

Numerical Functional Analysis

Carl de Boor

draft 30jul03

TABLE OF CONTENTS

Introduction

notation	2
--------------------	---

I. Preliminaries: Linear Algebra

linear space	5
linear map	6
special case: column maps, especially matrices	7
lss's often come as $\ker$ or ran	8
quotient space	9
linear functionals; dual	10
bidual	11
numerical representation; basis	11
Column maps: $\mathbb{F}^n$ into X	12
use of a basis	13
construction of a basis; dimension	13
basic wisdom	15
Row maps: X into $\mathbb{F}^m$	16
The interplay between column maps and row maps	16
the inverse of a basis	17
linear projectors	18
factorization and rank	19
The dual of a linear map	20
orthogonality	20
the duals of row maps and column maps	21
use of minimal factorization	21
tests for linear independence	22
Application: approximate evaluation of linear functionals; interpolation	23
inadequacy of rules	23
rule construction	24
interpolation	25
numerics	26

II. Preliminaries: Advanced Calculus

Topology	29
continuity	29
working with a collection of subsets	29
topology defined	29
equivalent topologies	30
open and closed sets	31
Metric space	33
metric	33
modulus of continuity	36
convergence of sequences	37
Application: Contraction maps and fixed point iteration	39
completeness	41

summary on contraction	43
Compactness and total boundedness	44
limit points	44
compactness	45
total boundedness	48
examples of compact sets	49
Tykhonov	51
III. Normed linear spaces	
definition	53
norm metric	53
boundedness and continuity	54
bounded below	55
any lm on a finite-dimensional domain is continuous	56
closed & bounded = compact iff finite-dimensional	57
computing the norm of a lm	58
Application of map norm: Approximate inverse	60
IV. The (continuous) dual	
hyperplanes and lfl's	63
elimination	64
a useful formula	64
λ is continuous iff $\ker \lambda$ is closed	65
error estimates	65
existence of ba from a hyperplane	65
Representation of bounded linear functionals	66
Application: Interpolation error and optimal recovery	71
model example	72
interpolation	72
model example (cont.)	72
Lebesgue inequality	73
segue into optimal recovery	73
best rule in the sense of Sard	73
GW-interval	74
a sharp estimate for $\int g$	74
Hahn-Banach	77
general HB	78
some uses of HB	79
the dual of a lm	79
Application: A sharp lower bound for $d(x, Y)$	80
bounded linear projectors and their norm	80
norm of a lfl on a lss	81
GW-interval is sharp	82
V. Baire category and consequences	
Pointwise convergence	85
norm convergence is rare	85

the next best thing is pointwise convergence	85
approximate identity	85
w-convergence, w*-convergence	86
bounded pointwise convergence	86
extension by continuity	87
fundamental sets	87
boundedness, though not necessary, is necessary	87
Baire category	88
Degree of approximation	89
Weierstraß	89
most of a Bs cannot be approximated well at all	89
degree of approximation	90
Hamel basis for inf.dim. Bs is uncountable	90
Schauder basis	91
Uniform boundedness	92
Banach-Steinhaus	92
Applications of uniform boundedness	92
Schauder basis	92
quadrature	92
equivalence theorems	93
divergence	93
Open mapping/closed graph	95
test for openness	95
almost solvable stably $\implies$ solvable stably	96
example: an ordinary differential equation	97
open mapping theorem	97
illustrations	98
equivalence of norms	98
completeness is necessary here	98
closed graphs	98
nls's of smooth functions	100
example: an ordinary differential equation initial value problem	102
VI. Convexity	
Convex sets	103
convexity is local linearity	103
convex hull	103
closure and interior of a convex set	104
Caratheodory	105
The Minkowski functional	106
Minkowski fl	106
The Separation Theorem	107
refinements	107
Application: Characterization of best approximation from a convex set	108
support fl	109

A constructive instance of Hahn-Banach (of practical interest)	109
application to quadrature	110
Characterization of ba from a lss	111
special case	111
special special case	112
Convex functionals	113
Krein-Milman	114
Unfinished business: $\text{ran } A^*$ closed $\implies \text{ran } A^* = (\ker A)^\perp$	115
VII. Inner Product Spaces	
definition	117
bilinear forms	117
ips is nls	118
parallelogram law	119
ba from convex sets	120
characterization of ba from lss	120
numerics	121
Riesz-Fischer	122
Application: Optimal interpolation	123
Application: Synge's hypercircle	124
Reproducing kernels	125
computing the representer	126
Rayleigh-Ritz-Galerkin	127
example: Poisson	128
example: second-order elliptic	129
abstract nonsense	130
numerics	131
generalization to bilinear fl on two Hilbert spaces	131
generalization to Bs	132
c.o.n. sets	132
VIII. Compact perturbations of the identity	
Projection methods	135
examples	135
residual reduction	136
analysis by perturbation from a simple case	136
Compact linear maps	137
examples	137
compactness and convergence	137
K compact $\implies$ dual of K compact	138
regularization	138
Compact perturbation of the identity	139
spectrum of a compact map	139
The standard compact perturbation argument	140
example: second kind Fredholm integral equation	141
Quadrature methods and uniform compactness	142

Nystrom's method	142
use better approximate inverses	142
uniformly compact strongly convergent perturbations of 1	143
back to Nystrom	144
The iterated projection method	145
example: Galerkin's method...	146
...using broken lines	146
splitting	147
example: m -th order linear ODE	147
superconvergence	148
PDEs are much tougher	149
IX. The spectrum of a lm	
point spectrum	151
Eigenstructure of a matrix	151
who cares about eigenstructure?	151
polynomials in a lm	151
A -invariant direct sum decompositions	152
primary decomposition	153
a finest A -invariant direct sum decomposition	154
Jordan form	154
Schur form	155
c dual of a map	155
derivation of the Schur form	155
spectral theorem for normal matrices	156
Eigenstructure of bounded lm 's	156
spectrum	156
resolvent	157
spectral radius	158
isolating parts of the spectrum	159
isolated eigenvalues	160
numerics	161
eigenstructure of a compact lm	161
Spectral Theorem for compact normal maps	162
The SVD for compact maps	164
X. Linearization and Newton's Method	
linearization	165
differentiation	165
examples	165
basic rules for Fréchet and Gateaux derivative	167
meanvalue estimates	167
Newton's method	168
a posteriori error estimates	171
infinite-dim. problems also need discretization	171
example: solving a second-order non-linear ode by collocation	172

implicit function theorem	173
Basic subjects not covered	175
Index	