Summability of Multi-Dimensional Trigonometric Fourier Series

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Abstract

We consider the summability of one- and multi-dimensional trigonometric Fourier series. The Fejér and Riesz summability methods are investigated in detail. Different types of summation and convergence are considered. We will prove that the maximal operator of the summability means is bounded from the Hardy space H_p to L_p, for all $p > p_0$, where p_0 depends on the summability method and the dimension. For $p = 1$, we obtain a weak type inequality by interpolation, which ensures the almost everywhere convergence of the summability means. Similar results are formulated for the more general θ-summability and for Fourier transforms.

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Keywords: Hardy spaces, H_p-atom, interpolation, Fourier series, circular, triangular, cubic and rectangular summability.

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1 Introduction

We will consider different summation methods for multi-dimensional trigonometric Fourier series. Basically two types of summations will be introduced. In the first one we take the sum in the partial sums and in the summability means over the balls of ℓ_q, it is called ℓ_q-summability. In the literature the cases $q = 1, 2, \infty$, i.e., the triangular, circular and cubic summability are investigated. In the second version of summation we take the sum over rectangles, it is called rectangular summability. In this case two types of convergence and maximal operators are considered, the restricted (convergence over the diagonal or more generally over a cone), and the unrestricted (convergence over $\mathbb{N}^d$). In each version, three well known summability methods, the Fejér, Riesz and Bochner-Riesz means will be investigated in detail. The Fejér summation is a special case of the Riesz method. We consider norm convergence and almost everywhere convergence of the summability means.

We introduce different types of Hardy spaces H_p and prove that the maximal operators of the summability means are bounded from H_p to L_p, whenever $p > p_0$ for some $p_0 < 1$. The critical index p_0 depends on the summability method and the dimension. For $p = 1$, we obtain a weak type inequality by interpolation, which implies the almost everywhere convergence of the summability means. The one-dimensional version of the almost everywhere convergence and the weak type inequality are proved usually with the help of a Calderon-Zygmund type decomposition lemma. However, in higher dimensions, this lemma can not be used for all cases investigated in this monograph. Our method, that can also be applied well in higher dimensions, can be regarded as a new method to prove the almost everywhere convergence and weak type inequalities.

Similar results are also formulated for summability of Fourier transforms. The so called θ-summability, which is a general summability method generated by a single function θ, and the Cesàro summability are also considered.

We will prove all results except the ones that can be found in the books Grafakos [43] and Weisz [94]. For example, the results about the circular Riesz summability below the critical index and the results about Hardy spaces and interpolation can be found in these books, so their proofs are omitted.

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2 Partial sums of one-dimensional Fourier series

In this and the next section we briefly present some theorems for one-dimensional Fourier series. Later we will give their generalization to higher dimensions in more details.

The set of the real numbers is denoted by $\mathbb{R}$, the set of the integers by $\mathbb{Z}$ and the set of the non-negative integers by $\mathbb{N}$. For a set $\mathbb{Y} \neq \emptyset$, let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \cdots \times \mathbb{Y}$ involving it d times ($d \geq 1, d \in \mathbb{N}$). We briefly write $L_p(\mathbb{T}^d)$ instead of the $L_p(\mathbb{T}^d, \lambda)$ **space** equipped with the norm (or quasi-norm)

$$\|f\|_p := \begin{cases} \left(\int_{\mathbb{T}^d} |f|^p \, \mathrm{d}\lambda \right)^{1/p}, & 0 < p < \infty; \\ \sup_{\mathbb{T}^d} |f|, & p = \infty, \end{cases}$$

where $\mathbb{T} := [-\pi, \pi]$ is the torus and λ is the Lebesgue measure. We use the notation $|I|$ for the Lebesgue measure of the set I. The **weak L_p space**, $L_{p,\infty}(\mathbb{T}^d)$ $(0 < p < \infty)$ consists of all measurable functions f for which

$$\|f\|_{p,\infty} := \sup_{\rho > 0} \rho \lambda(|f| > \rho)^{1/p} < \infty.$$

Note that $L_{p,\infty}(\mathbb{T}^d)$ is a quasi-normed space (see Bergh and Löfström [10]). It is easy to see that for each $0 < p < \infty$,

$$L_p(\mathbb{T}^d) \subset L_{p,\infty}(\mathbb{T}^d) \qquad \text{and} \qquad \|\cdot\|_{p,\infty} \leq \|\cdot\|_p.$$

The space of continuous functions with the supremum norm is denoted by $C(\mathbb{T}^d)$ and we will use $C_0(\mathbb{R}^d)$ for the space of continuous functions vanishing at infinity.

For an integrable function $f \in L_1(\mathbb{T})$, its kth **Fourier coefficient** is defined by

$$\widehat{f}(k) = \frac{1}{2\pi} \int_{\mathbb{T}} f(x) \mathrm{e}^{-\mathrm{i}kx} \, \mathrm{d}x \qquad (\mathrm{i} := \sqrt{-1}).$$

The formal trigonometric series

$$\sum_{k \in \mathbb{Z}} \widehat{f}(k) \mathrm{e}^{\mathrm{i}kx} \qquad (x \in \mathbb{T})$$

is called the **Fourier series** of f. This definition can be extended to distributions as well. Let $C^\infty(\mathbb{T})$ denote the set of all infinitely differentiable functions on $\mathbb{T}$. Then $f \in C^\infty(\mathbb{T})$ implies

$$\sup_{\mathbb{T}} |f^{(k)}| < \infty \qquad \text{for all } k \in \mathbb{N}.$$

We say that $f_n \to f$ in $C^\infty(\mathbb{T})$ if

$$\|f_n^{(k)} - f^{(k)}\|_\infty \to 0 \qquad \text{for all } k \in \mathbb{N}.$$

$C^\infty(\mathbb{T})$ is also denoted by $\mathcal{S}(\mathbb{T})$. A **distribution** $u : \mathcal{S}(\mathbb{T}) \to \mathbb{C}$ (briefly $u \in \mathcal{S}'(\mathbb{T})$) is a continuous linear functional on $\mathcal{S}(\mathbb{T})$, i.e., u is linear and

$$u(f_n) \to u(f) \quad \text{if} \quad f_n \to f \quad \text{in} \quad C^\infty(\mathbb{T}).$$

If $g \in L_p(\mathbb{T})$ $(1 \leq p \leq \infty)$, then

$$u_g(f) := \int_{\mathbb{T}} f g \, \mathrm{d}\lambda \qquad (f \in \mathcal{S}(\mathbb{T}))$$

is a distribution. So all functions from $L_p(\mathbb{T})$ $(1 \leq p \leq \infty)$ can be identified with distributions $u \in \mathcal{S}'(\mathbb{T})$. We say that the distributions u_j tend to the distribution u **in the sense of distributions** or in $\mathcal{S}'(\mathbb{T})$ if

$$u_j(f) \to u(f) \quad \text{for all} \quad f \in \mathcal{S}(\mathbb{T}) \quad \text{as} \quad j \to \infty.$$

The next definition extends the Fourier coefficients to distributions. For a distribution f, the kth **Fourier coefficient** is defined by $\widehat{f}(k) := f(e_{-k})$, where $e_k(x) := \mathrm{e}^{\imath kx}$ $(k \in \mathbb{Z})$ (see e.g. Edwards [27, p. 67]).

For $f \in L_1(\mathbb{T})$, the nth **partial sum** $s_n f$ of the Fourier series of f is introduced by

$$s_n f(x) := \sum_{|k| \leq n} \widehat{f}(k) \mathrm{e}^{\imath kx} = \frac{1}{2\pi} \int_{\mathbb{T}} f(x-u) D_n(u) \, \mathrm{d}u \qquad (n \in \mathbb{N}), \tag{2.1}$$

where

$$D_n(u) := \sum_{|k| \leq n} \mathrm{e}^{\imath ku}$$

is the nth **Dirichlet kernel** (see Figure 1). Using some simple trigonometric identities, we

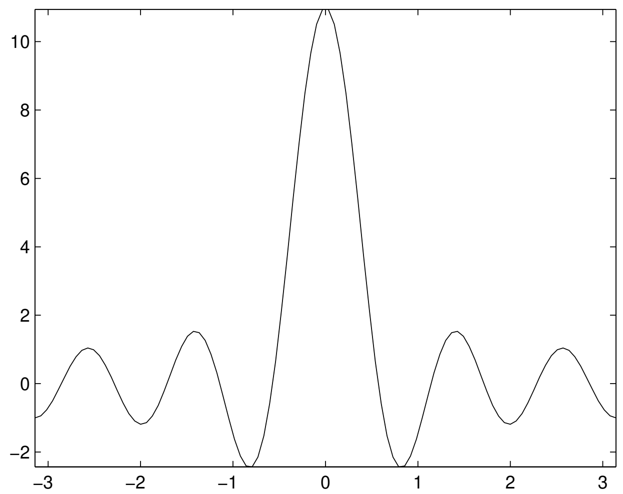

Figure 1: Dirichlet kernel D_n for $n = 5$.

obtain

$$\begin{aligned} D_n(u) &= 1 + 2\sum_{k=1}^{n} \cos(ku) \\ &= \frac{1}{\sin(u/2)} \Big(\sin(u/2) + 2\sum_{k=1}^{n} \cos(ku) \sin(u/2) \Big) \end{aligned}$$

$$= \frac{1}{\sin(u/2)}\Big(\sin(u/2)+\sum_{k=1}^{n}\Big(\sin((k+1/2)u)-\sin((k-1/2)u)\Big)\Big)$$

$$= \frac{\sin((n+1/2)u)}{\sin(u/2)}. \tag{2.2}$$

It is easy to see that $|D_n| \leq Cn$. The L_1-norms of D_n are not uniformly bounded, more exactly $\|D_n\|_1 \sim \log n$.

It is a basic question as to whether the function f can be reconstructed from the partial sums of its Fourier series. It can be found in most books about trigonometric Fourier series (e.g. Zygmund [110], Bary [5], Torchinsky [84] or Grafakos [43]) and is due to Riesz [71], that the partial sums converge to f in the L_p-norm if $1 < p < \infty$.

Theorem 2.1 *If $f \in L_p(\mathbb{T})$ for some $1 < p < \infty$, then*

$$\|s_n f\|_p \leq C_p \|f\|_p \qquad (n \in \mathbb{N})$$

and

$$\lim_{n\to\infty} s_n f = f \qquad \text{in the } L_p\text{-norm.}$$

The L_1-norms of D_n are not uniformly bounded, Theorem 2.1 is not true for $p = 1$ and $p = \infty$.

Let us define the **Riesz projection** with the formal series

$$P^+ f(x) \sim \sum_{k\in\mathbb{N}} \widehat{f}(k) e^{ikx}$$

and let

$$P_n^+ f(x) := \sum_{k=0}^{n} \widehat{f}(k) e^{ikx} \qquad (n \in \mathbb{N}).$$

Then Theorem 2.1 implies easily that $P^+ : L_p(\mathbb{T}) \to L_p(\mathbb{T})$ is bounded.

Theorem 2.2 *If $f \in L_p(\mathbb{T})$ for some $1 < p < \infty$, then*

$$\|P_n^+ f\|_p \leq C_p \|f\|_p \qquad (n \in \mathbb{N}).$$

Moreover,

$$P^+ f(x) = \sum_{k\in\mathbb{N}} \widehat{f}(k) e^{ikx} \qquad \text{in the } L_p\text{-norm}$$

and

$$\|P^+ f\|_p \leq C_p \|f\|_p.$$

Proof. Observe that

$$\sum_{k=0}^{2n} \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx} = \mathrm{e}^{\mathrm{i}nx} \sum_{k=-n}^{n} (\widehat{f(\cdot)\mathrm{e}^{-\mathrm{i}n(\cdot)}})(k)\mathrm{e}^{\mathrm{i}kx},$$

in other words, $|P_{2n}^{+} f| = |s_n(f(\cdot)\mathrm{e}^{-\mathrm{i}n(\cdot)})|$. Now the result follows from the Banach-Steinhaus theorem and from Theorem 2.1. ■

One of the deepest results in harmonic analysis is Carleson's theorem, that the partial sums of the Fourier series converge almost everywhere to $f \in L_p(\mathbb{T})$ $(1 < p \leq \infty)$ (see Carleson [18] and Hunt [51] or recently Grafakos [43]).

Theorem 2.3 *If $f \in L_p(\mathbb{T})$ for some $1 < p < \infty$, then*

$$\| \sup_{n\in\mathbb{N}} |s_n f| \|_p \leq C_p \|f\|_p$$

and if $1 < p \leq \infty$, then

$$\lim_{n\to\infty} s_n f = f \qquad \text{a.e.}$$

The inequality of Theorem 2.3 does not hold if $p = 1$ or $p = \infty$, and the almost everywhere convergence does not hold if $p = 1$. du Bois Reymond proved the existence of a continuous function $f \in C(\mathbb{T})$ and a point $x_0 \in \mathbb{T}$ such that the partial sums $s_n f(x_0)$ diverge as $n \to \infty$. Kolmogorov gave an integrable function $f \in L_1(\mathbb{T})$, whose Fourier series diverges almost everywhere or even everywhere (see Kolmogorov [54, 55], Zygmund [110] or Grafakos [43]).

Since there are many function spaces contained in $L_1(\mathbb{T})$ but containing $L_p(\mathbb{T})$ $(1 < p \leq \infty)$, it is natural to ask whether there is a largest subspace of $L_1(\mathbb{T})$ for which almost everywhere convergence holds. The next result, due to Antonov [2], generalizes Theorem 2.3.

Theorem 2.4 *If*

$$\int_{\mathbb{T}} |f(x)| \log^+ |f(x)| \log^+ \log^+ \log^+ |f(x)| \, \mathrm{d}x < \infty, \tag{2.3}$$

then

$$\lim_{n\to\infty} s_n f = f \qquad \text{a.e.}$$

Note that $\log^+ u = \max(0, \log u)$. It is easy to see that if $f \in L_p(\mathbb{T})$ $(1 < p \leq \infty)$, then f satisfies (2.3). If f satisfies (2.3), then of course $f \in L_1(\mathbb{T})$. For the converse direction, Konyagin [56] obtained the next result.

Theorem 2.5 *If the non-decreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the condition*

$$\phi(u) = o\Big(u\sqrt{\log u}/\sqrt{\log\log u}\Big) \qquad \text{as } u \to \infty,$$

then there exists an integrable function f such that

$$\int_{\mathbb{T}} \phi(|f(x)|)\,\mathrm{d}x < \infty$$

and

$$\limsup_{n\to\infty} s_n f(x) = \infty \qquad \textit{for all } x \in \mathbb{T},$$

i.e., the Fourier series of f diverges everywhere.

For example, if $\phi(u) = u\log^+\log^+ u$, then there exists a function f such that its Fourier series diverges everywhere and

$$\int_{\mathbb{T}} |f(x)|\log^+\log^+|f(x)|\,\mathrm{d}x < \infty.$$

3 Summability of one-dimensional Fourier series

Though Theorems 2.1 and 2.3 are not true for $p = 1$ and $p = \infty$, with the help of some summability methods they can be generalized for these endpoint cases. Obviously, summability means have better convergence properties than the original Fourier series. Summability is intensively studied in the literature. We refer at this time only to the books Stein and Weiss [80], Butzer and Nessel [15], Trigub and Belinsky [86], Grafakos [43] and Weisz [94] and the references therein.

The best known summability method is the Fejér method. In 1904 Fejér [39] investigated the arithmetic means of the partial sums, the so called Fejér means and proved that if the left and right limits $f(x-0)$ and $f(x+0)$ exist at a point x, then the Fejér means converge to $(f(x-0)+f(x+0))/2$. One year later Lebesgue [58] extended this theorem and obtained that every integrable function is Fejér summable at each Lebesgue point, thus a.e. The Riesz means are generalizations of the Fejér means. M. Riesz [71] proved that the Riesz means of a function $f \in L_1(\mathbb{T})$ converge almost everywhere to f as $n \to \infty$ (see also Zygmund [110, Vol. I, p.94]).

The **Fejér means** are defined by

$$\sigma_n f(x) := \frac{1}{n}\sum_{j=0}^{n-1} s_j f(x).$$

It is easy to see that

$$\sigma_n f(x) = \sum_{|k|\le n}\Big(1-\frac{|k|}{n}\Big)\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx} = \frac{1}{2\pi}\int_{\mathbb{T}} f(x-u)K_n(u)\,\mathrm{d}u,$$

where the **Fejér kernels** are given by

$$K_n(u) := \sum_{|k|\le n}\Big(1-\frac{|k|}{n}\Big)\mathrm{e}^{\mathrm{i}ku} = \frac{1}{n}\sum_{j=0}^{n-1} D_j(u)$$

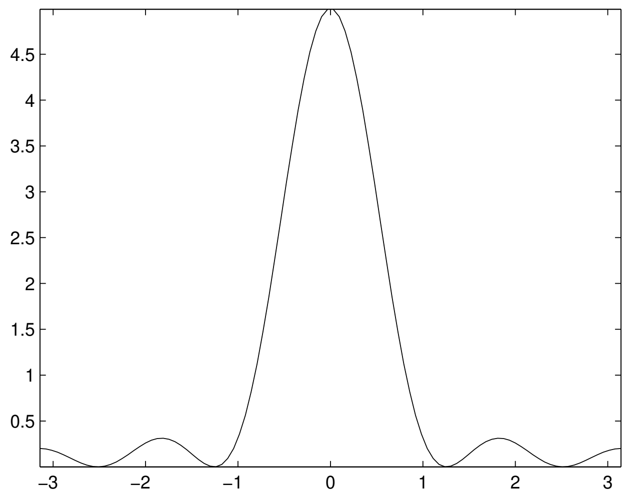

Figure 2: Fejér kernel K_n for $n = 5$.

