Quadrilateral Decomposition by Two-Ear Property Resulting in CAD Segmentation

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***Abstract*—We mainly aim at splitting a simply connected polygon into a set of convex quadrilaterals without inserting new boundary nodes. Our approach is based on repeatedly removing quadrilaterals from the given polygon. We derive theoretical results pertaining to quadrangulation of simply connected polygons from the usual 2-ear theorem. That yields a quadrangulation technique with $\mathcal{O}(n)$ number of quadrilaterals. Additionally, we supplement our theoretical methodology by practical results and CAD surface segmentation.

***Keywords*—Quadrilateral decomposition, quadrangulation, CAD.

I. Introduction

Quadrilateral decomposition is an important processing in computational geometry. It has several important applications in computer graphics, numerics and engineering [8], [7]. The main objective of this paper is to propose a methodoly for splitting a simply connected polygon into quadrilaterals. In fact, this is a completion of our previous work with Brunnett as described in [9], [10], [11]. Our main contribution can be summarized as follows:

- Quadrilateral decomposition without prior triangulation,
- Introducing two operations for quadrilateral removal,
- Theoretical proofs supporting the algorithm,
- The number of quadrilaterals is $\mathcal{O}(n)$,
- Practical implementation and CAD application.

Related works are as follows, Ramaswami *et al.* [8] used the *percolation algorithm* to transform a triangular mesh into a quadrilateral one by using graph-based approach. Lee and Lo [5] have used a method which needs the *merging front* that is initialized to be the polygonal boundary of the domain. It is also an indirect method which needs a background triangular mesh. It recursively tries to merge a triangle which is incident upon the merging front and another adjacent one. The authors failed to give any theoretical proof of their merging algorithm. The Q-Morph (or quad morphing) algorithm [7] uses a similar approach as the above method where the author uses an advancing front consisting of a set of edges that must be updated every time new quadrilaterals are formed. In [1], the authors first fill the domain with circles which are tangent to one another. The gaps in the domain are then bounded by a few circles. The *circle packing* method amounts to generate edges whose endpoints are centers of those circles.

The structure of this paper is as follows. First, we start by formulating the problem accurately and by introducing several definitions. Section III introduces two operations with which one can remove a single quadrilateral from a polygon. In Section IV, we use those two operations to design an algorithm for quadrangulation. The purpose of Section V is twofold. First, we show results from practical implementation of the theoretical methods. Second, we briefly describe how to apply our method to real CAD models.

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II. Problem Setting and Definitions

Let P be a simply connected polygon with an even number of boundary vertices $\{\mathbf{x}_k\}$ $k = 1, ..., n$. Our objective is to find a set of convex quadrilaterals $\{Q_i\}$ such that (see Fig. 1(a)):

(P0) $P = \bigcup_{i=1}^{m} Q_i$.

(P1) For $i, j = 1, ..., m$ $(i \neq j)$ the intersection $Q_i \cap Q_j$ is either empty or a single node or a complete edge.

(P2) Each vertex of a quadrilateral Q_i is either an element of $\{\mathbf{x}_k\}_{k=1}^{n}$ or it is strictly inside P. In other words, we do not allow boundary Steiner points.

We have required that the number of boundary nodes be even in order to guarantee [8] the solvability of this problem. To simplify the description of our approach, we introduce the following notations and definitions.

For two given points $\mathbf{a}$ and $\mathbf{b}$ in the plane, we will denote by $[\mathbf{a}, \mathbf{b}]$ and $]\mathbf{a}, \mathbf{b}[$ the closed and open line segments defined by

$$[\mathbf{a}, \mathbf{b}] \quad := \quad \{\lambda\mathbf{a} + (1-\lambda)\mathbf{b}, \qquad \lambda \in [0, 1] \subset \mathbf{R}\}, \quad (1)$$

$$]\mathbf{a}, \mathbf{b}[\quad := \quad \{\lambda\mathbf{a} + (1-\lambda)\mathbf{b}, \qquad \lambda \in]0, 1[\subset \mathbf{R}\}. \quad (2)$$

The line which passes through $\mathbf{a}$ and $\mathbf{b}$ splits the plane into two *half planes*:

$$(\mathbf{ab})^{+} \quad := \quad \{\mathbf{z} \in \mathbf{R}^2 : \quad \det(\vec{\mathbf{az}}, \vec{\mathbf{ab}}) > 0\}, \quad (3)$$

$$(\mathbf{ab})^{-} \quad := \quad \{\mathbf{z} \in \mathbf{R}^2 : \quad \det(\vec{\mathbf{az}}, \vec{\mathbf{ab}}) < 0\} \quad (4)$$

As in most papers in computational geometries, we suppose that the vertices of a polygon is given in counter-clockwise orientation. Let $\mathbf{x}_{i-1}$, $\mathbf{x}_i$, $\mathbf{x}_{i+1}$ be three consecutive vertices of a polygon P. The region

