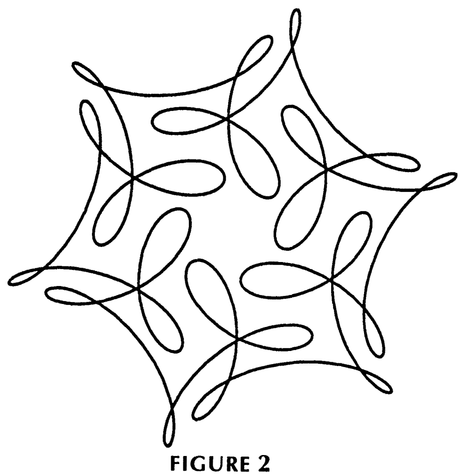


FIGURE 2

The symmetry of this curve, and a condition for the symmetry of any continuous curve, is illuminated by introducing complex notation, in terms of which the formulas above represent a terminating Fourier series. This connection enables the statement and proof of a general theorem relating the symmetry of a parametric curve to the frequencies present in its Fourier series.

1. Analysis of Examples In complex notation, the curve above is written

$$f(t) = x(t) + iy(t) = e^{it} + \frac{1}{2}e^{7it} + \frac{i}{3}e^{-17it}.$$

The source of the symmetry turns out to be this: 1, 7, and -17 are all congruent to 1 modulo 6. When t is advanced by one-sixth of 2π, each wheel has completed some number of complete turns, plus one-sixth of an additional turn, resulting in symmetry. Examine what happens when time is advanced by one-sixth of a period for a representative wheel:

$$e^{(6j+1)i(t+2\pi/6)} = e^{(6j+1)it}e^{2\pi i/6}.$$

This wheel is back where it started, but rotated one-sixth of the way around. When each wheel in the superposition has this same behavior, symmetry will result.

A similar result is obtained using any integer m instead of 6, and keeping track of all the wheels at once.

If

$$f(t) = \sum a_j e^{n_j it} \text{ with } n_j = b_j m + 1,$$

then

$$f\left(t + \frac{2\pi}{m}\right) = \sum a_j e^{n_j i(t+2\pi/m)} = \sum a_j e^{n_j it}e^{n_j i2\pi/m} = e^{2\pi i/m}f(t),$$

has m-fold symmetry. Imagine the trigonometric identities we have avoided by using complex rather than real notation! Notice that an infinite sum would behave the same way as long as it converges for each value of t.

That this is not the end of the story can be seen from another example:

$$f(t) = e^{-2it} + \frac{e^{5it}}{2} + \frac{e^{19it}}{4}.$$

The 7-fold symmetry apparent in FIGURE 3 arises because -2, 5, and 19 are all congruent to 5 modulo 7. Here there is an additional, perhaps less interesting, symmetry; since the coefficients are real, $f(-t)$ is the complex conjugate of $f(t)$. The analog of the computation above for a single wheel is now:

$$e^{(7j+5)i(t+2\pi/7)} = e^{(7j+5)it}e^{(5\cdot 2\pi i/7)}.$$

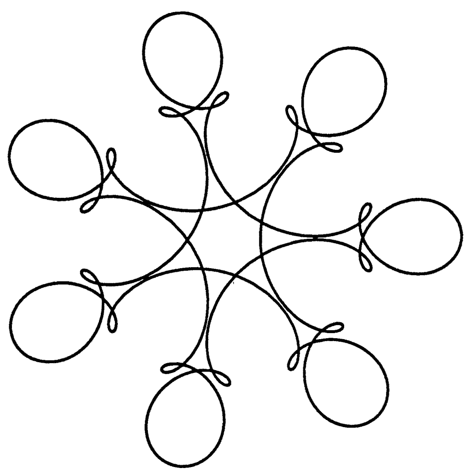


FIGURE 3

Here the wheel has rotated five-sevenths of a full rotation, after time has advanced one-seventh of the fundamental period 2π. If we superimpose several wheels with the same behavior, symmetry will result.

Imagine the plane divided into sectors of size $2\pi/7$, numbered $0, 1, \ldots 6$. After completing its first loop in sector 0, the curve moves on to trace the same pattern in sector 5, then in sector 10, which is more simply called sector 3, and so on. Proceeding by multiples of 5, reduced modulo 7, the curve fills in every sector.

This motivates a definition. We say that a function $f(t)$ exhibits *m-fold symmetry* if, for some integer k we have

$$f\left(t+\frac{2\pi}{m}\right)=e^{k2\pi i/m}f(t). \tag{1}$$

It also makes sense to require that k be prime modulo m, for if k times j were congruent to zero mod m, applying (1) j times would give:

$$f\left(t+j\frac{2\pi}{m}\right)=f(t).$$

This would be considered a periodicity of the function rather than symmetry. For instance, suppose we seek a 6-fold symmetry from a wheel with exponent $(2it)$. Advancing time by a sixth of the fundamental period gives:

$$e^{2i(t+2\pi/6)}=e^{2it}e^{2\cdot 2\pi/6}.$$

Numbering six sectors from 0 to 5, we find that the wheel traces its pattern in sectors 0, 2, and 4, returning to sector 0 without ever tracing in sectors 1, 3, and 5. This would be a 3-fold, rather than a 6-fold, symmetry.

The equations above show that m-fold symmetry occurs in a sum of this type when all the frequencies are congruent modulo m to the same number k, which must be a prime modulo m.