(see Figure 2). It is known ([84]) that

$$K_n(u) = \frac{1}{n}\Big(\frac{\sin nu/2}{u/2}\Big)^2.$$

We consider also a generalization of the Fejér means, the **Riesz means**,

$$\sigma_n^\alpha f(x) := \sum_{|k|\le n} \Big(1-\Big(\frac{|k|}{n}\Big)^\gamma\Big)^\alpha \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx} = \frac{1}{2\pi}\int_{\mathbb{T}} f(x-u)K_n^\alpha(u)\,\mathrm{d}u, \tag{3.1}$$

where

$$K_n^\alpha(u) := \sum_{|k|\le n} \Big(1-\Big(\frac{|k|}{n}\Big)^\gamma\Big)^\alpha \mathrm{e}^{\mathrm{i}ku}$$

are the **Riesz kernels**. Here, we suppose that $0<\alpha<\infty, 1\le\gamma<\infty$. Since the results are independent of γ, we omit the notation γ in $\sigma_n^\alpha f$ and K_n^α. If $\alpha=\gamma=1$, then we get the Fejér means. It is known that

$$|K_n^\alpha(u)| \le C\min(n, n^{-\alpha}u^{-\alpha-1}) \qquad (n\in\mathbb{N}, u\ne 0), \tag{3.2}$$

(see Zygmund [110], Stein and Weiss [80] or Weisz [94]). Note that this inequality follows from (10.7).

The **maximal Riesz operator** is defined by

$$\sigma_*^\alpha f := \sup_{n\in\mathbb{N}} |\sigma_n^\alpha f|.$$

Using (3.2), Zygmund and Riesz [110] proved

Theorem 3.1 *If $0 < \alpha < \infty$ and $f \in L_1(\mathbb{T})$, then*

$$\sup_{\rho>0} \rho\, \lambda(\sigma_*^\alpha f > \rho) \leq C\|f\|_1.$$

This theorem will be proved right after Theorem 16.1 in Section 16. The next density theorem, due to Marcinkiewicz and Zygmund [63], is fundamental for the almost everywhere convergence and is similar to the Banach-Steinhaus theorem about the norm convergence of operators.

Theorem 3.2 *Suppose that X is a normed space of measurable functions and $X_0 \subset X$ is dense in X. Let T and T_n $(n \in \mathbb{N})$ be linear operators such that*

$$Tf = \lim_{n\to\infty} T_n f \qquad \text{a.e. for every } f \in X_0.$$

If, for some $1 \leq p < \infty$,

$$\sup_{\rho>0} \rho\, \lambda(|Tf| > \rho)^{1/p} \leq C\|f\|_X \qquad (f \in X)$$

and

$$\sup_{\rho>0} \rho\, \lambda(T_* f > \rho)^{1/p} \leq C\|f\|_X \qquad (f \in X),$$

where

$$T_* f := \sup_{n\in\mathbb{N}} |T_n f| \qquad (f \in X),$$

then

$$Tf = \lim_{n\to\infty} T_n f \qquad \text{a.e. for every } f \in X.$$

Proof. Fix $f \in X$ and set

$$\xi := \limsup_{n\to\infty} |T_n f - Tf|.$$

It is sufficient to show that $\xi = 0$ a.e.

Choose $f_m \in X_0$ $(m \in \mathbb{N})$ such that $\|f - f_m\|_X \to 0$ as $m \to \infty$. Observe that

$$\begin{aligned} \xi &\leq \limsup_{n\to\infty} |T_n(f - f_m)| + \limsup_{n\to\infty} |T_n f_m - Tf_m| + |T(f_m - f)| \\ &\leq T^*(f_m - f) + |T(f_m - f)| \end{aligned}$$

for all $m \in \mathbb{N}$. Henceforth, for all $\rho > 0$ and $m \in \mathbb{N}$, we have

$$\begin{aligned} \lambda(\xi > 2\rho) &\leq \lambda(T^*(f_m - f) > \rho) + \lambda(|T(f_m - f)| > \rho) \\ &\leq C\rho^{-p}\|f_m - f\|_X^p + C\rho^{-p}\|f_m - f\|_X^p. \end{aligned}$$

Since $f_m \to f$ in X as $m \to \infty$, it follows that

$$\lambda(\xi > 2\rho) = 0$$

for all $\rho > 0$. So we can conclude that $\xi = 0$ a.e. $\blacksquare$

The weak type $(1,1)$ inequality of Theorem 3.1 and the density argument of Theorem 3.2 will imply the almost everywhere convergence of the Fejér and Riesz means. We apply Theorem 3.2 for $T = \mathcal{I}$, the identity function, $T_n = \sigma_n^\alpha$, $p = 1$ and $X = L_1(\mathbb{T})$. The dense set X_0 is the set of the trigonometric polynomials. It is easy to see that $\lim_{n\to\infty} \sigma_n^\alpha f = f$ everywhere for all $f \in X_0$. This implies the next well known theorem, which is due to Fejér [39] and Lebesgue [58] for $\alpha = 1$ and to Riesz [71] for other α's.

Corollary 3.3 *If $0 < \alpha < \infty$ and $f \in L_1(\mathbb{T})$, then*

$$\lim_{n\to\infty} \sigma_n^\alpha f = f \qquad \text{a.e.}$$

Using (3.2) and the density of trigonometric polynomials, the next corollary can be shown easily.

Corollary 3.4 *If $0 < \alpha < \infty$ and $f \in C(\mathbb{T})$, then*

$$\lim_{n\to\infty} \sigma_n^\alpha f = f \qquad \textit{uniformly.}$$

4 Multi-dimensional partial sums

For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$, set

$$u \cdot x := \sum_{k=1}^d u_k x_k, \qquad \|x\|_q := \begin{cases} \left(\sum_{k=1}^d |x_k|^q\right)^{1/q}, & 0 < q < \infty; \\ \sup_{i=1,\ldots,d} |x_i|, & q = \infty. \end{cases}$$

The d-dimensional trigonometric system is introduced as a Kronecker product by

$$\mathrm{e}^{\mathrm{i}k\cdot x} = \prod_{j=1}^d \mathrm{e}^{\mathrm{i}k_j x_j},$$

where $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$, $x = (x_1, \ldots, x_d) \in \mathbb{T}^d$. The multi-dimensional **Fourier coefficients** of an integrable function are given by

$$\widehat{f}(k) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) \mathrm{e}^{-\mathrm{i}k\cdot x}\, \mathrm{d}x \qquad (k \in \mathbb{N}^d).$$

The formal trigonometric series

$$\sum_{k\in\mathbb{Z}^d} \widehat{f}(k) \mathrm{e}^{\mathrm{i}k\cdot x} \qquad (x \in \mathbb{T}^d)$$

defines the multi-dimensional **Fourier series** of f.

We can generalize the partial sums (2.1) and summability means (3.1) for multi-dimensional functions basically in two ways. In the first version, we replace the $|\cdot|$ in (2.1) and

(3.1) by $\|\cdot\|_q$. In the literature the most natural choices $q=2$ (see e.g. Stein and Weiss [80, 78], Davis and Chang [25] and Grafakos [43]), $q=1$ (Berens, Li and Xu [7, 9, 8, 107], Weisz [99, 100]) and $q=\infty$ (Marcinkiewicz [62], Zhizhiashvili [109] and Weisz [94, 101]) are investigated. In the second generalization, we take the sum in each dimension, the so called rectangular partial sum (Zygmund [110] and Weisz [94]).

For $f \in L_1(\mathbb{T}^d)$, the nth **ℓ_q-partial sum** $s_n^q f$ $(n \in \mathbb{N})$ is given by

$$s_n^q f(x) := \sum_{k\in\mathbb{Z}^d,\, \|k\|_q \le n} \widehat{f}(k) \mathrm{e}^{\mathrm{i}k\cdot x} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-u) D_n^q(u)\, \mathrm{d}u$$

where

$$D_n^q(u) := \sum_{k\in\mathbb{Z}^d,\, \|k\|_q \le n} \mathrm{e}^{\mathrm{i}k\cdot u}$$

is the **ℓ_q-Dirichlet kernel**. The partial sums are called **triangular** if $q=1$, **circular** if $q=2$ and **cubic** if $q=\infty$ (see Figures 3–6).

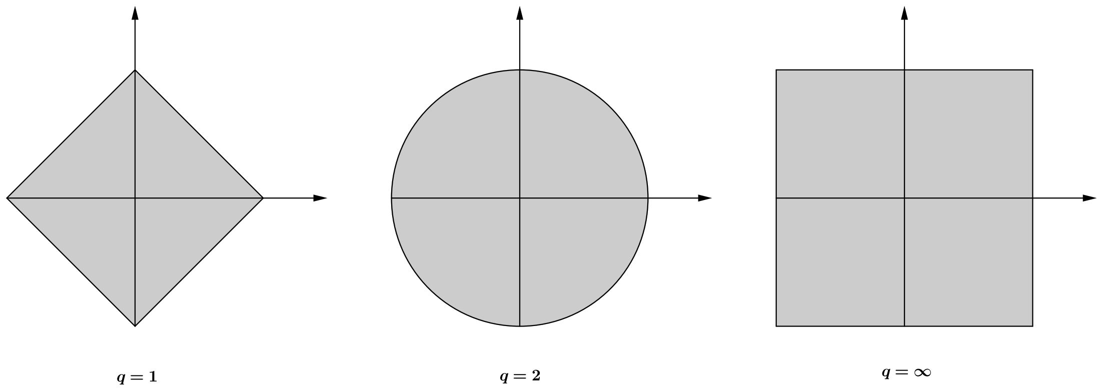

Figure 3: Regions of the ℓ_q-partial sums for $d=2$.

For $f \in L_1(\mathbb{T}^d)$, the nth **rectangular partial sum** $s_n f$ $(n \in \mathbb{N}^d)$ is introduced by

$$s_n f(x) := \sum_{|k_1|\le n_1} \cdots \sum_{|k_d|\le n_d} \widehat{f}(k) \mathrm{e}^{\mathrm{i}k\cdot x} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-u) D_n(u)\, \mathrm{d}u,$$

where

$$D_n(u) := \sum_{|k_1|\le n_1} \cdots \sum_{|k_d|\le n_d} \mathrm{e}^{\mathrm{i}k\cdot u}$$

is the **rectangular Dirichlet kernel** (see Figure 7).

By iterating the one-dimensional result, we get easily the next theorem.

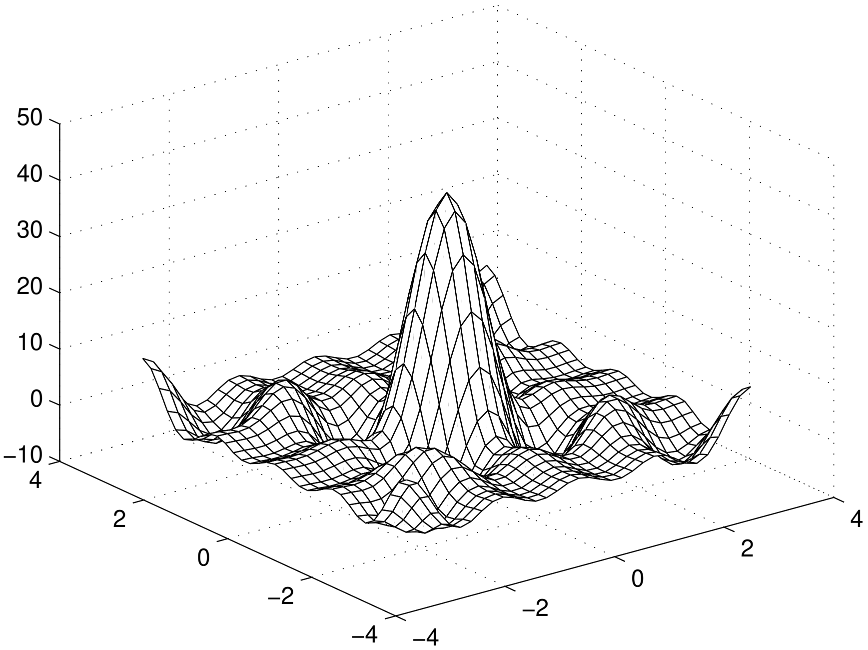

Figure 4: The Dirichlet kernel D_n^q with $d=2$, $q=1$, $n=4$.

Theorem 4.1 *If $f \in L_p(\mathbb{T}^d)$ for some $1<p<\infty$, then*

$$\|s_n f\|_p \le C_p \|f\|_p \qquad (n \in \mathbb{N}^d)$$

and

$$\lim_{n\to\infty} s_n f = f \qquad \textit{in the } L_p\textit{-norm.}$$

Here $n \to \infty$ means the Pringsheim convergence, i.e., $\min(n_1, \ldots, n_d) \to \infty$.

Proof. By Theorem 2.1,

$$\begin{aligned}\int_{\mathbb{T}} |s_n f(x)|^p \, \mathrm{d}x_1 &= \int_{\mathbb{T}} \Big| \int_{\mathbb{T}} \Big(\int_{\mathbb{T}^{d-1}} f(t) \\ &\quad (D_{n_2}(x_2+t_2) \cdots D_{n_d}(x_d+t_d)) \, \mathrm{d}t_2 \cdots \mathrm{d}t_d \Big) D_{n_1}(x_1+t_1) \, \mathrm{d}t_1 \Big|^p \mathrm{d}x_1 \\ &\le \int_{\mathbb{T}} \Big| \int_{\mathbb{T}^{d-1}} f(t)(D_{n_2}(x_2+t_2) \cdots D_{n_d}(x_d+t_d)) \, \mathrm{d}t_2 \cdots \mathrm{d}t_d \Big|^p \mathrm{d}t_1.\end{aligned}$$

Applying this inequality $(d-1)$-times, we get the desired inequality of Theorem 4.1. The convergence is a consequence of this inequality and of the density of trigonometric polynomials. ■

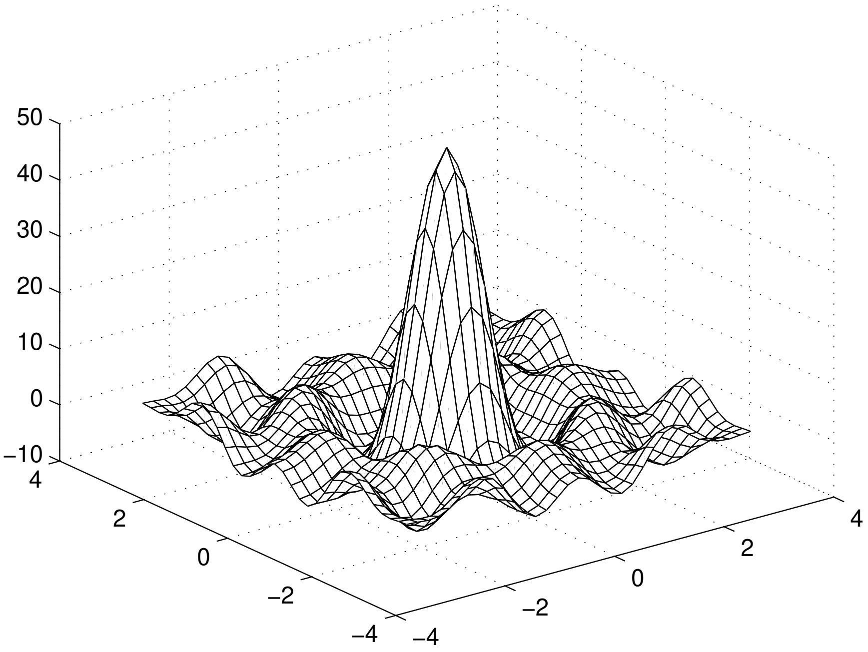

Figure 5: The Dirichlet kernel D_n^q with $d = 2$, $q = 2$, $n = 4$.

A similar result holds for the triangular and cubic partial sums.

Theorem 4.2 *If $q = 1, \infty$ and $f \in L_p(\mathbb{T}^d)$ for some $1 < p < \infty$, then*

$$\|s_n^q f\|_p \leq C_p \|f\|_p \qquad (n \in \mathbb{N})$$

and

$$\lim_{n\to\infty} s_n^q f = f \qquad \text{in the } L_p\text{-norm.}$$

If $q = 2$, then the same result is valid for $p = 2$.