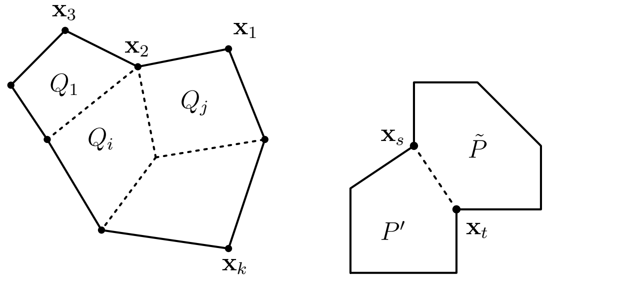
$$\mathcal{W}(\mathbf{x}_i) := (\mathbf{x}_{i-1}\mathbf{x}_i)^{-} \cap (\mathbf{x}_{i+1}\mathbf{x}_i)^{+} \quad (5)$$

is called the *wedge* of $\mathbf{x}_i$. The vertex $\mathbf{x}_i$ is called a *reflex* vertex if

$$\det(\mathbf{x}_i - \mathbf{x}_{i-1}, \mathbf{x}_{i+1} - \mathbf{x}_i) < 0. \quad (6)$$

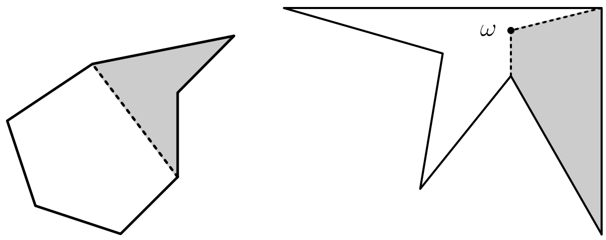
A point $\mathbf{a}$ is *visible* from a vertex $\mathbf{x}_k \in P$ if $]\mathbf{a}, \mathbf{x}_k[$ does not intersect any edge of P. The *kernel* $\ker(P)$ of P is the set of points inside P which are visible from all vertices of P

A *cut* within P is a line segment $[\mathbf{x}_p, \mathbf{x}_q]$ which is formed by two nonconsecutive vertices of P and which is inside P such that $]\mathbf{x}_p, \mathbf{x}_q[$ does not intersect any edge of P. Chopping off



(a) (b)

Fig. 1. (a) Quadrangulation (b) Chopping off a polygon from an initial one.



(a) (b)

Fig. 2. (a) Chop off a quadrilateral with one cut (b) Remove a quadrilateral with two cuts and one internal node.

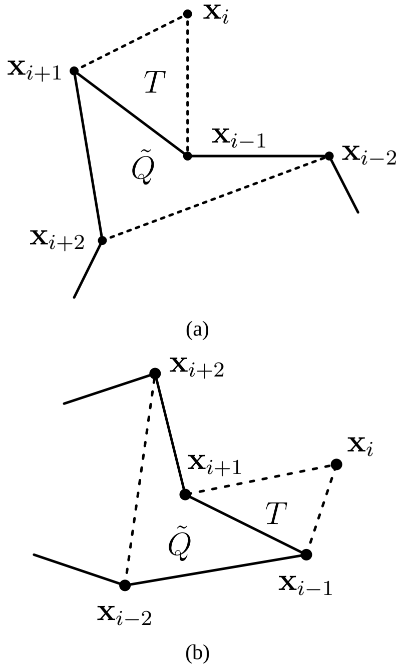


Fig. 3. The ear $[\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}]$ and the removed quadrilateral $\tilde{Q}$ are adjacent.

a subpolygon P' from P means introducing a cut e that splits P into P' and the remaining polygon $\tilde{P}$, i.e.

$$P = P' \cup \tilde{P}, \quad \text{and} \quad P' \cap \tilde{P} = e. \tag{7}$$

An *ear* T of a polygon P is a triangle formed by three consecutive vertices of P such that one edge of T is a cut. Since our method is based on the 2-ear theorem [6], we recall

Theorem 1[Meister, 1975]
Every simply connected polygon having at least four vertices has two nonoverlapping ears.

III. Two Operations for Quadrilateral Removal

We will use the next theorem to split a simply connected polygon into quadrilaterals without additional boundary vertices.

Theorem 2 Let P be a simply connected polygon having at least five vertices. Then, one of the next two operations can be applied:

(Op1) One can remove a quadrilateral which is not necessarily convex by inserting a single cut as in Fig. 2(a).

(Op2) There exists a point ω in the interior of P such that one can remove a *convex* quadrilateral by inserting two line segments emanating from ω to two vertices of P (Fig. 2(b)).