Knowing this, it is amusing to experiment by assembling terms to produce a curve of given symmetry.* For FIGURE 4, I chose frequencies $2, -16, -7, 29$ (all congruent to $2 \bmod 9$) to produce a curve with 9-fold symmetry; I then adjusted the coefficients to make the pattern more pleasing. They are 1, $i/2$, $1/5$, and $i/5$.

2. Symmetry and Fourier Series The discussion of examples virtually proves one direction of the following theorem. We choose $C^0[0, 2\pi]$ as a simple setting for proving the following theorem.

THEOREM. *If a continuous function f is not identically zero then f has m-fold symmetry, in the sense of satisfying* (1), *if, and only if, the nonzero coefficients in the Fourier series for f,*

$$f(t) \sim \sum_{n=-\infty}^{\infty} a_n e^{nit},$$

correspond to frequencies, n, which are all congruent to the same prime modulo m.

*Dr. Erich Neuwirth, of the University of Vienna, has kindly prepared a spreadsheet (Microsoft Excel 5.0) for experimenting with curves of this type. It can be downloaded from http://www.smc.univie.ac.at/~neuwirth/wheels.

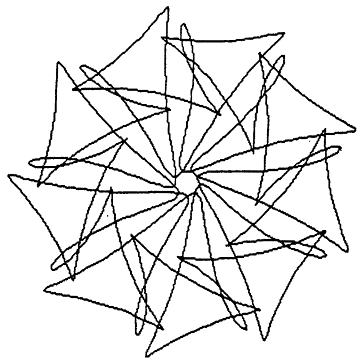


FIGURE 4

Proof. Since the Fourier series for f converges pointwise everywhere **[1]**, the discussion above shows that the sum of a series whose frequencies are all congruent to the same prime modulo m does exhibit m-fold symmetry. It remains to prove that functions with the given symmetry do indeed have Fourier series of the type discussed above, with the frequencies all congruent modulo m to a prime modulo m.

Assume that a continuous function f has the symmetry defined in equation (1), with k a prime modulo m. In the integral formula for the Fourier coefficients for f, we will break up the integral into a sum of m integrals:

$$2\pi a_n = \int_0^{2\pi} f(t)\, e^{-int}\, dt = \sum_{j=0}^{m-1} \int_{j2\pi/m}^{(j+1)2\pi/m} f(t)\, e^{-int}\, dt.$$

We make the change of variables, $u = t - j2\pi/m$ to make all the limits of integration range from 0 to $2\pi/m$. In term j of the resulting sum of integrals, we use (1) j times, obtaining:

$$\int_0^{2\pi/m} f\left(u + j\frac{2\pi}{m}\right) e^{-inu}\, e^{-inj2\pi/m}\, du = \int_0^{2\pi/m} f(u)\, e^{-inu}\, e^{ji(k-n)2\pi/m}\, du.$$

The m integrals are now identical and may be factored out. We find:

$$2\pi a_n = \Pi \int_0^{2\pi/m} f(u)\, e^{-inu}\, du \sum_{j=0}^{m-1} e^{ji(k-n)2\pi/m}.$$

The sum can be rewritten as

$$\sum_{j=0}^{m-1} \left(\omega^{(k-n)}\right)^j,$$

where ω is a primitive m-th root of unity. Such a sum is zero unless all the terms are one, in which case m divides $k - n$ and n is congruent to k modulo m. Thus the only frequencies with nonzero coefficients are those congruent to k modulo m.

3. Pedagogical Directions My intent in constructing the original example was to interest students in some pleasing curves that would be too difficult to sketch by hand. Showing students a few examples and assigning them the problem of discovering criteria for symmetry would make an interesting calculus project, serving as an effective advertisement for the power of complex notation.

Other questions present themselves. For any prime m, call

$$F_{k,m} = \{f\varepsilon L^2(0,2\pi) | a_n = 0 \text{ unless } n \equiv k \bmod m\}.$$

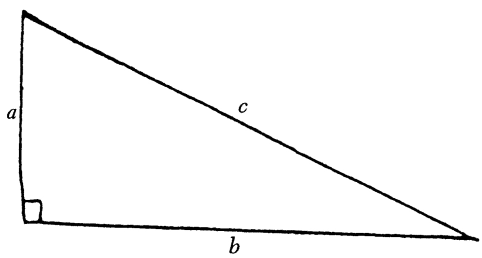
For each m, these spaces give an orthogonal decomposition of L^2 that generalizes the writing of a function as the sum of even and odd parts. Operators projecting onto such subspaces are similar to the Hardy projector. These would provide interesting examples for students first encountering Hilbert spaces.

REFERENCE

1. H. Dym and H. P. McKean, *Fourier Series and Integrals*, Academic Press, 1972.

Proof Without Words: Parametric Representation of Primitive Pythagorean Triples

$$\frac{a}{2}, b, c \in Z^+, (a,b) = 1$$



$$\frac{c+b}{a} = \frac{n}{m}, (n,m) = 1 \Rightarrow \frac{c-b}{a} = \frac{m}{n}$$

$$\Rightarrow \frac{c}{a} = \frac{n^2+m^2}{2nm}, \frac{b}{a} = \frac{n^2-m^2}{2nm}$$

$$\Rightarrow n \not\equiv m (\bmod 2)$$

$$\therefore (a,b,c) = (2nm, n^2-m^2, n^2+m^2)$$

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Note: For details and related results, see the authors' article "Pythagorean Triples: The Hyperbolic View," *College Math. J.*, May 1996.