Proof. The result for $q = \infty$ follows from Theorem 4.1. For $q = 2$, it is a basic result of Fourier analysis. If $q = 1$, then we will prove the result for $d = 2$, only. The general case can be proved in the same way. Observe that

$$\int_{\mathbb{T}^2} f(x,y) \mathrm{e}^{\mathrm{i}kx+\mathrm{i}ly} \,\mathrm{d}x \,\mathrm{d}y = 2 \int_{\mathbb{T}^2} f(u-v, u+v) \mathrm{e}^{\mathrm{i}u(k+l)+\mathrm{i}v(l-k)} \,\mathrm{d}u \,\mathrm{d}v. \tag{4.1}$$

If $|k| + |l| \leq n$ on the left hand side, then $|k + l| \leq n$ and $|l - k| \leq n$ on the right hand side, hence

$$s_n^1 f(x,y) = 2 s_n^\infty g(u,v), \tag{4.2}$$

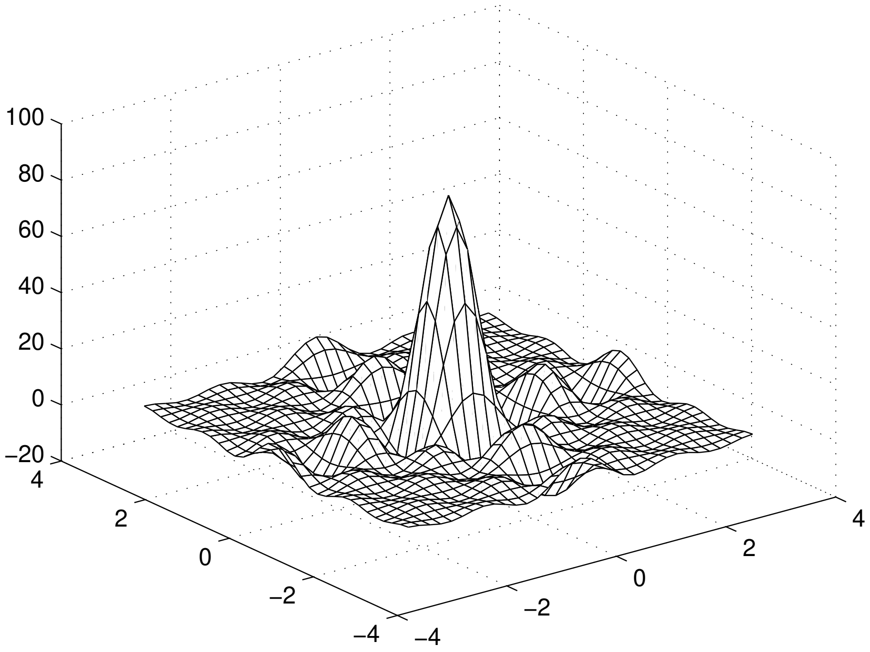

Figure 6: The Dirichlet kernel D_n^q with $d=2$, $q=\infty$, $n=4$.

where

$$g(u,v) := f(u-v,u+v), \quad x = u-v, \quad y = u+v.$$

Thus

$$\|s_n^1 f\|_p \leq 2^{1+1/p} \|s_n^\infty g\|_p \leq C_p \|g\|_p \leq C_p \|f\|_p$$

shows the result for $q=1$, too. ∎

Since the characteristic function of the unit ball is not an $L_p(\mathbb{R}^d)$ $(1<p\neq 2<\infty, d\geq 2)$ multiplier (see Fefferman [31] or Grafakos [43, p. 743]), we have

Theorem 4.3 *If $d \geq 2$, $q=2$ and $1<p\neq 2<\infty$, then the preceding theorem is not true.*

The analogue of Carleson's theorem does not hold in higher dimensions for the rectangular partial sums. However, it is true for the triangular and cubic partial sums (see Fefferman [29, 30] and Grafakos [43, p. 231]). Let us denote by

$$s_*^q f := \sup_{n\in\mathbb{N}} |s_n^q f|$$

the **maximal operator**.

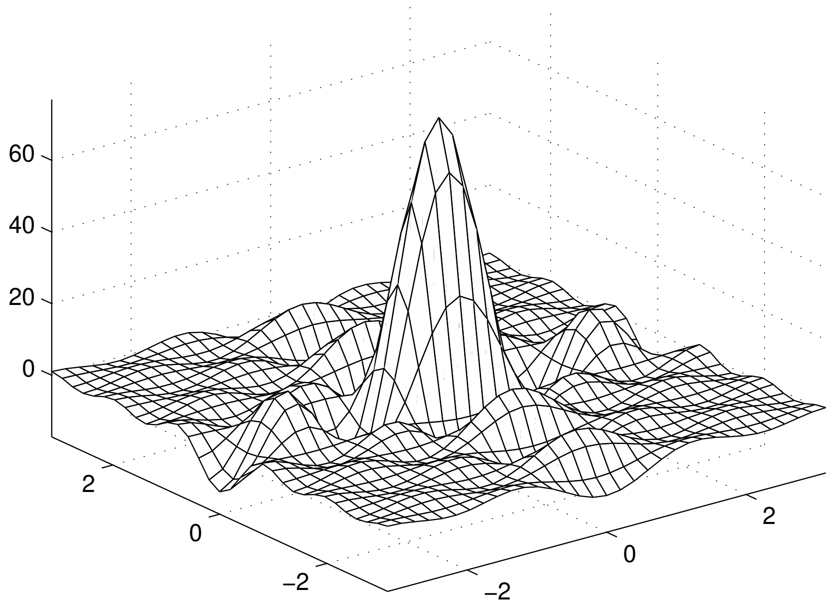

Figure 7: The rectangular Dirichlet kernel with $d = 2$, $n_1 = 3$, $n_2 = 5$.

Theorem 4.4 *If $q = 1, \infty$ and $f \in L_p(\mathbb{T}^d)$ for some $1 < p < \infty$, then*

$$\|s_*^q f\|_p \leq C_p \|f\|_p$$

and if $1 < p \leq \infty$, then

$$\lim_{n \to \infty} s_n^q f = f \qquad \text{a.e.}$$

Proof. We will prove the theorem for $d = 2$ only. The proof for higher dimensions is similar. Suppose first that $q = \infty$ and

$$\widehat{f}(k,l) = 0 \qquad \text{for } l < k \text{ or } k < 0. \tag{4.3}$$

Let

$$f_x(y) := f(x,y) \qquad (x, y \in \mathbb{T})$$

and observe by Fubini's theorem that f_x belongs to $L_p(\mathbb{T})$. Hence, by Theorem 2.3,

$$\|s_* f_x\|_p \leq C_p \|f_x\|_p \tag{4.4}$$

for a.e. $x \in \mathbb{T}$. Set

$$h_l(x) := \int_{\mathbb{T}} f_x(y) \mathrm{e}^{\imath l y} \, \mathrm{d}y \qquad (l \in \mathbb{N})$$

and observe that

$$\begin{aligned}
\|h_l\|_p &= \Big(\int_{\mathbb{T}} \Big|\int_{\mathbb{T}} f_x(y) e^{\mathrm{i}ly}\,\mathrm{d}y\Big|^p \,\mathrm{d}x\Big)^{1/p} \\
&\leq C_p\Big(\int_{\mathbb{T}}\int_{\mathbb{T}} |f_x(y)|^p \,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p} \\
&= C_p\|f\|_p.
\end{aligned}$$

Thus $h_l \in L_p(\mathbb{T})$. Since

$$\widehat{h}_l(k) = \int_{\mathbb{T}} h_l(x) e^{\mathrm{i}kx} = \widehat{f}(k,l),$$

it is clear by (4.3) that each h_l is a trigonometric polynomial. More precisely, $\widehat{h}_l(k)$ vanishes if $k < 0$ or $k > l$. Consequently,

$$\begin{aligned}
s_n f_x(y) &= \sum_{|l|\leq n} h_l(x) e^{\mathrm{i}ly} \\
&= \sum_{|l|\leq n} \Big(\sum_{k=0}^{l} \widehat{f}(k,l) e^{\mathrm{i}kx}\Big) e^{\mathrm{i}ly} \\
&= \sum_{0\leq k\leq l\leq n} \widehat{f}(k,l) e^{\mathrm{i}kx+\mathrm{i}ly} \\
&= s_n^{\infty} f(x,y).
\end{aligned}$$

Hence (4.4) implies

$$\|s_*^{\infty} f\|_p = \Big(\int_{\mathbb{T}}\int_{\mathbb{T}} |s_* f_x(y)|^p \,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p} \leq C_p\Big(\int_{\mathbb{T}}\int_{\mathbb{T}} |f_x(y)|^p \,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p} = C_p\|f\|_p,$$

which proves the theorem if (4.3) holds. Obviously, the same holds for functions f for which $\widehat{f}(k,l) = 0$ if $l > k$ or $l < 0$ and we could also repeat the proof for the other quadrants.

Let us define the projections

$$P_1^+ f(x,y) := \sum_{k\in\mathbb{N}}\sum_{l\in\mathbb{Z}} \widehat{f}(k,l) e^{\mathrm{i}kx+\mathrm{i}ly},$$

$$P_2^+ f(x,y) := \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{N}} \widehat{f}(k,l) e^{\mathrm{i}kx+\mathrm{i}ly},$$

$$Q_1 f(x,y) := \sum_{l\geq|k|} \widehat{f}(k,l) e^{\mathrm{i}kx+\mathrm{i}ly}$$

and

$$Qf(x,y) := \sum_{l\geq k\geq 0} \widehat{f}(k,l) e^{\mathrm{i}kx+\mathrm{i}ly}$$

(see Figure (8)).

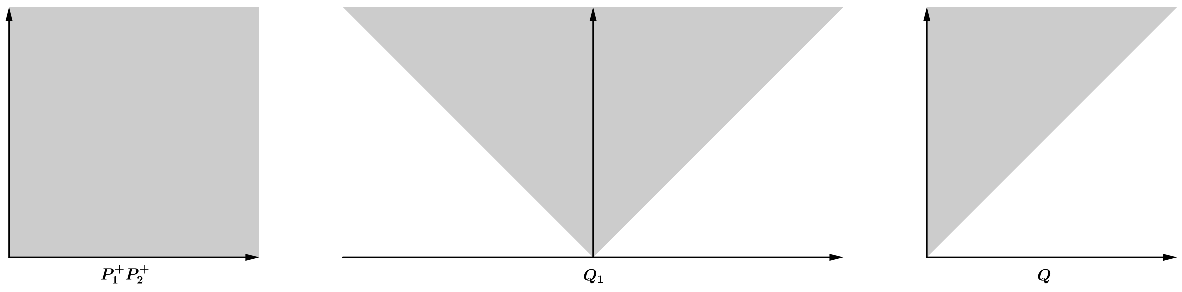

Figure 8: The projections $P_1^+P_2^+$, Q_1 and Q.

By (4.1) and Theorem 2.2, we conclude that $Q_1f(x,y) = 2P_1^+P_2^+g(u,v)$ and

$$\|Q_1f\|_p = 2^{1+1/p}\|P_1^+P_2^+g\|_p \le C_p\|P_2^+g\|_p \le C_p\|g\|_p \le C_p\|f\|_p,$$

where

$$g(u,v) := f(u-v, u+v), \quad x = u-v, \quad y = u+v$$

and $1 < p < \infty$. Thus Q_1 is a bounded projection on $L_p(\mathbb{T}^2)$ and so is $Q = Q_1P_1^+P_2^+$. Since Qf satisfies (4.3), we obtain

$$\|s_*^\infty(Qf)\|_p \le C_p\|Qf\|_p \le C_p\|f\|_p.$$

Each function f can be rewritten as the sum of eight similar projections, which implies the theorem for $q = \infty$.

Equality (4.2) implies

$$\|s_*^1f\|_p \le 2^{1+1/p}\|s_*^\infty g\|_p \le C_p\|g\|_p \le C_p\|f\|_p,$$

which also shows the result for $q = 1$. ■

The generalization of Theorem 2.4 for higher dimensions was proved by Antonov [3].

Theorem 4.5 *If $q = \infty$ and*

$$\int_{\mathbb{T}^d} |f(x)|(\log^+|f(x)|)^d \log^+\log^+\log^+|f(x)| \,\mathrm{d}x < \infty,$$

then

$$\lim_{n\to\infty} s_n^q f = f \qquad \text{a.e.}$$

Theorem 4.4 does not hold for circular partial sums (Stein and Weiss [80, p. 268]).

Theorem 4.6 *If $q=2$ and $p<2d/(d+1)$, then there exists a function $f \in L_p(\mathbb{T}^d)$ whose circular partial sums $s_n^q f$ diverge almost everywhere.*

In other words, for a general function in $L_p(\mathbb{T}^d)$ $(p<2)$ almost everywhere convergence of the circular partial sums is not true if the dimension is sufficiently large. It is an open problem, whether Theorem 4.4 holds for $p=2$ and for circular partial sums. As in the one-dimensional case, Theorem 4.1, Theorem 4.2 and the inequality in Theorem 4.4 do not hold for $p=1$ and $p=\infty$.

5 ℓ_q-summability

As we mentioned before, we define the **ℓ_q-Fejér and Riesz means** of an integrable function $f \in L_1(\mathbb{T}^d)$ by

$$\sigma_n^q f(x) := \sum_{k\in\mathbb{Z}^d,\, \|k\|_q \le n} \Big(1-\frac{\|k\|_q}{n}\Big)\widehat{f}(k)e^{ik\cdot x} = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f(x-u)K_n^q(u)\,du,$$

and

$$\begin{aligned}\sigma_n^{q,\alpha} f(x) &:= \sum_{k\in\mathbb{Z}^d,\, \|k\|_q \le n} \Big(1-\Big(\frac{\|k\|_q}{n}\Big)^\gamma\Big)^\alpha \widehat{f}(k)e^{ik\cdot x} \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f(x-u)K_n^{q,\alpha}(u)\,du, \end{aligned} \tag{5.1}$$

where

$$K_n^q(u) := \sum_{k\in\mathbb{Z}^d,\, \|k\|_q \le n} \Big(1-\frac{\|k\|_q}{n}\Big)e^{ik\cdot u}$$

and

$$K_n^{q,\alpha}(u) := \sum_{k\in\mathbb{Z}^d,\, \|k\|_q \le n} \Big(1-\Big(\frac{\|k\|_q}{n}\Big)^\gamma\Big)^\alpha e^{ik\cdot u}$$

are the **ℓ_q-Fejér and Riesz kernels** (see Figures 9–13).

Observe that if $q=1,\infty$ (in this case $\|k\|_q$ is an integer), then

$$K_n^q(u) = \sum_{\|k\|_q \le n}\sum_{j=\|k\|_q}^{n-1}\frac{1}{n}e^{ik\cdot u} = \frac{1}{n}\sum_{j=0}^{n-1} D_j^q(u).$$

Moreover,

$$|K_n^{q,\alpha}| \le Cn^d \qquad (n\in\mathbb{N}^d). \tag{5.2}$$

We will always suppose that $0\le\alpha<\infty$, $1\le\gamma<\infty$. In the case $q=2$ let $\gamma\in\mathbb{N}$. If $\alpha=0$, we get the partial sums and if $q=\gamma=2, \alpha>0$ the means are called **Bochner-Riesz**

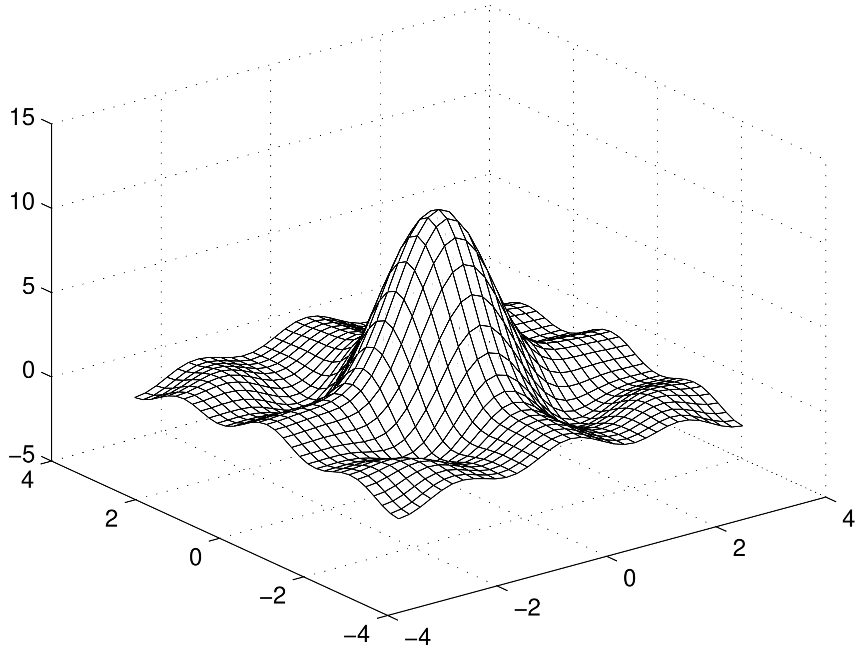

Figure 9: The Riesz kernel $K_n^{q,\alpha}$ with $d = 2$, $q = 1$, $n = 4$, $\alpha = 1$, $\gamma = 1$.

means. The cubic summability (when $q = \infty$) is also called **Marcinkiewicz summability**. Obviously, the ℓ_q-Fejér means are the arithmetic means of the ℓ_q-partial sums when $q = 1, \infty$:

$$\sigma_n^q f(x) = \frac{1}{n} \sum_{k=0}^{n-1} s_k^q f(x).$$

The proofs of the results presented later are very different for the cases $q = 1, 2, \infty$, because the kernel functions are very different. To demonstrate this, we present a few details about the kernels in this section. For the triangular Dirichlet kernel, we need the notion of the divided difference, which is usually used in numerical analysis. The nth **divided difference** of a function f at the (pairwise distinct) nodes $x_1, \ldots, x_n \in \mathbb{R}$ is introduced inductively as

$$[x_1]f := f(x_1), \qquad [x_1, \ldots, x_n]f := \frac{[x_1, \ldots, x_{n-1}]f - [x_2, \ldots, x_n]f}{x_1 - x_n}. \tag{5.3}$$

One can see that the difference is a symmetric function of the nodes. It is known (see e.g. DeVore and Lorentz [26, p. 120]) that

$$[x_1, \ldots, x_n]f = \sum_{k=1}^{n} \frac{f(x_k)}{\prod_{j=1, j \neq k}^{n} (x_k - x_j)}. \tag{5.4}$$

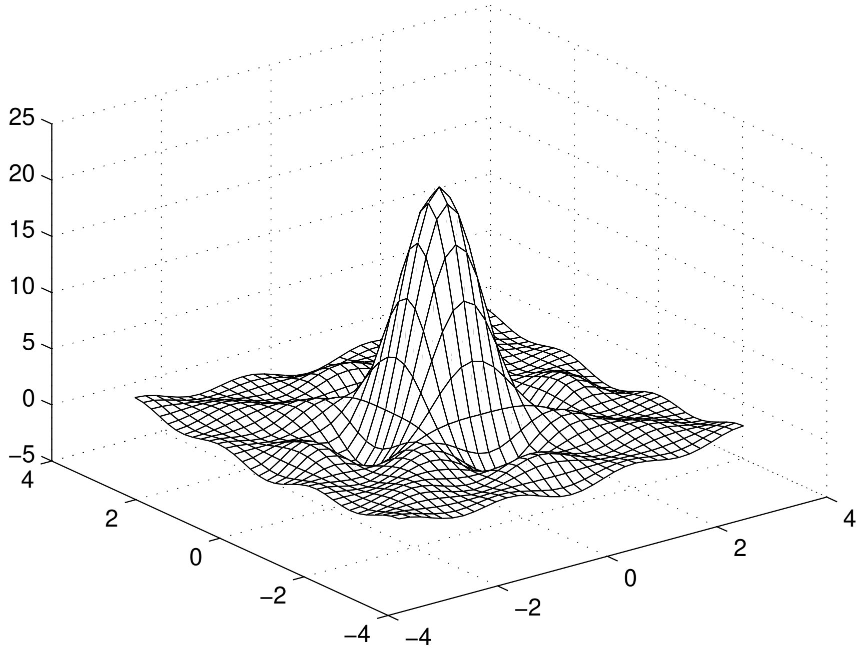

Figure 10: The Riesz kernel $K_n^{q,\alpha}$ with $d = 2$, $q = \infty$, $n = 4$, $\alpha = 1$, $\gamma = 1$.