Proof
It is proved by induction with respect to the number n of vertices of P. For $n = 5$, use the theorem of Meister to chop off one triangle from P and the remaining four vertices form the quadrilateral which can be discarded from P by using operation (Op1). Let us suppose that the claim holds for every polygon having n vertices. Now we consider a polygon P with $n+1$ vertices. We will search for a quadrilateral Q which can be discarded from P.

First, we apply the 2-ear theorem to chop off a triangle $T := [\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}]$ from P and we denote by $\tilde{P}$ the remaining polygon which must have n vertices. That is,

$$\tilde{P} := [\mathbf{x}_0, \cdots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \cdots, \mathbf{x}_n]. \tag{8}$$

After applying the hypothesis of induction to the new polygon $\tilde{P}$, we obtain a quadrilateral $\tilde{Q}$. Three different cases have to be distinguished.

Case 1: If the quadrilateral $\tilde{Q}$ is not incident upon the edge $[\mathbf{x}_{i-1}, \mathbf{x}_{i+1}]$ of T, then we simply need to define $Q := \tilde{Q}$.

Case 2: Suppose that $[\mathbf{x}_{i-1}, \mathbf{x}_{i+1}]$ is an edge of $\tilde{Q}$ and all vertices of $\tilde{Q}$ are elements of $\tilde{P}$ (Fig. 3 and Fig. 4). We have to investigate three subcases.

Case 2.a: Assume that $\tilde{Q} = [\mathbf{x}_{i-2}, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}]$ as in Fig. 3. Observe that with respect to the quadrilateral $\tilde{Q}$, $\mathbf{x}_{i-1}$ is visible from $\mathbf{x}_{i+2}$ (Fig. 3(a)) or $\mathbf{x}_{i+1}$ is visible from $\mathbf{x}_{i-2}$ (Fig. 3(b)) (In fact, you can apply the 2-ear theorem to $\tilde{Q}$). In the first situation, define $Q := [\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}, \mathbf{x}_{i-1}]$. In the other case, define $Q := [\mathbf{x}_{i+1}, \mathbf{x}_{i-2}, \mathbf{x}_{i-1}, \mathbf{x}_i]$. In other words, we have just applied operation (Op1) to P in case 2.a.

Case 2.b: Assume now that $\tilde{Q} = [\mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}, \mathbf{x}_{i+3}]$ as in Fig. 4. If the quadrilateral $\tilde{Q}$ is convex, then we apply (Op1) by defining

$$Q := [\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}]. \tag{9}$$

In the situation that $\tilde{Q}$ is nonconvex, we distinguish four configurations with respect to the position of the reflex vertex within $\tilde{Q}$:

(i) If the vertex $\mathbf{x}_{i+2}$ is the reflex vertex as depicted in Fig. 4(a), we define $Q := [\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}]$.

(ii) If $\mathbf{x}_{i-1}$ is reflex (Fig. 4(b)), we define $Q := [\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}]$.

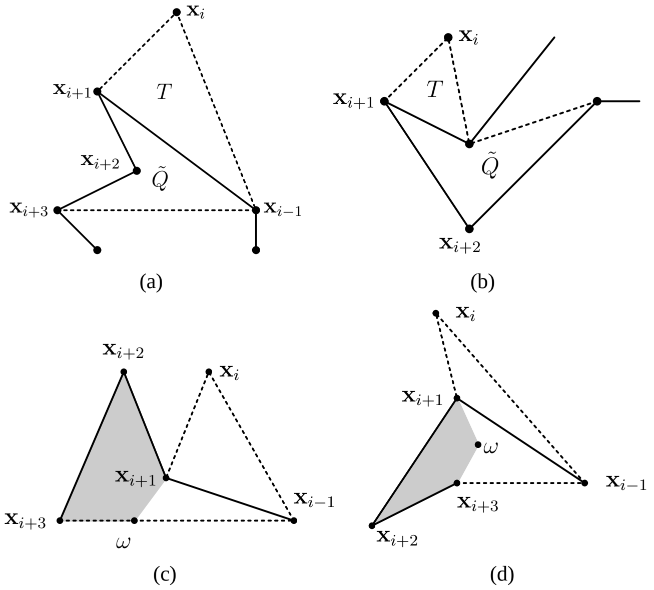


Fig. 4. (a)$\mathbf{x}_{i+2}$ is reflex in $\tilde{Q}$ (b)$\mathbf{x}_{i-1}$ is reflex in $\tilde{Q}$ (c)$\mathbf{x}_{i+1}$ is reflex (d)$\mathbf{x}_{i+3}$ is the reflex vertex in $\tilde{Q}$

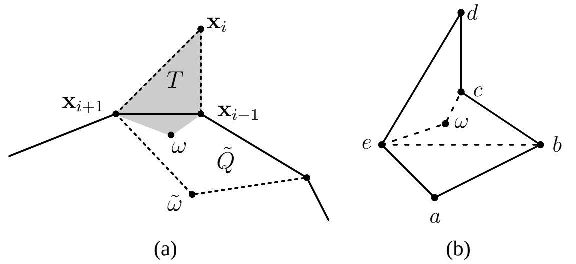


Fig. 5. (a)Introducing a Steiner point ω in $Q \cap \mathcal{W}(\mathbf{x}_i)$ (b) Case of a pentagon.

