ANSWERS to HOMEWORK assignment 6, due Wednesday, 14 April 03

(1) Prove that the splines E_n of a previous homework satisfy the recurrence

$$E_{n+1} = \int_0^1 E_n(\cdot + t)\,\mathrm{d}t/(-2).$$

For the record, the function

$$\mathcal{E}_n := E(\cdot - (n+1)/2)/E(-(n+1)/2)$$

is called the **Euler spline** of degree n. It has simple knots, at $(n/2) + \mathbb{Z}$, and satisfies $\mathcal{E}_n(m) = (-1)^m$, all $m \in \mathbb{Z}$.

By construction, $DE_{n+1} = E_n$, hence, for $x \in [0\,..\,1)$,

$$\int_0^1 E_n(x+\cdot) = \int_x^{x+1} E_n = \int_x^1 E_n + \int_1^{1+x} E_n = E_{n+1}(1) - E_{n+1}(x) + E_n(1+x) - E_{n+1}(1),$$

and this equals $-2E_{n+1}(x)$ since $E_{n+1}(1+x) = -E_{n+1}(x)$.

(2) A **monospline** of degree k with knot sequence $\mathbf{t}$ is, by definition, any element of $\Pi_k + S_{k,\mathbf{t}}$.

Show that monosplines with simple knots occur naturally as the Peano kernel in a quadrature rule based on function values.

The Peano kernel for a rule that is exact for polynomials of degree $< k$ is the function ψ for which the rule error at a smooth function f can be written as $\int \psi D^k$. If the rule is for $\mu := \int_a^b \cdot$, and of the form $\lambda_{U,w} := \sum_{u\in U} w(u)\delta_u$, then, applying $\mu - \lambda_{U,w}$ to both sides of the Taylor identity

$$f = \sum_{j<k} D^j f(a)(\cdot - a)^j/j! + \int_a^b (\cdot - s)_+^{k-1} D^k f(s)\,\mathrm{d}s/(k-1)!,$$

we find that the quadrature error at f is

$$\int_a^b \psi D^k f,$$

with

$$\psi(s) = (\mu - \sum_{u\in U} w(u)\delta_u)(\cdot - s)^{k-1}/(k-1)!.$$

Further, on $[a\,..\,b]$,

$$\mu(\cdot - s)_+^{k-1}/(k-1)! = (\cdot - s)_+^k/k!|_a^b = (b-s)^k/k!,$$

i.e., a polynomial of degree k in s, while

$$\lambda_{U,w}(\cdot - s)_+^{k-1}/(k-1)! = \sum_{u\in U} w(u)(u-s)_+^{k-1}/(k-1)!,$$

i.e., a spline of order k in s with simple knots at the $u \in U$.

(3) With $\mu := \int_0^1 \cdot$, show that the 1-periodic function $\mathcal{B}_k$ that agrees with p_k^μ on $[0\,..\,1)$ is a monospline of degree k. For the record, $\mathcal{B}_k$ is the **Bernoulli spline** of degree k.

Since B_k is 1-periodic and piecewise polynomial of degree k, it is sufficient to prove that, for $j = 0, \ldots, k-2$,

$$D^j B_k(1-) = D^j p_k^\mu(1) = D^j p_k^\mu(0) = D^j B_k(1+).$$

But the only equality in doubt here, the middle one, follows directly from the fact that, for that range of j,

$$0 = \mu D^{j+1} p_k^\mu = D^j p_k^\mu(1) - D^j p_k^\mu(0).$$

(4) Following Glaeser (and Schoenberg), a (polynomial) spline is called **perfect** if its highest nontrivial derivative is absolutely constant. For example, the Euler spline $\mathcal{E}_n$ of a previous homework is perfect.

Prove that $M(\cdot|t_0, \ldots, t_k)$ is perfect if $t_j = \cos(k-j)\pi/k$, all j.

See `http://www.cs.wisc.edu/~deboor/toast/pages008.html`.