Moreover, if f is $(n-1)$-times continuously differentiable on $[a, b]$ and $x_i \in [a, b]$, then there exists $\xi \in [a, b]$ such that

$$[x_1, \ldots, x_n]f = \frac{f^{(n-1)}(\xi)}{(n-1)!}. \tag{5.5}$$

To give an explicit form of the triangular Dirichlet kernel, we will need the following trigonometric identities.

Lemma 5.1 *For all $n \in \mathbb{N}$ and $0 \leq x, y \leq \pi$,*

$$\sum_{k=0}^{n} \epsilon_k \cos(ky) \sin((n-k+1/2)x)$$
$$= \sin(x/2) \frac{\cos(x/2)\cos((n+1/2)x) - \cos(y/2)\cos((n+1/2)y)}{\cos x - \cos y} \tag{5.6}$$

and

$$\sum_{k=0}^{n} \epsilon_k \cos(ky) \cos((n-k+1/2)x)$$
$$= \cos(x/2) \frac{\sin(y/2)\sin((n+1/2)y) - \sin(x/2)\sin((n+1/2)x)}{\cos x - \cos y}, \tag{5.7}$$

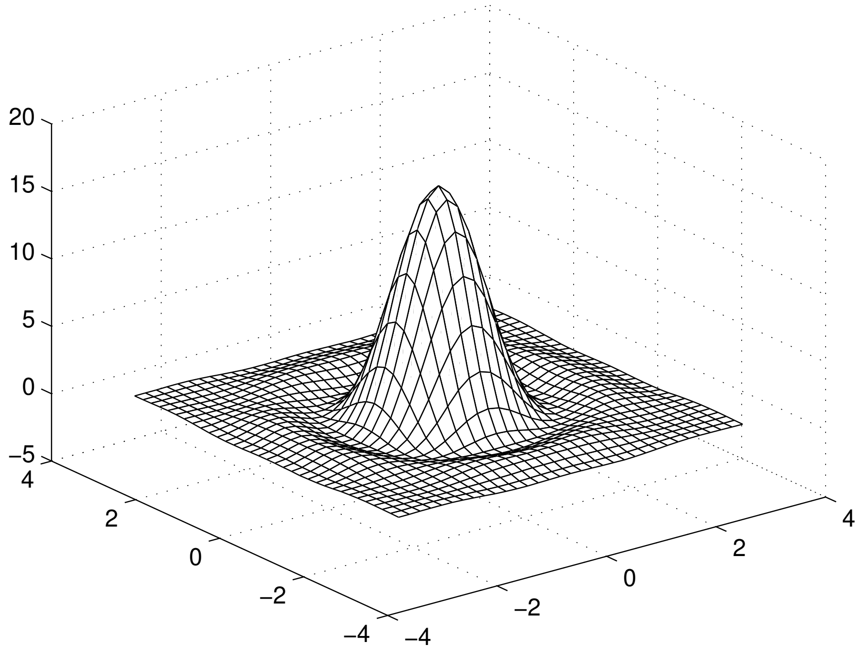

Figure 11: The Riesz kernel $K_n^{q,\alpha}$ with $d=2$, $q=2$, $n=4$, $\alpha=1$, $\gamma=1$.

where $\epsilon_0 := 1/2$ *and* $\epsilon_k := 1$, $k \geq 1$.

Proof. By trigonometric identities,

$$\begin{aligned}
&\sum_{k=0}^{n} \epsilon_k \cos(ky)\sin((n-k+1/2)x) \\
&= \sin((n+1/2)x)\sum_{k=0}^{n} \epsilon_k \cos(ky)\cos(kx) - \cos((n+1/2)x)\sum_{k=0}^{n} \epsilon_k \cos(ky)\sin(kx) \\
&= \frac{1}{2}\sin((n+1/2)x)\sum_{k=0}^{n}\Big(\epsilon_k \cos(k(x-y)) + \epsilon_k \cos(k(x+y)\Big) \\
&\quad - \frac{1}{2}\cos((n+1/2)x)\sum_{k=0}^{n}\Big(\epsilon_k \sin(k(x-y)) + \epsilon_k \sin(k(x+y))\Big).
\end{aligned}$$

Similarly to (2.2), we can show that

$$\sum_{k=0}^{n} \epsilon_k \sin(kx) = \frac{\cos(x/2) - \cos((n+1/2)x)}{2\sin(x/2)}.$$

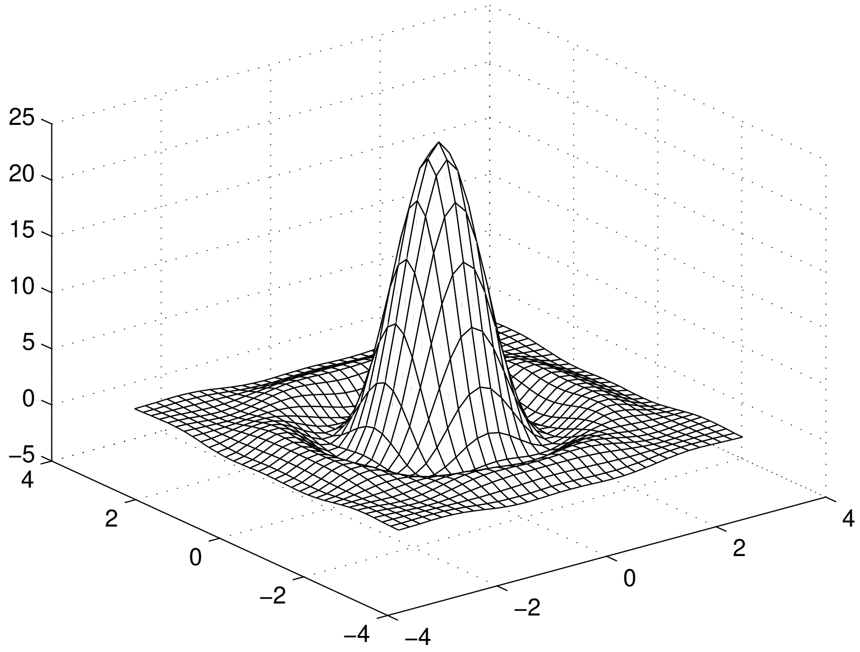

Figure 12: The Bochner-Riesz kernel $K_n^{q,\alpha}$ with $d=2$, $q=2$, $n=4$, $\alpha=1$, $\gamma=2$.

Using this and (2.2), we conclude

$$\begin{aligned}
&\sum_{k=0}^{n} \epsilon_k \cos(ky)\sin((n-k+1/2)x) \\
&= \frac{1}{4}\sin((n+1/2)x)\Big(\frac{\sin((n+1/2)(x-y))}{\sin((x-y)/2)} + \frac{\sin((n+1/2)(x+y))}{\sin((x+y)/2)}\Big) \\
&\quad - \frac{1}{4}\cos((n+1/2)x)\Big(\frac{\cos((x-y)/2)-\cos((n+1/2)(x-y))}{\sin((x-y)/2)} \\
&\quad + \frac{1}{4}\frac{\cos((x+y)/2)-\cos((n+1/2)(x+y))}{\sin((x+y)/2)}\Big).
\end{aligned}$$

Now after some computation, we obtain (5.6).

Formula (5.7) can be shown in the same way. ∎

Define the function G_n by

$$G_n(\cos x) := (-1)^{[(d-1)/2]} 2\cos(x/2)(\sin x)^{d-2} \operatorname{soc}((n+1/2)x)$$

where the soc function is defined by

$$\operatorname{soc} x := \begin{cases} \cos x, & \text{if } d \text{ is even;} \\ \sin x, & \text{if } d \text{ is odd.} \end{cases}$$

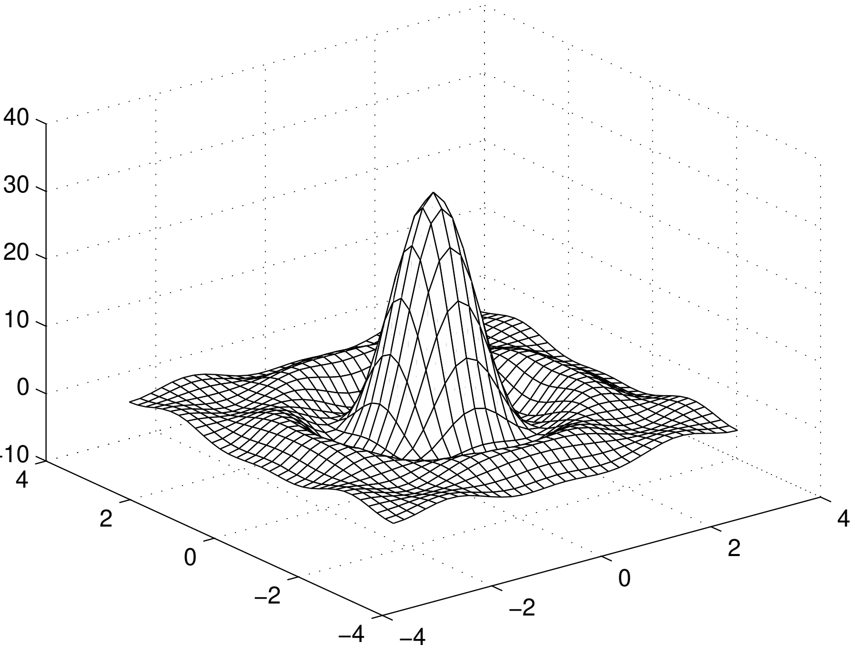

Figure 13: The Bochner-Riesz kernel $K_n^{q,\alpha}$ with $d = 2$, $q = 2$, $n = 4$, $\alpha = 1/2$, $\gamma = 2$.

The following representation of the triangular Dirichlet kernel was proved by Herriot [48] and Berens and Xu [8, 107].

Lemma 5.2 *For $x \in \mathbb{T}^d$,*

$$\begin{aligned} D_n^1(x) &= [\cos x_1, \ldots, \cos x_d] G_n \qquad (5.8) \\ &= (-1)^{[(d-1)/2]} 2 \sum_{k=1}^{d} \frac{\cos(x_k/2)(\sin x_k)^{d-2} \mathrm{soc}\,((n+1/2)x_k)}{\prod_{j=1, j\neq k}^{d} (\cos x_k - \cos x_j)}. \end{aligned}$$

Proof. First, we note that the second equality follows from the definition of G_n and from the property of the divided difference described in (5.4). We prove the lemma by induction. Let us denote the Dirichlet kernel in this proof by $D_{d,n}^1(x) = D_n^1(x)$. We have seen in (2.2) that in the one-dimensional case

$$D_n^1(x) = D_{1,n}^1(x) = \frac{\sin((n+1/2)x)}{\sin(x/2)} = 2\cos(x/2)(\sin x)^{-1} \sin((n+1/2)x),$$

thus (5.8) holds for $d = 1$. Suppose the lemma is true for integers up to d and let d be even. It is easy to see that

$$D_{d+1,n}^1(x) = 2^{d+1} \sum_{j \in \mathbb{N}^d,\, \|j\|_1 \leq n} \epsilon_{j_1} \cos(j_1 x_1) \cdots \epsilon_{j_{d+1}} \cos(j_{d+1} x_{d+1})$$

$$
\begin{aligned}
&= 2\sum_{l=0}^{n} \epsilon_l \cos(lx_{d+1}) D_{d,n-l}(x_1,\ldots,x_d) \\
&= (-1)^{[(d-1)/2]} 4 \sum_{k=1}^{d} \frac{\cos(x_k/2)(\sin x_k)^{d-2}}{\prod_{j=1,j\neq k}^{d}(\cos x_k - \cos x_j)} \\
&\quad \sum_{l=0}^{n} \epsilon_l \cos(lx_{d+1}) \cos((n-l+1/2)x_k),
\end{aligned}
$$

where $\epsilon_0 := 1/2$ and $\epsilon_l := 1$, $l \geq 1$. Using (5.7), we obtain

$$
\begin{aligned}
D^1_{d+1,n}(x) &= -(-1)^{[(d-1)/2]} 4 \sum_{k=1}^{d} \frac{\cos(x_k/2)(\sin x_k)^{d-2}}{\prod_{j=1,j\neq k}^{d+1}(\cos x_k - \cos x_j)} \\
&\quad \cos(x_k/2)\sin(x_k/2)\sin((n+1/2)x_k) \\
&\quad + (-1)^{[(d-1)/2]} 4 \sum_{k=1}^{d} \frac{\cos(x_k/2)(\sin x_k)^{d-2}}{\prod_{j=1,j\neq k}^{d+1}(\cos x_k - \cos x_j)} \\
&\quad \cos(x_k/2)\sin(x_{d+1}/2)\sin((n+1/2)x_{d+1}) \\
&= -(-1)^{[(d-1)/2]} 2 \Big(\sum_{k=1}^{d} \frac{\cos(x_k/2)(\sin x_k)^{d-1}\sin((n+1/2)x_k)}{\prod_{j=1,j\neq k}^{d+1}(\cos x_k - \cos x_j)} \\
&\quad - \sin(x_{d+1}/2)\sin((n+1/2)x_{d+1}) \sum_{k=1}^{d} \frac{(1+\cos x_k)(\sin x_k)^{d-2}}{\prod_{j=1,j\neq k}^{d+1}(\cos x_k - \cos x_j)} \Big).
\end{aligned}
$$

If d is even, then the function $h(t) := (1+t)(1-t^2)^{(d-2)/2}$ is a polynomial of degree $d-1$. Then, by (5.5),

$$
\begin{aligned}
0 &= [\cos x_1, \ldots, \cos x_{d+1}]h \\
&= \sum_{k=1}^{d} \frac{(1+\cos x_k)(\sin x_k)^{d-2}}{\prod_{j=1,j\neq k}^{d+1}(\cos x_k - \cos x_j)} + \frac{(1+\cos x_{d+1})(\sin x_{d+1})^{d-2}}{\prod_{j=1,j\neq d+1}^{d+1}(\cos x_{d+1} - \cos x_j)}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
D^1_{d+1,n}(x) &= -(-1)^{[(d-1)/2]} 2 \Big(\sum_{k=1}^{d} \frac{\cos(x_k/2)(\sin x_k)^{d-1}\sin((n+1/2)x_k)}{\prod_{j=1,j\neq k}^{d+1}(\cos x_k - \cos x_j)} \\
&\quad + \sin(x_{d+1}/2)\sin((n+1/2)x_{d+1}) \frac{(1+\cos x_{d+1})(\sin x_{d+1})^{d-2}}{\prod_{j=1,j\neq d+1}^{d+1}(\cos x_{d+1} - \cos x_j)} \Big) \\
&= (-1)^{[d/2]} 2 \sum_{k=1}^{d+1} \frac{\cos(x_k/2)(\sin x_k)^{d-1}\sin((n+1/2)x_k)}{\prod_{j=1,j\neq k}^{d+1}(\cos x_k - \cos x_j)},
\end{aligned}
$$

which proves the result if d is even.

If d is odd, the lemma can be proved similarly. ■

We explicitly write out the result for $d=2$:

$$\begin{aligned} D_n^1(x) &= [\cos x_1, \cos x_2]G_n \\ &= \frac{[\cos x_1]G_n - [\cos x_2]G_n}{\cos x_1 - \cos x_2} \\ &= 2\frac{\cos(x_1/2)\cos((n+1/2)x_1) - \cos(x_2/2)\cos((n+1/2)x_2)}{\cos x_1 - \cos x_2}. \end{aligned} \tag{5.9}$$

The cubic Dirichlet kernels ($q=\infty$) are

$$D_n^\infty(x) = \prod_{i=1}^d D_n^\infty(x_i) = \prod_{i=1}^d \frac{\sin((n+1/2)x_i)}{\sin(x_i/2)}.$$

If $q=2$, then the continuous version of the Dirichlet kernel (see Section 11)

$$D_t^2(x) := \int_{\{\|v\|_2 \le t\}} e^{ix\cdot v}\, dv$$

can be expressed as

$$D_t^2(x) = \|x\|_2^{-d/2} t^{d/2} J_{d/2}(2\pi\|x\|_2 t),$$

where

$$J_k(t) := \frac{(t/2)^k}{\sqrt{\pi}\,\Gamma(k+1/2)} \int_{-1}^1 e^{its}(1-s^2)^{k-1/2}\, ds \qquad (k > -1/2, t > 0)$$

are the **Bessel functions** (see Subsection 6.2.4).