(iii) In the case that $\mathbf{x}_{i+1}$ is the reflex vertex of $\tilde{Q}$, take any node ω on the open segment $]\mathbf{x}_{i+3}, \mathbf{x}_{i-1}[$ in the wedge of $\mathbf{x}_{i+2}$ as illustrated in Fig. 4(c) and apply (Op2) by defining

$$Q := [\mathbf{x}_{i+2}, \mathbf{x}_{i+3}, \omega, \mathbf{x}_{i+1}]. \tag{10}$$

Observe that in this case Q is convex because ω is visible from $\mathbf{x}_{i+1}$, $\mathbf{x}_{i+2}$, $\mathbf{x}_{i+3}$.

(iv) If $\mathbf{x}_{i+3}$ is the reflex vertex of $\tilde{Q}$, then Q is defined as in (10) but the internal Steiner point ω is chosen within the interior of the triangle (Fig. 4(d)) $[\mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \mathbf{x}_{i+3}]$ and within the wedge of $\mathbf{x}_{i+2}$.

Case 2.c: If $\tilde{Q} = [\mathbf{x}_{i-3}, \mathbf{x}_{i-2}, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}]$, then proceed analogously to case 2.b.

Case 3: Finally we consider the situation that the segment $[\mathbf{x}_{i-1}, \mathbf{x}_{i+1}]$ is an edge of $\tilde{Q}$ which has a vertex $\tilde{\omega}$ that is not a vertex of $\tilde{P}$ (Fig. 5(a)). Then $\tilde{Q}$ is convex and we choose any point ω within the interior of $\tilde{Q}$ and within the wedge of $\mathbf{x}_i$. Due to the convexity of $\tilde{Q}$, both $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i+1}$ must be visible from the node ω. Therefore, we may define

$$Q := [\mathbf{x}_i, \mathbf{x}_{i+1}, \omega, \mathbf{x}_{i-1}] \tag{11}$$

as a quadrilateral which can be removed from P by using operation (Op2). **Q.E.D.**

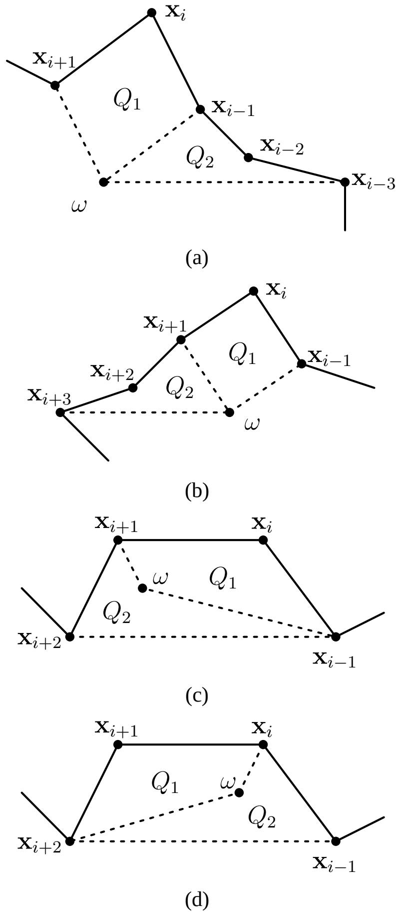


Fig. 6. Q_1 and Q_2 which are discarded by using operations (Op1) and (Op2) are adjacent.

IV. Quadrilateral Decomposition

After applying operation (Op1) to a polygon having n vertices, the number of vertices of the remaining polygon is reduced to $(n-2)$. However, applying operation (Op2) as illustrated in Fig. 2(b) does not reduce the number of vertices. Thus, it is not obvious that the recursive application of the above theorem splits a polygon into a set of quadrilaterals. Therefore we note

Addition to Theorem 2

The internal node ω can be chosen in such a way that if a quadrilateral Q_1 has been removed from P via operation (Op2), then there is a quadrilateral Q_2 adjacent to Q_1 that can be removed from $P \setminus Q_1$ via operation (Op1). Thus, we have four cases which are illustrated in Fig. 6:

$$Q_1 = [\omega, \mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}] \quad Q_2 = [\omega, \mathbf{x}_{i-3}, \mathbf{x}_{i-2}, \mathbf{x}_{i-1}], \tag{12}$$

$$Q_1 = [\omega, \mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}] \quad Q_2 = [\omega, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}, \mathbf{x}_{i+3}], \tag{13}$$

$$Q_1 = [\mathbf{x}_i, \mathbf{x}_{i+1}, \omega, \mathbf{x}_{i-1}] \quad Q_2 = [\mathbf{x}_{i-1}, \omega, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}], \tag{14}$$

$$Q_1 = [\mathbf{x}_{i+1}, \mathbf{x}_{i+2}, \omega, \mathbf{x}_i] \quad Q_2 = [\mathbf{x}_i, \omega, \mathbf{x}_{i+2}, \mathbf{x}_{i-1}]. \tag{15}$$

In cases (12) and (13), the union $Q_1 \cup Q_2$ is a hexagon, while it is a quadrilateral in (14) and (15).

