title: A sharp bound for the error in linear interpolation at the vertices of a d-dimensional simplex to a Lipschitz function.

Recently, I came across a proof that the error in linear interpolation at the vertices $v \in V$ of a simplex in $\mathbb{R}^d$ to a Lipschitz-continuous function f with Lipschitz constant M is bounded by $3MR$ on that simplex, with R the radius of the sphere containing the vertex set V. The proof was quite complicated, particularly in light of the fact that, with ℓ_v the linear polynomial which is 1 at v and vanishes on $V\backslash v$, we have, for any $x \in \mathrm{conv}(V)$ and with $|.|$ the Euclidean norm in $\mathbb{R}^d$,

$$\begin{aligned}
|f(x) - \sum_{v\in V} f(v)\ell_v(x)| &= |\sum_{v\in V}(f(x) - f(v))\ell_v(x)| \\
&\leq \sum_{v\in V} |f(x) - f(v)|\ell_v(x) \\
&\leq M \sum_{v\in V} |x - v|\ell_v(x) \leq M \max_{v\in V} |x - v|,
\end{aligned}$$

hence the error on the simplex is trivially bounded by $2Mr$, with r the radius of the smallest ball containing V.

However, the other proof came with the conjecture that the bound should be MR. I made up a proof that it is, in fact, Mr (it is easy to see that it couldn't be smaller, by interpolating to the function $x \mapsto \|x-c\|$, with c the center of that smallest ball containing V), and then used email to ask various people whether they had seen this result.

None of those I asked had, but Yuri Brudnyi responded that he could prove the bound Mr by an inductive argument (my proof was sort-of inductive; it showed that any critical point of

$$x \mapsto \sum_{v\in V} |x - v|\ell_v(x)$$

in the interior of the simplex necessarily equals c). More than that, he said that his former student, Pavel Shvartsman (now at the Technion), had a simpler proof of a stronger result. When I recently visited the Technion, Pavel Shvartsman showed me his lovely argument. Here it is:

Assuming without loss that, after an appropriate shift, the center c of that smallest ball containing V is 0, simply replace the final step in the above chain by the following:

$$\begin{aligned}
\sum_{v\in V} |x - v|\ell_v(x) = \sum_{v\in V} \sqrt{|x|^2 - 2\langle x, v\rangle + |v|^2}\, \ell_v(x) &\leq \sum_{v\in V} \sqrt{|x|^2 - 2\langle x, v\rangle + r^2}\, \ell_v(x) \\
&\leq \sqrt{|x|^2 - 2\langle x, \sum_{v\in V} v\ell_v(x)\rangle + r^2} \\
&= \sqrt{-|x|^2 + r^2},
\end{aligned}$$

the last (and crucial) inequality being justified by the fact that the map $v \mapsto \sqrt{|x|^2 + r^2 - 2\langle x, v\rangle}$ is concave on that ball.

Thus, the error at x is bounded by $M\sqrt{-|x|^2 + r^2}$, and the bound Mr is a trivial consequence of that.

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