6 Norm convergence of the ℓ_q-summability means

We introduce the **reflection** and **translation** operators by

$$\tilde{f}(x) := f(-x), \qquad T_x f(t) := f(t-x).$$

A Banach space B consisting of measurable functions on $\mathbb{T}^d$ is called a **homogeneous Banach space** if

(i) for all $f \in B$ and $x \in \mathbb{T}^d$, $T_x f \in B$ and $\|T_x f\|_B = \|f\|_B$,

(ii) the function $x \mapsto T_x f$ from $\mathbb{T}^d$ to B is continuous for all $f \in B$,

(iii) $\|f\|_1 \le C\|f\|_B$ for all $f \in B$.

For an introduction to homogeneous Banach spaces, see Katznelson [53]. It is easy to see that the spaces $L_p(\mathbb{T}^d)$ $(1 \le p < \infty)$, $C(\mathbb{T}^d)$, the Lorentz spaces $L_{p,q}(\mathbb{T}^d)$ $(1 < p < \infty, 1 \le q < \infty)$ and the Hardy space $H_1(\mathbb{T}^d)$ are homogeneous Banach spaces. The definition of the Fejér and Riesz means can be extended to distributions. Note first of all that

$$\sigma_n^{q,\alpha} f := f * K_n^{q,\alpha} \qquad (n \in \mathbb{N}), \tag{6.1}$$

where $*$ denotes **convolution**, i.e., if $f,g \in L_1(\mathbb{R}^d)$, then

$$f * g(x) := \int_{\mathbb{T}^d} f(t)g(x-t)\,\mathrm{d}t = \int_{\mathbb{T}^d} f(x-t)g(t)\,\mathrm{d}t.$$

Obviously, the convolution is well defined for all $g \in L_1(\mathbb{T}^d)$ and $f \in L_p(\mathbb{T}^d)$ $(1 \le p \le \infty)$ or $f \in B$, where B is a homogeneous Banach space. The convolution can be extended to distributions as follows. It is easy to see that

$$\int_{\mathbb{T}^d} (f*g)(x)h(x)\,\mathrm{d}x = \int_{\mathbb{T}^d} f(t)(\tilde{g}*h)(t)\,\mathrm{d}t$$

for all $f,g,h \in \mathcal{S}(\mathbb{T}^d)$. For a distribution $u \in \mathcal{S}'(\mathbb{T}^d)$ and $g \in \mathcal{S}(\mathbb{T}^d)$, let us define the convolution $u*g$ by

$$u*g(h) := u(\tilde{g}*h) \qquad (h \in \mathcal{S}(\mathbb{T}^d)). \tag{6.2}$$

It is easy to see that $u*g$ is indeed a distribution. Moreover,

$$\begin{aligned} u*g(h) = u(\tilde{g}*h) &= u\Big(\int_{\mathbb{T}^d} \tilde{g}(\cdot - x)h(x)\,\mathrm{d}x\Big) \\ &= u\Big(\int_{\mathbb{T}^d} T_x\tilde{g}(\cdot)h(x)\,\mathrm{d}x\Big). \end{aligned}$$

The Riemann sums of the last integral are easily shown to converge in the topology of $\mathcal{S}(\mathbb{T}^d)$. Since u is continuous,

$$u*g(h) = \int_{\mathbb{T}^d} u(T_x\tilde{g})h(x)\,\mathrm{d}x.$$

Thus the distribution $u*g$ equals the function $x \mapsto u(T_x\tilde{g})$ $(g \in \mathcal{S}(\mathbb{T}^d))$. One can show that this function is a C^∞ function.

We can see easily that

$$\int_{\mathbb{T}^d} (f*g)(x)h(x)\,\mathrm{d}x = \int_{\mathbb{T}^d} (f*\tilde{h})(t)\tilde{g}(t)\,\mathrm{d}t$$

for all $f,g,h \in \mathcal{S}(\mathbb{T}^d)$. If $g \in L_1(\mathbb{T}^d)$, then we define $u*g$ by

$$u*g(h) := \langle u*\tilde{h}, \tilde{g}\rangle = \int_{\mathbb{T}^d} (u*\tilde{h})(x)\tilde{g}(x)\,\mathrm{d}x \qquad (h \in \mathcal{S}(\mathbb{T}^d)). \tag{6.3}$$

The last integral is well defined, because $u*\tilde{h} \in L_\infty(\mathbb{T}^d)$ and $g \in L_1(\mathbb{T}^d)$. We can show that $u*g$ is a distribution if $u \in H_p^\square(\mathbb{T}^d)$ (for the definition of the Hardy space $H_p^\square(\mathbb{T}^d)$ see Section 7). Moreover, if $\lim_{k\to\infty} u_k = u$ in the $H_p^\square$-norm, then $\lim_{k\to\infty} u_k * g = u*g$ in the sense of distributions. For more details, see Stein [78]. Consequently, since $K_n^{q,\alpha}$ is integrable, we obtain that $\sigma_n^{q,\alpha} f$ is well defined in (6.1) for all distributions $f \in \mathcal{S}'(\mathbb{T}^d)$.

It was proved in Berens, Li and Xu [7], Oswald [68] and Weisz [100, 101] for $q = 1, \infty$ and in Bochner [11] (see also Stein and Weiss [80]) for $q = 2$ that the L_1-norms of the Riesz kernels are uniformly bounded. Moreover, this theorem was proved by Li and Xu [59] for Jacobi polynomials.

Theorem 6.1 *If $q = 1, \infty$ and $\alpha > 0$, then*

$$\int_{\mathbb{T}^d} |K_n^{q,\alpha}(x)| \, dx \le C \qquad (n \in \mathbb{N}).$$

If $q = 2$, then the same holds for $\alpha > (d-1)/2$.

We will prove this theorem in Subsection 6.2. This implies easily

Theorem 6.2 *If $q = 1, \infty$, $\alpha > 0$ and B is a homogeneous Banach space on $\mathbb{T}^d$, then*

$$\|\sigma_n^{q,\alpha} f\|_B \le C \|f\|_B \qquad (n \in \mathbb{N})$$

and

$$\lim_{n\to\infty} \sigma_n^{q,\alpha} f = f \qquad \text{in the } B\text{-norm for all } f \in B.$$

If $q = 2$, then the same holds for $\alpha > (d-1)/2$.

Proof. Observe that

$$\|\sigma_n^{q,\alpha} f\|_B \le \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \|f(\cdot - u)\|_B K_n^{q,\alpha}(u) \, du = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \|f\|_B K_n^{q,\alpha}(u) \, du.$$

Since the trigonometric polynomials are dense in B (see Katznelson [53]), the theorem follows from Theorem 6.1. ■

Originally the theorem was proved in the case $q = 1, \infty$ only for $1 \le \alpha < \infty$. However, the analogous result for Fourier transforms holds for every α. Now Theorem 6.2 follows from the transference theorem (see Grafakos [43, pp. 220-226]).

Since the $L_p(\mathbb{T}^d)$ $(1 \le p < \infty)$ spaces are homogeneous Banach spaces, Theorem 6.2 holds for these spaces, too. The situation is more complicated and not completely solved if $q = 2$ and $\alpha \le (d-1)/2$. It is clear by the Banach-Steinhaus theorem that $\lim_{n\to\infty} \sigma_n^{q,\alpha} f = f$ in the L_p-norm for all $f \in L_p(\mathbb{T}^d)$ if and only if the operators $\sigma_n^{q,\alpha}$ are uniformly bounded from $L_p(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$. We note that each operator $\sigma_n^{q,\alpha}$ is bounded on $L_p(\mathbb{T}^d)$, because $K_n^{q,\alpha} \in L_1(\mathbb{T}^d)$.

6.1 Further results for the Bochner-Riesz means

The following results are all proved in the book of Grafakos [43, Chapter 10], so we do not prove them here. Figures 14–18 show the regions where the operators $\sigma_n^{2,\alpha}$ are uniformly bounded or unbounded.

Theorem 6.3 *Suppose that $d \ge 2$ and $q = \gamma = 2$. If $0 \le \alpha \le (d-1)/2$ and*

$$p \le \frac{2d}{d+1+2\alpha} \qquad \text{or} \qquad p \ge \frac{2d}{d-1-2\alpha},$$

then the Bochner-Riesz operators $\sigma_n^{2,\alpha}$ are not uniformly bounded on $L_p(\mathbb{T}^d)$ (see Figure 14).

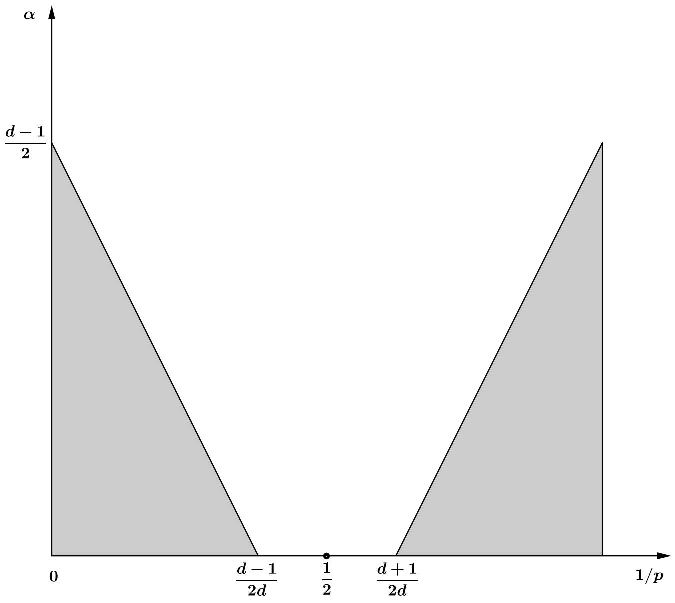

Figure 14: Uniform unboundedness of $\sigma_n^{2,\alpha}$.

The following result about the uniform boundedness of $\sigma_n^{2,\alpha}$ was proved by Stein [80, p. 276].

Theorem 6.4 *Suppose that $d \geq 2$ and $q = \gamma = 2$. If $0 < \alpha \leq \frac{d-1}{2}$ and*

$$\frac{2(d-1)}{d-1+2\alpha} < p < \frac{2(d-1)}{d-1-2\alpha},$$

then the Bochner-Riesz operators $\sigma_n^{2,\alpha}$ are uniformly bounded on $L_p(\mathbb{T}^d)$ (see Figure 15).

Carleson and Sjölin [19] solved completely the uniform boundedness of the Bochner-Riesz operators for $d = 2$. They are uniformly bounded for p's which are excluded from Theorem 6.3 (other proofs were given by Fefferman [32] and Hörmander [50]).

Theorem 6.5 *Suppose that $d = 2$ and $q = \gamma = 2$. If $0 < \alpha \leq 1/2$ and*

$$\frac{4}{3+2\alpha} < p < \frac{4}{1-2\alpha},$$

then the Bochner-Riesz operators $\sigma_n^{2,\alpha}$ are uniformly bounded on $L_p(\mathbb{T}^d)$ (see Figure 16).

Fefferman [28] generalized this result to higher dimensions.

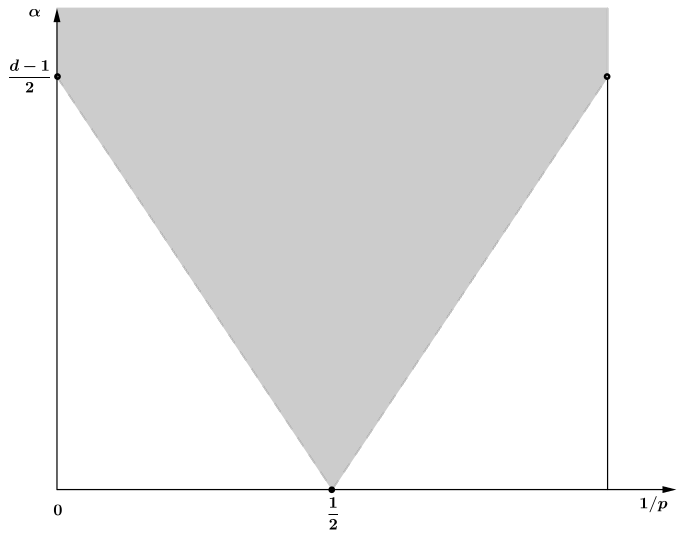

Figure 15: Uniform boundedness of $\sigma_n^{2,\alpha}$ when $d \geq 3$.

Theorem 6.6 *Suppose that $d \geq 3$ and $q = \gamma = 2$. If $\frac{d-1}{2(d+1)} \leq \alpha \leq \frac{d-1}{2}$ and*

$$\frac{2d}{d+1+2\alpha} < p < \frac{2d}{d-1-2\alpha},$$

then the Bochner-Riesz operators $\sigma_n^{2,\alpha}$ are uniformly bounded on $L_p(\mathbb{T}^d)$ (see Figure 17).

Combining Theorems 6.6 and 6.4 and using analytic interpolation (see e.g. Stein and Weiss [80, p. 276, p. 205]), we obtain

Theorem 6.7 *Suppose that $d \geq 3$ and $q = \gamma = 2$. If $0 < \alpha < \frac{d-1}{2(d+1)}$ and*

$$\frac{2(d-1)}{d-1+4\alpha} < p < \frac{2(d-1)}{d-1-4\alpha},$$

then the Bochner-Riesz operators $\sigma_n^{2,\alpha}$ are uniformly bounded on $L_p(\mathbb{T}^d)$ (see Figure 17).

It is still an open question as to whether the operators $\sigma_n^{2,\alpha}$ are uniformly bounded or unbounded in the region of Figure 18.

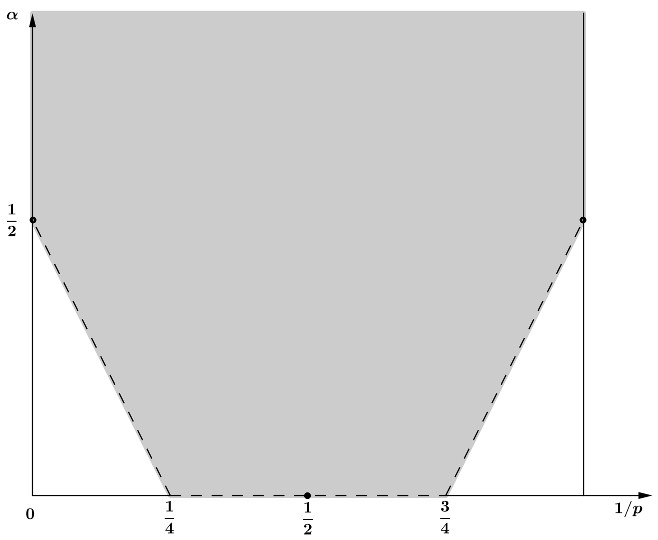

Figure 16: Uniform boundedness of $\sigma_n^{2,\alpha}$ when $d=2$.

6.2 Proof of Theorem 6.1

In this section, we will prove Theorem 6.1. Since the kernel functions and hence the proofs are very different for different q's, we prove the theorem in four subsections. If $q=1,\infty$, then we suppose here that $\alpha=\gamma=1$. For other parameters, see Subsection 10.1. For $q=1$, we prove the theorem separately for $d=2$ and $d\geq 3$.

6.2.1 Proof for $q=1$ in the two-dimensional case

In this section, we write (x,y) instead of the two-dimensional vector x. Recall from (5.9) that

$$\begin{aligned} D_k^1(x,y) &= 2\frac{\cos(x/2)\cos((k+1/2)x)-\cos(y/2)\cos((k+1/2)y)}{\cos x-\cos y} \\ &= -\frac{\cos(x/2)\cos((k+1/2)x)-\cos(y/2)\cos((k+1/2)y)}{\sin((x-y)/2)\sin((x+y)/2)}. \end{aligned} \tag{6.4}$$

In what follows, we may suppose that $\pi>x>y>0$. We denote the **characteristic function** of a set H by 1_H, i.e.,

$$1_H(x):=\begin{cases}1, & \text{if } x\in H;\\ 0, & \text{if } x\notin H.\end{cases}$$

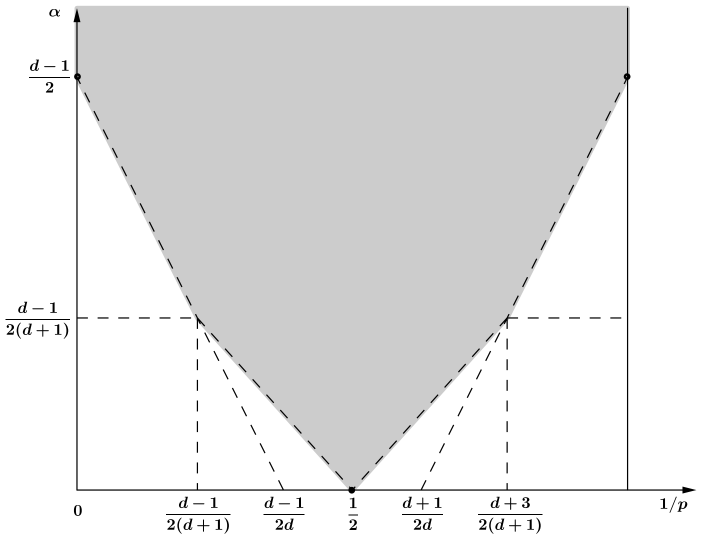

Figure 17: Uniform boundedness of $\sigma_n^{2,\alpha}$ when $d \geq 3$.