Proof

This is proved by using induction in a very similar manner as in the preceding theorem where we consider a polygon with

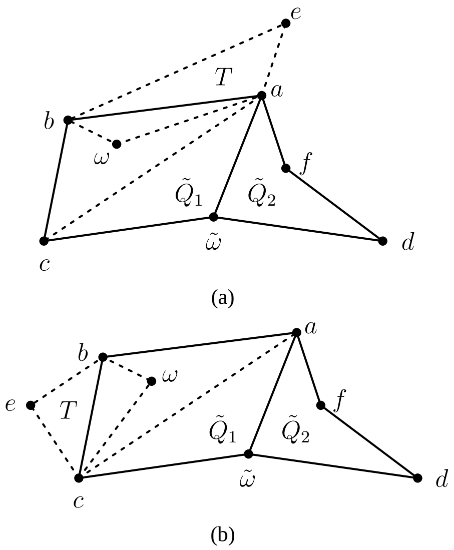


Fig. 7. (a)Ear T incident upon edge $[a, b]$ of $\tilde{Q}_1$, (b)Ear T incident upon edge $[b, c]$ of $\tilde{Q}_1$.

$(n + 1)$ vertices and remove one ear T in order to obtain an auxiliary polygon with n vertices.

For the case $n = 5$, suppose that the vertices are $[a, b, c, d, e]$ as Fig. 5(b). Discard an ear which is supposedly $[a, b, e]$. For the remaining quadrilateral, choose a point $\omega \in \mathcal{W}(d) \cap (ec)^+$. Therefore, we have $Q_1 := [e, \omega, c, d]$ and $Q_2 := [c, \omega, e, b]$.

Suppose in the hypothesis of induction that there are two quadrilaterals $\tilde{Q}_1$ and $\tilde{Q}_2$ and an internal node $\tilde{\omega}$ fulfilling relation (12) or (13) or (14) or (15). Let us prove that after removing an ear T, we can find two quadrilaterals Q_1 and Q_2 and a point ω satisfying those relations. The trivial case consists of an ear which is neither incident upon $\tilde{Q}_1$ nor upon $\tilde{Q}_2$. In such a case we have $Q_1 := \tilde{Q}_1$, $Q_2 := \tilde{Q}_2$ and $\omega := \tilde{\omega}$. We distinguish two nontrivial cases according to whether $\tilde{Q}_1 \cup \tilde{Q}_2$ is a hexagon or a quadrilateral.

Case A: Suppose $\tilde{Q}_1 \cup \tilde{Q}_2$ is a (non-convex) hexagon. Without loss of generality, we assume that $\tilde{Q}_1$ and $\tilde{Q}_2$ satisfy (12). Let us denote the vertices of $\tilde{Q}_1$ by $[a, b, c, \tilde{\omega}]$, those of $\tilde{Q}_2$ by $[d, f, a, \tilde{\omega}]$. Let e be the vertex of T which does not belong to $\tilde{Q}_1 \cup \tilde{Q}_2$ as illustrated in Fig. 7. Let us consider four subcases with respect to the position of the ear T.

Case A.1: If the ear T is incident upon the edge $[a, b]$ of $\tilde{Q}_1$ as illustrated in Fig. 7(a), consider the diagonal $[c, a]$ which must be inside the quadrilateral $\tilde{Q}_1$ because $\tilde{Q}_1$ is convex. Choose then a point $\omega \in \tilde{Q}_1$ which is in $(ca)^-$ and which is visible from the vertex e. Define $Q_1 := [\omega, a, e, b]$ and $Q_2 := [c, a, \omega, b]$ which are adjacent and which form a quadrilateral union.

Case A.2: If the ear T is incident upon the edge $[b, c]$ of $\tilde{Q}_1$ as in Fig. 7(b), we proceed as in case A.1 but we define now $Q_1 := [e, c, \omega, b]$ and $Q_2 := [b, \omega, c, a]$.

Case A.3: Suppose now that the ear T is incident upon the edge $[a, f]$ as in Fig. 8. If $\tilde{\omega} \in \mathcal{W}(e)$ (see Fig. 8(a)), we simply define $\omega := \tilde{\omega}$, $Q_2 := \tilde{Q}_1$ and $Q_1 := [a, \omega, f, e]$ which is a convex quadrilateral. Else, choose $w \in \mathcal{W}(e)$ such that ω is visible from c (see Fig. 8(b)). Define afterwards $Q_1 :=$

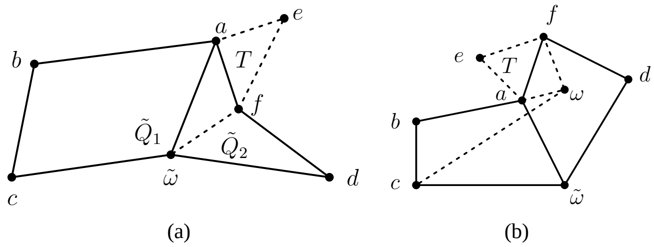


Fig. 8. Ear T incident upon edge $[a, f]$ of $\tilde{Q}_2$

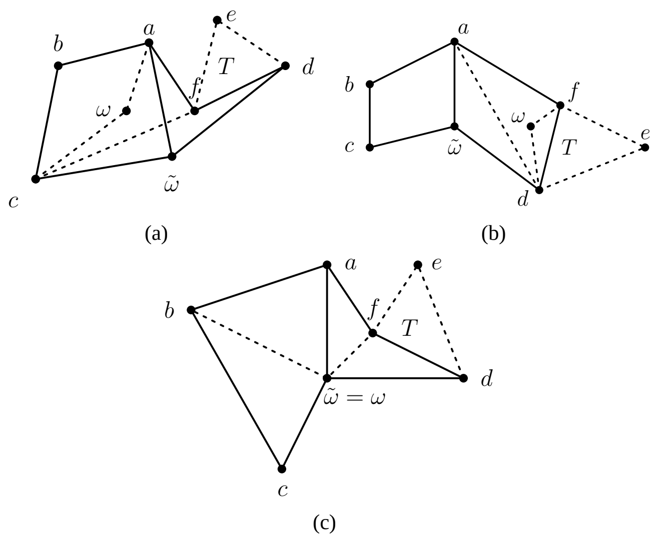


Fig. 9. The ear T is incident upon the edge $[f, d]$ of $\tilde{Q}_2$

$[\omega, f, e, a]$ and $Q_2 := [b, c, \omega, a]$.

Case A.4: Suppose that T is incident upon the edge $[f, d]$.

(i) If c and f are mutually visible in $\tilde{Q}_1 \cup \tilde{Q}_2$ as in Fig.9(a), choose $\omega \in \tilde{Q}_1$ such that $\omega \in (ca)^+ \cap (cf)^-$ and define $Q_1 := [\omega, a, b, c]$ and $Q_2 := [f, a, \omega, c]$.

(ii) Suppose that a and d are mutually visible in $\tilde{Q}_2$ (Fig.9(b)). That means, f must be in $(da)^+$. Choose $\omega \in \tilde{Q}_2$ such that $\omega \in \mathcal{W}(e)$ and define $Q_1 := [e, f, \omega, d]$ and $Q_2 := [a, d, \omega, f]$.

(iii) Suppose that none of (i) and (ii) occurs (Fig.9(c)). Since a is not visible from d in $\tilde{Q}_2$, f must be in $(da)^-$. We set $\omega := \tilde{\omega}$, $Q_1 := [\omega, f, a, b]$ and $Q_2 := [f, \omega, d, e]$.

Case B: Suppose $\tilde{Q}_1 \cup \tilde{Q}_2$ is a quadrilateral. Without loss of generality, let $\tilde{Q}_1$ and $\tilde{Q}_2$ satisfy (14). Let us denote by $[a, b, c, d]$ that union as depicted in Fig. 10. We consider three subcases according to the incidence of the ear T on $[a, b, c, d]$

Case B.1: Suppose that the ear T is incident upon the edge $[b, c]$ as shown in Fig. 10(a). We consider the diagonal $[a, c]$ of the union $[a, b, c, d]$ and we choose $\omega \in \tilde{Q}_1$ such that $\omega \in (ac)^+ \cap \mathcal{W}(e)$. Then we define $Q_1 := [b, e, c, \omega]$ and $Q_2 := [a, b, \omega, c]$.

Case B.2: Suppose that the ear T is incident upon the edge $[a, b]$ as in Fig. 10(b). Generate $\omega \in \tilde{Q}_1$ such that $\omega \in (ac)^+ \cap \mathcal{W}(e)$ and define $Q_1 := [e, b, \omega, a]$ and $Q_2 := [a, \omega, b, c]$.