Lemma 6.8 *For* $0 < \beta < 1$,

$$\begin{aligned} |K_n^1(x,y)| &\leq C(x-y)^{-3/2}(y^{-1/2}1_{\{y\leq\pi/2\}} + (\pi-x)^{-1/2}1_{\{y>\pi/2\}}), &(6.5)\\ &\leq Cn^{-1}(x-y)^{-1-\beta}(y^{\beta-2}1_{\{y\leq\pi/2\}} + (\pi-x)^{\beta-2}1_{\{y>\pi/2\}}), &(6.6)\\ &\leq Cy^{-2}1_{\{y\leq\pi/2\}} + C(\pi-x)^{-2}1_{\{y>\pi/2\}}. &(6.7)\end{aligned}$$

Proof. In (6.4), we use that

$$\sin(x \pm y)/2 \sim x \pm y \quad \text{if} \quad y \leq \pi/2$$

and

$$\sin(x-y)/2 \sim x-y, \quad \sin(x+y)/2 \sim 2\pi - x - y \quad \text{if} \quad y > \pi/2.$$

The facts $x+y > x-y$, $x+y > y$ and $2\pi - x - y > x - y$, $2\pi - x - y > \pi - x$ imply (6.5). Using (6.4) and the formulas

$$\sum_{k=0}^{n-1} \cos(k+1/2)t = \frac{\sin(nt)}{2\sin(t/2)}, \qquad \sum_{k=0}^{n-1} \sin(k+1/2)t = \frac{1-\cos(nt)}{2\sin(t/2)}, \tag{6.8}$$

we conclude

$$|K_n^1(x,y)| \leq Cn^{-1}(x-y)^{-1}(x+y)^{-1}y^{-1} \leq Cn^{-1}(x-y)^{-1-\beta}y^{\beta-2}$$

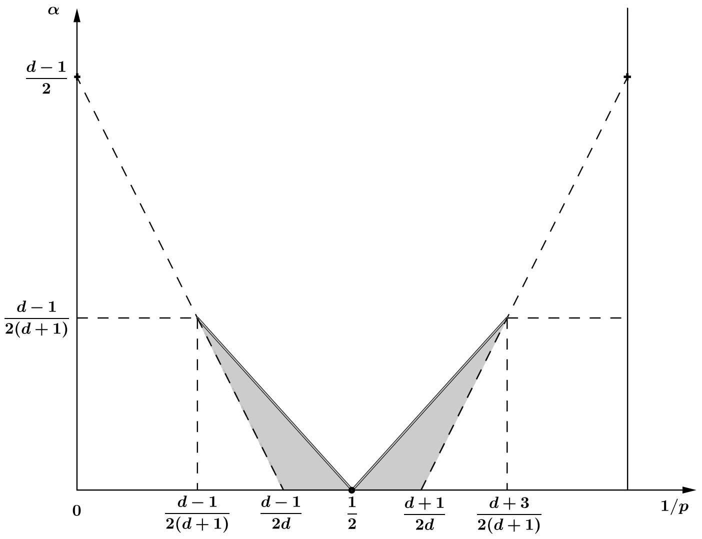

Figure 18: Open question of the uniform boundedness of $\sigma_n^{2,\alpha}$ when $d \geq 3$.

if $y \leq \pi/2$, which is exactly (6.6). The inequality for $y > \pi/2$ can be proved in the same way.

Lagrange's mean value theorem and (6.4) imply that there exists $x > \xi > y$ such that

$$D_k^1(x,y) = -\frac{H_k'(\xi)(x-y)}{\sin((x-y)/2)\sin((x+y)/2)},$$

where

$$H_k(t) := \cos(t/2)\cos((k+1/2)t).$$

Then

$$|K_n^1(x,y)| \leq Cn^{-1}(x-y)(n+1)(x-y)^{-1}(x+y)^{-1}y^{-1} \leq Cy^{-2}$$

shows (6.7) if $y \leq \pi/2$. The case $y > \pi/2$ is similar. ∎

In the next lemma, we estimate the partial derivatives of the kernel function.

Lemma 6.9 *If $0 < \beta < 1$, then for $j = 1, 2$,*

$$|\partial_j K_n^1(x,y)| \leq C(x-y)^{-1-\beta}(y^{\beta-2}1_{\{y\leq\pi/2\}} + (\pi-x)^{\beta-2}1_{\{y>\pi/2\}}). \qquad (6.9)$$

Proof. By Lagrange's mean value theorem and (6.4),

$$\begin{aligned}\partial_1 D_k^1(x,y) &= \frac{1}{2}(\sin(x/2)\cos((k+1/2)x)+\cos(x/2)(2k+1)\sin((k+1/2)x))\\ &\quad \sin((x-y)/2)^{-1}\sin((x+y)/2)^{-1}\\ &\quad +\frac{1}{2}H_k'(\xi)(x-y)(\sin((x-y)/2)^{-2}\sin((x+y)/2)^{-1}\cos((x-y)/2)\\ &\quad +\sin((x-y)/2)^{-1}\sin((x+y)/2)^{-2}\cos((x+y)/2)),\end{aligned}$$

where $y<\xi<x$ is a suitable number. Using the methods above,

$$|\partial_1 K_n^1(x,y)| \le C(x-y)^{-1}(x+y)^{-1}y^{-1}+C(x+y)^{-2}y^{-1} \le C(x-y)^{-1-\beta}y^{\beta-2},$$

which proves (6.9) if $y\le \pi/2$. The case $y>\pi/2$ can be shown similarly. ∎

Now we are ready to prove that the L_1-norm of the kernel functions are uniformly bounded.

Proof of Theorem 6.1 for $q=1$ and $d=2$. It is enough to integrate the kernel function over the set $\{(x,y):0<y<x<\pi\}$. Let us decompose this set into the union $\cup_{i=1}^{10}A_i$, where

$$\begin{aligned}A_1 &:= \{(x,y):0<x\le 2/n, 0<y<x<\pi, y\le \pi/2\},\\ A_2 &:= \{(x,y):2/n<x<\pi, 0<y\le 1/n, y\le\pi/2\},\\ A_3 &:= \{(x,y):2/n<x<\pi, 1/n<y\le x/2, y\le\pi/2\},\\ A_4 &:= \{(x,y):2/n<x<\pi, x/2<y\le x-1/n, y\le\pi/2\},\\ A_5 &:= \{(x,y):2/n<x<\pi, x-1/n<y<x, y\le\pi/2\}\\ A_6 &:= \{(x,y):y>\pi/2, \pi-2/n\le y<\pi, 0<y<x<\pi\},\\ A_7 &:= \{(x,y):\pi/2<y<\pi-2/n, \pi-1/n<x<\pi\},\\ A_8 &:= \{(x,y):\pi/2<y<\pi-2/n, (\pi+y)/2<x\le\pi-1/n\},\\ A_9 &:= \{(x,y):\pi/2<y<\pi-2/n, y+1/n<x\le(\pi+y)/2\},\\ A_{10} &:= \{(x,y):\pi/2<y<\pi-2/n, y<x\le y+1/n\}.\end{aligned}$$

The sets A_i can be seen on Figure 19.

Inequality (5.2) implies

$$\int_{A_1\cup A_6}|K_n^1(x,y)|\,\mathrm{d}x\,\mathrm{d}y \le C.$$

By (6.5),

$$\begin{aligned}\int_{A_2\cup A_7}|K_n^1(x,y)|\,\mathrm{d}x\,\mathrm{d}y &\le C\int_{2/n}^{\pi}\int_0^{1/n}(x-1/n)^{-3/2}y^{-1/2}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad +\int_{\pi-1/n}^{\pi}\int_{\pi/2}^{\pi-2/n}(\pi-1/n-y)^{-3/2}(\pi-x)^{-1/2}\,\mathrm{d}x\,\mathrm{d}y\\ &\le C.\end{aligned}$$

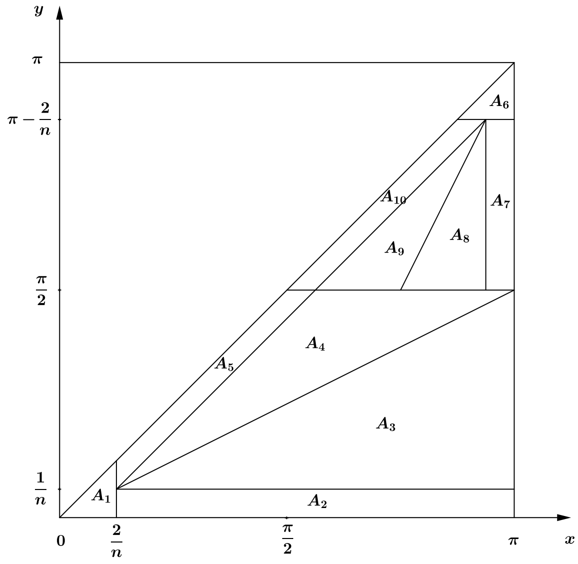

Figure 19: The sets A_i.

Since $x-y>x/2$ on the set A_3 and $x-y>(\pi-y)/2$ on the set A_8, we get from (6.6) that

$$\begin{aligned}\int_{A_3\cup A_8}|K_n^1(x,y)|\,\mathrm{d}x\,\mathrm{d}y &\leq Cn^{-1}\int_{2/n}^{\pi}\int_{1/n}^{x/2}x^{-1-\beta}y^{\beta-2}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad + Cn^{-1}\int_{\pi/2}^{\pi-2/n}\int_{(\pi+y)/2}^{\pi-1/n}(\pi-y)^{-1-\beta}(\pi-x)^{\beta-2}\,\mathrm{d}x\,\mathrm{d}y\\ &\leq C.\end{aligned}$$

Observe that $y>x/2$ on A_4 and $\pi-x>(\pi-y)/2$ on the set A_9, hence (6.6) implies

$$\begin{aligned}\int_{A_4\cup A_9}|K_n^1(x,y)|\,\mathrm{d}x\,\mathrm{d}y &\leq Cn^{-1}\int_{2/n}^{\pi}\int_{x/2}^{x-1/n}(x-y)^{-1-\beta}x^{\beta-2}\,\mathrm{d}y\,\mathrm{d}x\\ &\quad + Cn^{-1}\int_{\pi/2}^{\pi-2/n}\int_{y+1/n}^{(\pi+y)/2}(x-y)^{-1-\beta}(\pi-y)^{\beta-2}\,\mathrm{d}x\,\mathrm{d}y\\ &\leq C.\end{aligned}$$

Finally, by (6.7),

$$\int_{A_5\cup A_{10}}|K_n^1(x,y)|\,\mathrm{d}x\,\mathrm{d}y$$

$$\leq C\int_{1/n}^{\pi}\int_{y}^{y+1/n} y^{-2}\,\mathrm{d}x\,\mathrm{d}y + C\int_{\pi/2}^{\pi-1/n}\int_{x-1/n}^{x}(\pi-x)^{-2}\,\mathrm{d}y\,\mathrm{d}x$$
$$\leq C,$$

which completes the proof of the theorem. ■

6.2.2 Proof for $q = 1$ in higher dimensions $(d \geq 3)$

We also need another representation of the kernel function D_n^1. If we apply the inductive definition of the divided difference in (5.3) to D_n^1, then in the denominator, we have to choose the factors from the following table:

$$\begin{array}{ccccc}
\cos x_1 - \cos x_d & & & & \\
\cos x_1 - \cos x_{d-1} & \cos x_2 - \cos x_d & & & \\
\dots & & & & \\
\cos x_1 - \cos x_{d-k+1} & \cos x_2 - \cos x_{d-k+2} & \dots & \cos x_k - \cos x_d & \\
\dots & & & & \\
\cos x_1 - \cos x_2 & \cos x_2 - \cos x_3 & \dots & & \cos x_{d-1} - \cos x_d.
\end{array}$$

Observe that the kth row contains k terms and the differences of the indices in the kth row is equal to $d-k$, more precisely, if $\cos x_{i_k} - \cos x_{j_k}$ is in the kth row, then $j_k - i_k = d-k$. We choose exactly one factor from each row. First, we choose $\cos x_1 - \cos x_d$ and then from the second row $\cos x_1 - \cos x_{d-1}$ or $\cos x_2 - \cos x_d$. If we have chosen the $(k-1)$th factor from the $(k-1)$th row, say $\cos x_j - \cos x_{j+d-k+1}$, then we have to choose the next one from the kth row as either the one below the $(k-1)$th factor (it is equal to $\cos x_j - \cos x_{j+d-k}$) or its right neighbor (it is equal to $\cos x_{j+1} - \cos x_{j+d-k+1}$).

Definition 6.10 *If the sequence of integer pairs $((i_n, j_n) : n = 1, \dots, d-1)$ has the following properties, then we say that it is in $\mathcal{I}$. Let $i_1 = 1$, $j_1 = d$, (i_n) is non-decreasing and (j_n) is non-increasing. If (i_n, j_n) is given, then let $i_{n+1} = i_n$ and $j_{n+1} = j_n - 1$ or $i_{n+1} = i_n + 1$ and $j_{n+1} = j_n$.*

Observe that the difference $\cos x_{i_k} - \cos x_{j_k}$ is in the kth row of the table $(k = 1, \dots, d-1)$. So the factors we have just chosen can be written as $\prod_{l=1}^{d-1}(\cos x_{i_l} - \cos x_{j_l})$. In other words,

$$\begin{aligned}
&D_n^1(x)\\
&= \sum_{(i_l,j_l)\in\mathcal{I}} (-1)^{i_{d-1}-1}\prod_{l=1}^{d-2}(\cos x_{i_l} - \cos x_{j_l})^{-1}[\cos x_{i_{d-1}}, \cos x_{j_{d-1}}]G_n\\
&= \sum_{(i_l,j_l)\in\mathcal{I}} (-1)^{i_{d-1}-1}\\
&\quad \prod_{l=1}^{d-1}(\cos x_{i_l} - \cos x_{j_l})^{-1}(G_n(\cos x_{i_{d-1}}) - G_n(\cos x_{j_{d-1}})) \qquad (6.10)\\
&=: \sum_{(i_l,j_l)\in\mathcal{I}} D^1_{n,(i_l,j_l)}(x).
\end{aligned}$$

This proves

Lemma 6.11 *We have*

$$
\begin{aligned}
& K_n^1(x) \\
= & \sum_{(i_l,j_l)\in\mathcal{I}} \frac{(-1)^{i_{d-1}-1}}{n} \prod_{l=1}^{d-1} (\cos x_{i_l} - \cos x_{j_l})^{-1} \sum_{k=0}^{n-1} (G_k(\cos x_{i_{d-1}}) - G_k(\cos x_{j_{d-1}})) \\
=: & \sum_{(i_l,j_l)\in\mathcal{I}} K^1_{n,(i_l,j_l)}(x).
\end{aligned}
$$

We may suppose that $\pi > x_1 > x_2 > \cdots > x_d > 0$. We will need the following sharp estimations of the kernel functions.

Lemma 6.12 *For all* $0 < \beta < \frac{2}{d-1}$,

$$
\begin{aligned}
|K^1_{n,(i_l,j_l)}(x)| \leq\ & \frac{C}{n} \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1-\beta} x_{j_{d-1}}^{\beta(d-1)-2} 1_{\{x_{j_{d-1}} \leq \pi/2\}} \\
& + \frac{C}{n} \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1-\beta} (\pi - x_{i_{d-1}})^{\beta(d-1)-2} 1_{\{x_{j_{d-1}} > \pi/2\}}. \qquad (6.11)
\end{aligned}
$$

Proof. Using the formulas in (6.8), we conclude

$$
|K^1_{n,(i_l,j_l)}(x)| \leq \prod_{l=1}^{d-1} \frac{(\sin x_{i_{d-1}})^{d-2} (\sin(x_{i_{d-1}}/2))^{-1} + (\sin x_{j_{d-1}})^{d-2} (\sin(x_{j_{d-1}}/2))^{-1}}{n \sin((x_{i_l} - x_{j_l})/2) \sin((x_{i_l} + x_{j_l})/2)}.
$$

If $x_{j_{d-1}} \leq \pi/2$, then $(x_{i_l} + x_{j_l})/2 \leq 3\pi/4$ and so

$$
|K^1_{n,(i_l,j_l)}(x)| \leq \frac{C}{n} \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1} (x_{i_l} + x_{j_l})^{-1} (x_{i_{d-1}}^{d-3} + x_{j_{d-1}}^{d-3}).
$$

Since $x_{i_l} + x_{j_l} > x_{i_l} - x_{j_l}$ and $x_{i_l} + x_{j_l} > x_{i_{d-1}} > x_{j_{d-1}}$, we can see that

$$
\begin{aligned}
|K^1_{n,(i_l,j_l)}(x)| &\leq \frac{C}{n} \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1-\beta} (x_{i_{d-1}}^{d-3+(\beta-1)(d-1)} + x_{j_{d-1}}^{d-3+(\beta-1)(d-1)}) \\
&\leq \frac{C}{n} \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1-\beta} x_{j_{d-1}}^{\beta(d-1)-2}
\end{aligned}
$$

for all $0 < \beta < \frac{2}{d-1}$.