Case B.3: If the ear T is incident upon the edge $[c, d]$ like in Fig. 10(c), we proceed as in case B.1 but define now $Q_1 := [e, d, \omega, c]$ and $Q_2 := [c, \omega, d, b]$. **Q.E.D.**

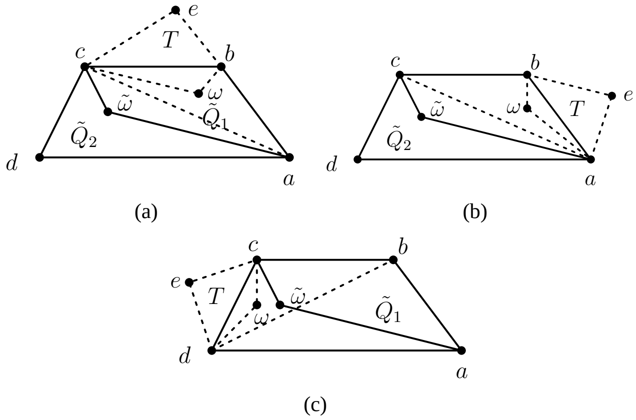


Fig. 10. $\tilde{Q}_1 \cup \tilde{Q}_2$ is a quadrilateral

As a consequence of those statements, the next algorithm terminates because in at most two iterations the number of vertices decrements by two. The number of resulting quadrilaterals is of order $\mathcal{O}(n)$ for a polygon having n vertices.

	Algorithm: Quadrangulation of P
1:	While (the number of vertices of $P > 4$)
2:	Use (Op1) to chop off a quad Q if possible.
3:	Else use (Op2) to chop off a quad Q.
4:	$P := P \setminus Q$.
5:	Output(Q).
6:	End While
7:	Output(P).

We describe now how to convert a quadrangulation which has non-convex quadrilaterals into another one which contains only convex quadrilaterals. Note that two adjacent quadrilaterals q and p form either a single quadrilateral or a hexagon. In the first case, the quadrilaterals q and p share two edges and it is possible that the union $q \cup p$ is a nonconvex or a convex quadrilateral. In the second case, only one edge is shared by q and p. Now, let us recall the following result about hexagon quadrangulations.

Theorem[Bremner, 2001]
Every hexagon (which may include reflex vertices) can be decomposed into a set of convex quadrilaterals by using at most three internal Steiner points.

Bremner [2] proved this theorem but he did not specify the way of exactly choosing the internal Steiner points. We did that specification in [9]. Based on those facts, the next two steps perform the conversion into convex quadrangulation:

Step1: For every nonconvex quadrilateral p having a neighboring quadrilateral q such that $p \cup q$ is a quadrilateral, replace p by $p \cup q$ and remove the quadrilateral q from the quadrangulation. We repeat this step until such a union does not exist any more. After this step, there can only exist nonconvex quadrilaterals whose union with a neighboring quadrilateral forms a hexagon.

Step2: We merge a nonconvex quadrilateral q with a neighboring quadrilateral p in order to have a hexagon $q \cup p$. If we have the choice then we select a nonconvex neighbor p. Then, we re-quadrilate the resulting hexagon by using the hexagon quadrangulation method from the above theorem in order to obtain a local *convex* quadrangulation $\mathcal{Q}_{\rm loc}$. Afterwards, we substitute the union $q \cup p$ by $\mathcal{Q}_{\rm loc}$ in the quadrangulation.

V. Practical Results and Application to CAD

In this section, we want to describe practical applications of the former theory. First, we will see quadrangulations of simply connected polygons. Then, its application to CAD models is briefly described. In fact, we implemented the former theory in C/C++ in order to see its practical behavior. In Figs. 11(a)–11(e), some quadrangulations of a few polygons are displayed. We do not use any new boundary nodes at all as we have already forecast in the theory.

There are many possible applications of the previous quadrangulation technique but we used it in CAD segmentation. The goal is to split the surface boundary of a CAD model into four-sided patches Γ_k (Fig. 12). The main steps of the segmentation is summarized below and their details are found in [9], [11]. The CAD surfaces are collection of trimmed surfaces [3] which are images of $\mathcal{D}_i \subset \mathbf{R}^2$ by bivariate parametrizations ψ_i. The first step consists in approximating the parameter domains $\mathcal{D}_i$ by polygons $P^{(i)}$. Note that if we take too few vertices in the polygonal approximation, the resulting polygon may have imperfections such as different edges which intersect. But if the polygonal approximation is too fine, then it ends up with too many four-sided patches. We have used adaptive method to solve that. Afterwards, we decompose each polygon $P^{(i)}$ into convex quadrilaterals $q_{k,i}$. For simply connected polygons, we use the quadrangulation of the previous sections. For multiply connected ones, we have to insert internal cuts and to generalize the previous results to more complicated polygons [9]. Four-sided domains $Q_{k,i}$ are obtained from $q_{k,i}$ where we replace each straight boundary edge of $q_{k,i}$ by the corresponding curve portion of $\mathcal{D}_i$ such as $\mathcal{D}_i = \bigcup_k Q_{k,i}$. The final four-sided patches Γ_k are therefore the images by ψ_i of the 2D domains $Q_{k,i}$. Careful operations must be done to remove boundary intersection caused by the curve replacement as detailed in [9]. As a result, two CAD surface segmentations can be observed in Fig. 12.

VI. Conclusion

We presented two operations for discarding a quadrilateral from a simply connected polygon. We proved that one of those operations can always be applied. From that fact, we designed an algorithm for generating a quadrangulation from a polygon. The algorithm was then implemented in order to obtain interesting practical results.

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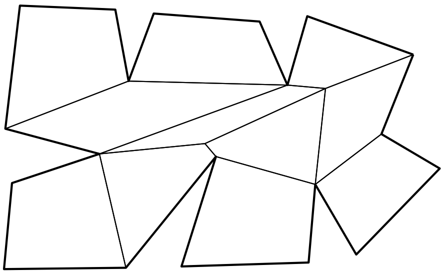
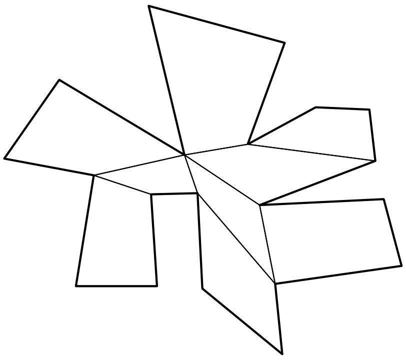
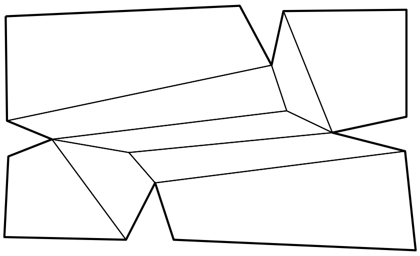


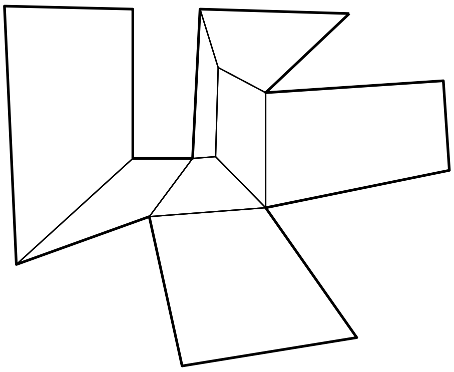
(a)



(b)



(c)



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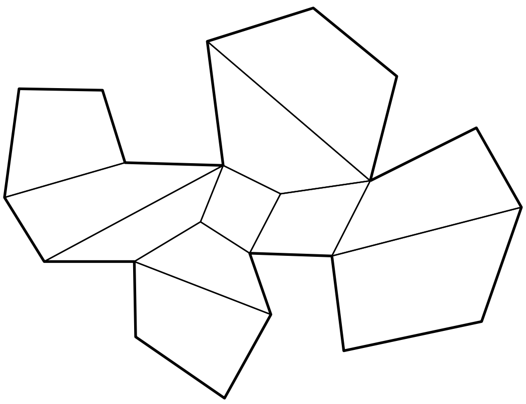
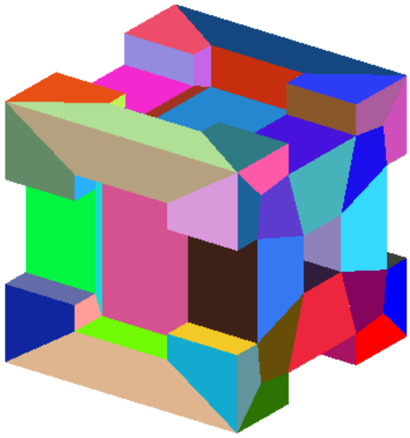
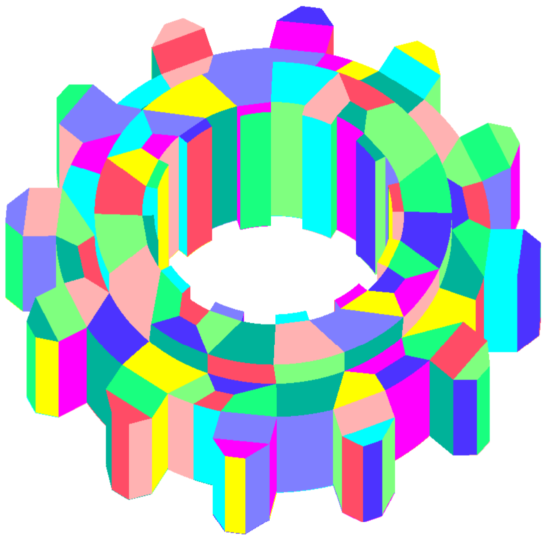


(e)

Fig. 11. Decomposition of polygons



(a)



(b)

Fig. 12. Segmentation of CAD surfaces

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