If $x_{j_{d-1}} > \pi/2$, then $(x_{i_l} + x_{j_l})/2 > \pi/4$ and

$$|K^1_{n,(i_l,j_l)}(x)| \leq \frac{C}{n}\prod_{l=1}^{d-1}(x_{i_l} - x_{j_l})^{-1}(2\pi - x_{i_l} - x_{j_l})^{-1}((\pi - x_{i_{d-1}})^{d-3} + (\pi - x_{j_{d-1}})^{d-3}).$$

Observe that $2\pi - x_{i_l} - x_{j_l} > x_{i_l} - x_{j_l}$ and $2\pi - x_{i_l} - x_{j_l} > \pi - x_{j_l} > \pi - x_{j_{d-1}} > \pi - x_{i_{d-1}}$. Thus

$$\begin{aligned} |K^1_{n,(i_l,j_l)}(x)| &\leq \frac{C}{n}\prod_{l=1}^{d-1}(x_{i_l} - x_{j_l})^{-1-\beta} \\ &\quad ((\pi - x_{i_{d-1}})^{d-3+(\beta-1)(d-1)} + (\pi - x_{j_{d-1}})^{d-3+(\beta-1)(d-1)}) \\ &\leq \frac{C}{n}\prod_{l=1}^{d-1}(x_{i_l} - x_{j_l})^{-1-\beta}(\pi - x_{i_{d-1}})^{\beta(d-1)-2} \end{aligned}$$

if $0 < \beta < \frac{2}{d-1}$. ■

Lemma 6.13 *For all* $0 < \beta < \frac{2}{d-2}$,

$$\begin{aligned} |K^1_{n,(i_l,j_l)}(x)| &\leq C\prod_{l=1}^{d-2}(x_{i_l} - x_{j_l})^{-1-\beta}x_{j_{d-1}}^{\beta(d-2)-2}1_{\{x_{j_{d-1}}\leq\pi/2\}} \\ &\quad + C\prod_{l=1}^{d-2}(x_{i_l} - x_{j_l})^{-1-\beta}(\pi - x_{i_{d-1}})^{\beta(d-2)-2}1_{\{x_{j_{d-1}}>\pi/2\}}. \end{aligned} \tag{6.12}$$

Proof. Lagrange's mean value theorem and (6.10) imply that there exists $x_{i_{d-1}} > \xi > x_{j_{d-1}}$, such that

$$D^1_{k,(i_l,j_l)}(x) = (-1)^{i_{d-1}-1}\prod_{l=1}^{d-1}(\cos x_{i_l} - \cos x_{j_l})^{-1}H_k'(\xi)(x_{i_{d-1}} - x_{j_{d-1}}),$$

where

$$H_k(t) := (-1)^{[(d-1)/2]}2\cos(t/2)(\sin t)^{d-2}\mathrm{soc}\,(k+1/2)t.$$

Then

$$\begin{aligned} |K^1_{n,(i_l,j_l)}(x)| &\leq C\prod_{l=1}^{d-1}\frac{(\sin\xi)^{d-2} + n(\sin\xi)^{d-2}}{n\sin((x_{i_l} - x_{j_l})/2)\sin((x_{i_l} + x_{j_l})/2)\sin(\xi/2)}(x_{i_{d-1}} - x_{j_{d-1}}) \\ &\quad + C\prod_{l=1}^{d-1}\frac{(\sin\xi)^{d-3}}{\sin((x_{i_l} - x_{j_l})/2)\sin((x_{i_l} + x_{j_l})/2)}(x_{i_{d-1}} - x_{j_{d-1}}). \end{aligned}$$

Beside (6.8), we have used that $|\sum_{k=0}^{n-1} \mathrm{soc}\,(k+1/2)t| \leq n$. In the case $x_{j_{d-1}} \leq \pi/2$,

$$\begin{aligned} |K^1_{n,(i_l,j_l)}(x)| &\leq C\prod_{l=1}^{d-1}(x_{i_l}-x_{j_l})^{-1}(x_{i_l}+x_{j_l})^{-1}(x_{i_{d-1}}-x_{j_{d-1}})\xi^{d-3} \\ &\leq C\prod_{l=1}^{d-2}(x_{i_l}-x_{j_l})^{-1-\beta}\xi^{d-4+(\beta-1)(d-2)} \\ &\leq C\prod_{l=1}^{d-2}(x_{i_l}-x_{j_l})^{-1-\beta}x_{j_{d-1}}^{\beta(d-2)-2} \end{aligned}$$

for all $0<\beta<\frac{2}{d-2}$.

Similarly, if $x_{j_{d-1}} > \pi/2$, then $(x_{i_l}+x_{j_l})/2 > \pi/4$ and

$$\begin{aligned} |K^1_{n,(i_l,j_l)}(x)| &\leq C\prod_{l=1}^{d-1}(x_{i_l}-x_{j_l})^{-1}(2\pi-x_{i_l}-x_{j_l})^{-1}(x_{i_{d-1}}-x_{j_{d-1}})(\pi-\xi)^{d-3} \\ &\leq C\prod_{l=1}^{d-2}(x_{i_l}-x_{j_l})^{-1-\beta}(\pi-\xi)^{d-4+(\beta-1)(d-2)} \\ &\leq C\prod_{l=1}^{d-2}(x_{i_l}-x_{j_l})^{-1-\beta}(\pi-x_{i_{d-1}})^{\beta(d-2)-2} \end{aligned}$$

if $0<\beta<\frac{2}{d-2}$. ■

In the next lemma, we estimate the partial derivatives of the kernel function.

Lemma 6.14 *If $0<\beta<\frac{2}{d-1}$, then for all $q=1,\ldots,d$,*

$$\begin{aligned} |\partial_q K^1_{n,(i_l,j_l)}(x)| &\leq C\prod_{l=1}^{d-1}(x_{i_l}-x_{j_l})^{-1-\beta}x_{j_{d-1}}^{\beta(d-1)-2}1_{\{x_{j_{d-1}}\leq\pi/2\}} \\ &\quad + C\prod_{l=1}^{d-1}(x_{i_l}-x_{j_l})^{-1-\beta}(\pi-x_{i_{d-1}})^{\beta(d-1)-2}1_{\{x_{j_{d-1}}>\pi/2\}}. \end{aligned} \tag{6.13}$$

Proof. Let $m_i=0,1$ and $\delta_{m_1,\ldots,m_d}=0,\pm1$ be suitable numbers. (6.10) implies that the partial derivative of $D^1_{k,(i_l,j_l)}$,

$$\begin{aligned} \partial_q D^1_{k,(i_l,j_l)}(x) &= (-1)^{i_{d-1}-1}\sum_{m_1+\cdots+m_d=1}\delta_{m_1,\ldots,m_d} \\ &\quad \prod_{l=1}^{d-1}\partial_q^{m_l}((\cos x_{i_l}-\cos x_{j_l})^{-1})\partial_q^{m_d}(G_k(\cos x_{i_{d-1}})-G_k(\cos x_{j_{d-1}})). \end{aligned}$$

If we differentiate in the first $(d-1)$ factors, then

$$\sum_{m_1+\cdots+m_{d-1}=1}\prod_{l=1}^{d-1}(\cos x_{i_l}-\cos x_{j_l})^{-1-m_l}(-\sin y_l)^{m_l}(G_k(\cos x_{i_{d-1}})-G_k(\cos x_{j_{d-1}}))$$
$$=\sum_{m_1+\cdots+m_{d-1}=1}\prod_{l=1}^{d-1}(\cos x_{i_l}-\cos x_{j_l})^{-1-m_l}(-\sin y_l)^{m_l}H_k'(\xi)(x_{i_{d-1}}-x_{j_{d-1}}),$$

where $y_l=x_{i_l}$ or $y_l=-x_{j_l}$. If $x_{j_{d-1}}\le \pi/2$, then $|\sin y_l|\le |y_l|\le x_{i_l}+x_{j_l}$ and, as in the proof of Lemma 6.13,

$$\Big|\frac{1}{n}\sum_{k=0}^{n-1}\sum_{m_1+\cdots+m_{d-1}=1}\prod_{l=1}^{d-1}(\cos x_{i_l}-\cos x_{j_l})^{-1-m_l}(\sin y_l)^{m_l}(G_k(\cos x_{i_{d-1}})-G_k(\cos x_{j_{d-1}}))\Big|$$
$$\begin{aligned}
&\le C\sum_{m_1+\cdots+m_{d-1}=1}\prod_{l=1}^{d-1}(x_{i_l}-x_{j_l})^{-1-m_l}(x_{i_l}+x_{j_l})^{-1}\Big|\frac{1}{n}\sum_{k=0}^{n-1}H_k'(\xi)(x_{i_{d-1}}-x_{j_{d-1}})\Big|\\
&\le C\sum_{m_1+\cdots+m_{d-1}=1}\prod_{l=1}^{d-1}(x_{i_l}-x_{j_l})^{-1}(x_{i_l}+x_{j_l})^{-1}\xi^{d-3}\\
&\le C\prod_{l=1}^{d-1}(x_{i_l}-x_{j_l})^{-1-\beta}\xi^{(\beta-1)(d-1)+d-3}\\
&\le C\prod_{l=1}^{d-1}(x_{i_l}-x_{j_l})^{-1-\beta}x_{j_{d-1}}^{\beta(d-1)-2},
\end{aligned}$$

whenever $0<\beta<\frac{2}{d-1}$.

If $m_d=1$ and, say $i_{d-1}=q$, then

$$\prod_{l=1}^{d-1}(\cos x_{i_l}-\cos x_{j_l})^{-1}\partial_q(G_k(\cos x_{i_{d-1}})-G_k(\cos x_{j_{d-1}}))$$
$$=\prod_{l=1}^{d-1}(\cos x_{i_l}-\cos x_{j_l})^{-1}H_k'(\cos x_{i_{d-1}})$$

and

$$\Big|\frac{1}{n}\sum_{k=0}^{n-1}\prod_{l=1}^{d-1}(\cos x_{i_l}-\cos x_{j_l})^{-1}\partial_q(G_k(\cos x_{i_{d-1}})-G_k(\cos x_{j_{d-1}}))\Big|$$
$$\begin{aligned}
&\le C\prod_{l=1}^{d-1}(x_{i_l}-x_{j_l})^{-1}(x_{i_l}+x_{j_l})^{-1}\Big|\frac{1}{n}\sum_{k=0}^{n-1}H_k'(\cos x_{i_{d-1}})\Big|\\
&\le C\prod_{l=1}^{d-1}(x_{i_l}-x_{j_l})^{-1}(x_{i_l}+x_{j_l})^{-1}\xi^{d-3}
\end{aligned}$$

$$\leq \quad C \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1-\beta} x_{j_{d-1}}^{\beta(d-1)-2}.$$

Consequently,

$$|\partial_q K^1_{n,(i_l,j_l)}(x)| = \Big| \frac{1}{n} \sum_{k=0}^{n-1} \partial_q D^1_{k,(i_l,j_l)}(x) \Big| \leq C \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1-\beta} x_{j_{d-1}}^{\beta(d-1)-2}$$

if $x_{j_{d-1}} \leq \pi/2$ and $0 < \beta < \frac{2}{d-1}$. The case $x_{j_{d-1}} > \pi/2$ can be proved similarly. ∎

Now we show that the L_1-norm of the kernel functions are uniformly bounded.

Proof of Theorem 6.1 for $q = 1$ and $d \geq 3$. We may suppose again that $\pi > x_1 > x_2 > \cdots > x_d > 0$. If $x_1 \leq 16/n$ or $\pi - x_d \leq 16/n$, then (5.2) implies

$$\int_{\{16/n \geq x_1 > x_2 > \cdots > x_d > 0\}} |K^1_n(x)| \,\mathrm{d}x + \int_{\{\pi > x_1 > x_2 > \cdots > x_d \geq \pi - 16/n\}} |K^1_n(x)| \,\mathrm{d}x \leq C.$$

Hence it is enough to integrate over

$$\mathcal{S} := \{x \in \mathbb{T}^d : \pi > x_1 > x_2 > \cdots > x_d > 0, x_1 > 16/n, x_d < \pi - 16/n\}.$$

For a sequence $(i_l, j_l) \in \mathcal{I}$, let us define the set $\mathcal{S}_{(i_l,j_l),k}$ by

$$\mathcal{S}_{(i_l,j_l),k} := \begin{cases} x \in \mathcal{S} : x_{i_l} - x_{j_l} > 4/n, l = 1, \ldots, k-1, x_{i_k} - x_{j_k} \leq 4/n, & \text{if } k < d; \\ x \in \mathcal{S} : x_{i_l} - x_{j_l} > 4/n, l = 1, \ldots, d-1, & \text{if } k = d \end{cases}$$

and

$$\mathcal{S}_{(i_l,j_l),k,1} := \begin{cases} x \in \mathcal{S}_{(i_l,j_l),k} : x_{j_k} > 4/n, x_{j_{d-1}} \leq \pi/2, & \text{if } k < d; \\ x \in \mathcal{S}_{(i_l,j_l),k} : x_{j_{d-1}} > 4/n, x_{j_{d-1}} \leq \pi/2, & \text{if } k = d, \end{cases}$$

$$\mathcal{S}_{(i_l,j_l),k,2} := \begin{cases} x \in \mathcal{S}_{(i_l,j_l),k} : x_{j_k} \leq 4/n, x_{j_{d-1}} \leq \pi/2, & \text{if } k < d; \\ x \in \mathcal{S}_{(i_l,j_l),k} : x_{j_{d-1}} \leq 4/n, x_{j_{d-1}} \leq \pi/2, & \text{if } k = d, \end{cases}$$

$$\mathcal{S}_{(i_l,j_l),k,3} := \begin{cases} x \in \mathcal{S}_{(i_l,j_l),k} : \pi - x_{i_k} > 4/n, x_{j_{d-1}} > \pi/2, & \text{if } k < d; \\ x \in \mathcal{S}_{(i_l,j_l),k} : \pi - x_{i_{d-1}} > 4/n, x_{j_{d-1}} > \pi/2, & \text{if } k = d, \end{cases}$$

$$\mathcal{S}_{(i_l,j_l),k,4} := \begin{cases} x \in \mathcal{S}_{(i_l,j_l),k} : \pi - x_{i_k} \leq 4/n, x_{j_{d-1}} > \pi/2, & \text{if } k < d; \\ x \in \mathcal{S}_{(i_l,j_l),k} : \pi - x_{i_{d-1}} \leq 4/n, x_{j_{d-1}} > \pi/2, & \text{if } k = d. \end{cases}$$

Then

$$\int_{\mathbb{T}^d} |K^1_n(x)| 1_{\mathcal{S}}(x) \,\mathrm{d}x \leq \sum_{k=1}^{d} \sum_{m=1}^{4} \int_{\mathbb{T}^d} |K^1_{n,(i_l,j_l)}(x)| 1_{\mathcal{S}_{(i_l,j_l),k,m}(x)} \,\mathrm{d}x.$$

We estimate the right hand side in four steps.

Step 1. First, we consider the set $\mathcal{S}_{(i_l,j_l),k,1}$ and let $1 \leq k \leq d-1$. Since $x_{i_{d-1}} - x_{j_{d-1}} \leq x_{i_l} - x_{j_l}$, (6.12) implies

$$\begin{aligned}\int_{\mathbb{T}^d} K^1_{n,(i_l,j_l)}(x) 1_{\mathcal{S}_{(i_l,j_l),k,1}}(x)\,\mathrm{d}x &\leq C \int_{\mathbb{T}^d} \prod_{l=1}^{d-2} (x_{i_l} - x_{j_l})^{-1-\beta} x_{j_{d-1}}^{\beta(d-2)-2} 1_{\mathcal{S}_{(i_l,j_l),k,1}}(x)\,\mathrm{d}x \\ &\leq C \int_{\mathbb{T}^d} \prod_{l=1}^{k-1} (x_{i_l} - x_{j_l})^{-1-\beta} \prod_{l=k}^{d-2} (x_{i_l} - x_{j_l})^{-1-\beta+1/(d-k)} \\ &\quad (x_{i_{d-1}} - x_{j_{d-1}})^{1/(d-k)-1} x_{j_{d-1}}^{\beta(d-2)-2} 1_{\mathcal{S}_{(i_l,j_l),k,1}}(x)\,\mathrm{d}x.\end{aligned}$$

First, we choose the indices j_{d-1} $(= i'_d)$, i_{d-1} $(= i'_{d-1})$ and then i_{d-2} if $i_{d-2} \neq i_{d-1}$ or j_{d-2} if $j_{d-2} \neq j_{d-1}$. (Exactly one case of these two cases is satisfied.) If we repeat this process, then we get an injective sequence $(i'_l, l = 1, \ldots, d)$. We integrate the term $x_{i_1} - x_{j_1}$ in $x_{i'_1}$, the term $x_{i_2} - x_{j_2}$ in $x_{i'_2}$, $\ldots$, and finally the term $x_{i_{d-1}} - x_{j_{d-1}}$ in $x_{i'_{d-1}}$ and $x_{j_{d-1}}$ in $x_{i'_d}$. Since $x_{i_l} - x_{j_l} > 4/n$ $(l = 1, \ldots, k-1)$, $x_{i_l} - x_{j_l} \leq 4/n$ $(l = k, \ldots, d-1)$, $x_{j_{d-1}} \geq x_{j_k} > 4/n$ and we can choose β such that $\beta < 1/(d-1)$, we have

$$\begin{aligned}&\int_{\mathbb{T}^d} K^1_{n,(i_l,j_l)}(x) 1_{\mathcal{S}_{(i_l,j_l),k,1}}(x)\,\mathrm{d}x \\ &\leq C \prod_{l=1}^{k-1} (1/n)^{-\beta} \prod_{l=k}^{d-2} (1/n)^{-\beta+1/(d-k)} (1/n)^{1/(d-k)} (1/n)^{\beta(d-2)-1} \\ &\leq C.\end{aligned}$$

Step 2. For $k = d$, we use (6.11) to obtain

$$\begin{aligned}&\int_{\mathbb{T}^d} |K^1_{n,(i_l,j_l)}(x)| 1_{\mathcal{S}_{(i_l,j_l),d,1}}(x)\,\mathrm{d}x \\ &\leq Cn^{-1} \int_{\mathbb{T}^d} \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1-\beta} x_{j_{d-1}}^{\beta(d-1)-2} 1_{\mathcal{S}_{(i_l,j_l),d,1}}(x)\,\mathrm{d}x \\ &\leq Cn^{-1} \prod_{l=1}^{d-1} (1/n)^{-\beta} (1/n)^{\beta(d-1)-1} \\ &\leq C\end{aligned}$$

if $\beta < 1/(d-1)$.

Step 3. Now let us consider the set $\mathcal{S}_{(i_l,j_l),k,2}$ for $k = 1, \ldots, d-1$. Then $x_{i_k} - x_{j_k} \leq 4/n$ and so $x_{j_{d-1}} \leq x_{i_k} \leq 8/n$ and this holds also for $k = d$. Observe that $k \neq 1$, because $i_1 = 1$ and $x_1 > 16/n$ in $\mathcal{S}$. It follows from (6.12) that

$$\begin{aligned}&\int_{\mathbb{T}^d} K^1_{n,(i_l,j_l)}(x) 1_{\mathcal{S}_{(i_l,j_l),k,2}}(x)\,\mathrm{d}x \\ &\leq C \int_{\mathbb{T}^d} \prod_{l=1}^{d-2} (x_{i_l} - x_{j_l})^{-1-\beta} x_{j_{d-1}}^{\beta(d-2)-2} 1_{\mathcal{S}_{(i_l,j_l),k,2}}(x)\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\le C \int_{\mathbb{T}^d} \prod_{l=1}^{k-1} (x_{i_l} - x_{j_l})^{-1-\beta} \prod_{l=k}^{d-2} (x_{i_l} - x_{j_l})^{-1-\beta+(1-\epsilon)/(d-k-1)} \\
&\quad (x_{i_{d-1}} - x_{j_{d-1}})^{\epsilon-1} x_{j_{d-1}}^{\beta(d-2)-2} 1_{\mathcal{S}_{(i_l,j_l),k,2}}(x)\, \mathrm{d}x \\
&\le C \prod_{l=1}^{k-1} (1/n)^{-\beta} \prod_{l=k}^{d-2} (1/n)^{-\beta+(\alpha-\epsilon)/(d-k-1)} (1/n)^{\epsilon} (1/n)^{\beta(d-2)-\alpha} \\
&\le C,
\end{aligned}
$$

whenever $0 < \epsilon < 1$ and $1/(d-2) < \beta < (1-\epsilon)/(d-3)$, which implies $\epsilon < 1/(d-2)$.

Step 4. For the set $\mathcal{S}_{(i_l,j_l),d,2}$, we obtain similarly to Step 2 that

$$
\begin{aligned}
&\int_{\mathbb{T}^d} |K^1_{n,(i_l,j_l)}(x)| 1_{\mathcal{S}_{(i_l,j_l),d,2}}(x)\, \mathrm{d}x \\
&\quad \le Cn^{-1} \int_{\mathbb{T}^d} \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1-\beta} x_{j_{d-1}}^{\beta(d-1)-2} 1_{\mathcal{S}_{(i_l,j_l),d,2}}(x)\, \mathrm{d}x \\
&\quad \le Cn^{-1} \prod_{l=1}^{d-1} (1/n)^{-\beta} (1/n)^{\beta(d-1)-1} \\
&\quad \le C
\end{aligned}
$$

if $\beta > 1/(d-1)$. We can prove the corresponding inequalities for the sets $\mathcal{S}_{(i_l,j_l),k,3}$ and $\mathcal{S}_{(i_l,j_l),k,4}$ in the same way. ∎

6.2.3 Proof for $q = \infty$

Recall that

$$
D_n^{\infty}(x) = \prod_{i=1}^{d} \frac{\sin((n+1/2)x_i)}{\sin(x_i/2)}.
$$

Let $\epsilon := (\epsilon_1, \ldots, \epsilon_d)$ with $\epsilon_1 := 1$ and $\epsilon_j := \pm 1$, $j = 2, \ldots, d$ and $\epsilon' := (\epsilon_1, \ldots, \epsilon_{d-1})$. The sums $\sum_{\epsilon}$ and $\sum_{\epsilon'}$ mean $\sum_{\epsilon_j=\pm 1, j=2,\ldots,d}$ and $\sum_{\epsilon_j=\pm 1, j=2,\ldots,d-1}$, respectively. We may suppose that $x_1 > x_2 > \cdots > x_d > 0$. Applying the identities

$$
\sin a \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b)), \qquad \cos a \sin b = \frac{1}{2}(\sin(a+b) - \sin(a-b)),
$$

we obtain

$$
\begin{aligned}
\prod_{i=1}^{d} \sin((k+1/2)x_i) &= 2^{-d+1} \sum_{\epsilon'} \pm \Big(\mathrm{soc}\, \Big((k+1/2) \Big(\sum_{j=1}^{d-1} \epsilon_j x_j + x_d \Big) \Big) \\
&\quad - \mathrm{soc}\, \Big((k+1/2) \Big(\sum_{j=1}^{d-1} \epsilon_j x_j - x_d \Big) \Big) \Big).
\end{aligned}
$$

Thus

$$\begin{aligned}
K_n^\infty(x) &= \frac{1}{n}\sum_{k=0}^{n-1}\prod_{i=1}^{d}\frac{\sin((k+1/2)x_i)}{\sin(x_i/2)} \\
&= \sum_{\epsilon'} \pm 2^{-d+1}\prod_{i=1}^{d}(\sin(x_i/2))^{-1}\frac{1}{n}\sum_{k=0}^{n-1} \\
&\quad \Big(\mathrm{soc}\,((k+1/2)(\sum_{j=1}^{d-1}\epsilon_j x_j + x_d)) - \mathrm{soc}\,((k+1/2)(\sum_{j=1}^{d-1}\epsilon_j x_j - x_d))\Big) \\
&=: \sum_{\epsilon'} K_{n,\epsilon'}^\infty(x). \qquad (6.14)
\end{aligned}$$

It is easy to see that

$$|K_n^\infty(x)| \le Cn^d \qquad \text{and} \qquad |K_{n,\epsilon'}^\infty(x)| \le C\prod_{i=1}^{d} x_i^{-1}. \qquad (6.15)$$

For a fixed $\epsilon = (\epsilon_1,\ldots,\epsilon_d)$, we consider those $x \in \mathbb{T}^d$ for which $|\sum_{j=1}^d \epsilon_j x_j| \le \pi$ and $|\sum_{j=1}^{d-1}\epsilon_j x_j| \le \pi$. If this were not the case, say if $\sum_{j=1}^d \epsilon_j x_j$ were in the interval $(2k\pi,(2k+1)\pi)$ or in $((2k+1)\pi,(2k+2)\pi)$ for a fixed $k \in \mathbb{N}$, then we should write in the definitions and theorems below $\sum_{j=1}^d \epsilon_j x_j - 2k\pi$ or $(2k+2)\pi - \sum_{j=1}^d \epsilon_j x_j$ instead of $\sum_{j=1}^d \epsilon_j x_j$. The same holds for $\sum_{j=1}^{d-1}\epsilon_j x_j$. So, for simplicity, we will always suppose that $|\sum_{j=1}^d \epsilon_j x_j| \le \pi$ and $|\sum_{j=1}^{d-1}\epsilon_j x_j| \le \pi$.

Lemma 6.15 *We have*

$$|K_{n,\epsilon'}^\infty(x)| \le C\sum_{\epsilon_d} n^{-1}\Big(\prod_{i=1}^d x_i^{-1}\Big)\Big|\sum_{j=1}^d \epsilon_j x_j\Big|^{-1}.$$

Proof. Use (6.14) and the trigonometric identities (6.8). ■

Let us introduce the sets

$$\mathcal{S} := \{x \in \mathbb{T}^d : \pi > x_1 > x_2 > \cdots > x_d > 0, x_1 > 32/n\},$$

$$\mathcal{S}_{\epsilon'} := \{x \in \mathbb{T}^d : \Big|\sum_{j=1}^{d-1}\epsilon_j x_j\Big| < d\cdot 16/n\},$$

$$\mathcal{S}' := \{x \in \mathbb{T}^d : \exists\ \epsilon, \Big|\sum_{j=1}^{d}\epsilon_j x_j\Big| < d\cdot 16/n\},$$

$$\mathcal{S}_{\epsilon,1} := \{x \in \mathbb{T}^d : |\sum_{j=1}^{d} \epsilon_j x_j| < 4x_1\},$$

$$\mathcal{S}_{\epsilon',d} := \{x \in \mathbb{T}^d : \Big|\sum_{j=1}^{d-1} \epsilon_j x_j\Big| < 4x_d\},$$

$$\mathcal{S}_k := \{x \in \mathcal{S} : x_1 > x_2 > \cdots > x_k \geq 4/n > x_{k+1} > \cdots > x_d > 0\},$$

$k = 1, \ldots, d$. Recall that $\epsilon_1 = 1$ and $\epsilon_j = \pm 1$, $j = 2, \ldots, d$.

Lemma 6.16 *For all $x \in \mathcal{S}_k \setminus \mathcal{S}_{\epsilon'}$, $x \in \mathcal{S}_k \setminus \mathcal{S}'$ $(k = 1, \ldots, d-1)$ and $x \in \mathcal{S}^c_{\epsilon',d}$,*

$$|K^{\infty}_{n,\epsilon'}(x)| \leq C \sum_{\epsilon_d} \Big(\prod_{i=1}^{d-1} x_i^{-1}\Big) \Big|\sum_{j=1}^{d} \epsilon_j x_j\Big|^{-1}.$$

Proof. By Lagrange's mean value theorem,

$$\operatorname{soc}\Big((k+1/2)\Big(\sum_{j=1}^{d-1} \epsilon_j x_j + x_d\Big)\Big) - \operatorname{soc}\Big((k+1/2)\Big(\sum_{j=1}^{d-1} \epsilon_j x_j - x_d\Big)\Big)$$
$$= \operatorname{soc}'\Big((k+1/2)u\Big)(2k+1)x_d,$$

where $u \in (\sum_{j=1}^{d-1} \epsilon_j x_j - x_d, \sum_{j=1}^{d-1} \epsilon_j x_j + x_d)$. If $x \in \mathcal{S}^c_{\epsilon',d}$, then

$$|\sum_{j=1}^{d-1} \epsilon_j x_j + \epsilon_d x_d| \geq 3x_d.$$

In the case $\sum_{j=1}^{d-1} \epsilon_j x_j + x_d \geq 0$, we have

$$\sum_{j=1}^{d-1} \epsilon_j x_j - x_d \geq x_d \qquad \text{and so} \quad |u| > |\sum_{j=1}^{d-1} \epsilon_j x_j - x_d|.$$

If $\sum_{j=1}^{d-1} \epsilon_j x_j + x_d < 0$, then

$$\sum_{j=1}^{d-1} \epsilon_j x_j - x_d < 0 \quad \text{and} \quad |u| > |\sum_{j=1}^{d-1} \epsilon_j x_j + x_d|.$$

In both cases

$$|u|^{-1} \leq |\sum_{j=1}^{d-1} \epsilon_j x_j + x_d|^{-1} + |\sum_{j=1}^{d-1} \epsilon_j x_j - x_d|^{-1}.$$

The lemma can be proved in the same way if $x \in \mathcal{S}_k \setminus \mathcal{S}_{\epsilon'}$ or $x \in \mathcal{S}_k \setminus \mathcal{S}'$ $(k = 1, \ldots, d-1)$.
■

Lemma 6.17 *For all $l = 1, \ldots, d$ and $x \in \mathcal{S}$,*

$$\begin{aligned} |\partial_l K_{n,\epsilon'}^\infty(x)| &\leq C \sum_{\epsilon_d} \Big(\prod_{i=1}^d x_i^{-1} \Big) \Big| \sum_{j=1}^d \epsilon_j x_j \Big|^{-1} \\ &\quad + C \sum_{\epsilon_d} \Big(\prod_{i=1}^d x_i^{-1} \Big) x_d^{-1} 1_{\cup_{k=1}^{d-1}(\mathcal{S}_k \cap \mathcal{S}_{\epsilon'}) \cup (\mathcal{S}_d \cap \mathcal{S}_{\epsilon',d})}(x). \end{aligned}$$

Proof. Since

$$\frac{\partial}{\partial x_l} \frac{\operatorname{soc}((k+1/2)x_l)}{\sin(x_l/2)} = \frac{(k+1/2)\operatorname{soc}'((k+1/2)x_l)}{\sin(x_l/2)} - \frac{\operatorname{soc}((k+1/2)x_l)\cos(x_l/2)}{2\sin^2(x_i/2)},$$

we obtain by Lemmas 6.15 and 6.16 that

$$\begin{aligned} |\partial_l K_{n,\epsilon'}^\infty(x)| &\leq C \sum_{\epsilon_d} \Big(\prod_{i=1}^d x_i^{-1} \Big) \Big| \sum_{j=1}^d \epsilon_j x_j \Big|^{-1} \\ &\quad + C \sum_{\epsilon_d} \Big(\prod_{i=1}^{d-1} x_i^{-1} \Big) x_l^{-1} \Big| \sum_{j=1}^d \epsilon_j x_j \Big|^{-1} \big(\sum_{k=1}^{d-1} 1_{\mathcal{S}_k \setminus \mathcal{S}_{\epsilon'}}(x) + 1_{\mathcal{S}_d \setminus \mathcal{S}_{\epsilon',d}}(x) \big) \\ &\quad + C \sum_{\epsilon_d} \Big(\prod_{i=1}^d x_i^{-1} \Big) x_l^{-1} \big(\sum_{k=1}^{d-1} 1_{\mathcal{S}_k \cap \mathcal{S}_{\epsilon'}}(x) + 1_{\mathcal{S}_d \cap \mathcal{S}_{\epsilon',d}}(x) \big). \end{aligned}$$

Now $x_l > x_d$ finishes the proof. ∎

Proof of Theorem 6.1 for $q = \infty$. If $x_1 \leq 32/n$, then (6.15) implies

$$\int_{\{32/n \geq x_1 > x_2 > \cdots > x_d > 0\}} |K_n^\infty(x)| \, \mathrm{d}x \leq C.$$

It is enough to estimate the integrals

$$\begin{aligned} \int_{\mathcal{S}} |K_n^\infty(x)| \, \mathrm{d}x &\leq \sum_{k=1,d} \int_{\mathcal{S}_k \cap \mathcal{S}'} |K_n^\infty(x)| \, \mathrm{d}x + \sum_{\epsilon'} \sum_{k=2}^{d-1} \int_{\mathcal{S}_k \cap \mathcal{S}_{\epsilon'}} |K_{n,\epsilon'}^\infty(x)| \, \mathrm{d}x \\ &\quad + \sum_{k=1,d} \int_{\mathcal{S}_k \setminus \mathcal{S}'} |K_n^\infty(x)| \, \mathrm{d}x + \sum_{\epsilon'} \sum_{k=2}^{d-1} \int_{\mathcal{S}_k \setminus \mathcal{S}_{\epsilon'}} |K_{n,\epsilon'}^\infty(x)| \, \mathrm{d}x. \end{aligned} \tag{6.16}$$

Step 1. It is easy to see in the first sum that if $x \in \mathcal{S}'$, i.e., $|\sum_{j=1}^d \epsilon_j x_j| < d \cdot 16/n$, then x_1 must be in an interval of length $d \cdot 32/n$. Since $x_k \geq 4/n > x_{k+1}$ on $\mathcal{S}_k$, we have

$$\int_{\mathcal{S}_1 \cap \mathcal{S}'} |K_n^\infty(x)| \, \mathrm{d}x \leq C n^d \int_{\mathcal{S}_1 \cap \mathcal{S}'} \mathrm{d}x \leq C.$$

If $k=d$, then (6.15) implies

$$\int_{\mathcal{S}_d\cap\mathcal{S}'}|K_{n,\epsilon'}^\infty(x)|\,\mathrm{d}x\le C\int_{\mathcal{S}_d\cap\mathcal{S}'}\prod_{i=1}^d x_i^{-1}\,\mathrm{d}x\le C\int_{\mathcal{S}_d\cap\mathcal{S}'}\prod_{i=2}^d x_i^{-1-1/(d-1)}\,\mathrm{d}x\le Cn^{-1}n.$$

Step 2. Let us investigate the second sum in (6.16). First, we multiply by $1_{\mathcal{S}_{\epsilon',d}}(x)$ in the integrand. If $x\in\mathcal{S}_{\epsilon',d}$, then x_1 is in an interval of length $8x_d$. By (6.15),

$$\begin{aligned}\int_{\mathcal{S}_k\cap\mathcal{S}_{\epsilon'}\cap\mathcal{S}_{\epsilon',d}}|K_{n,\epsilon'}^\infty(x)|\,\mathrm{d}x &\le C\int_{\mathcal{S}_k\cap\mathcal{S}_{\epsilon'}\cap\mathcal{S}_{\epsilon',d}}\prod_{i=1}^d x_i^{-1}\,\mathrm{d}x\\ &\le C\int_{\mathcal{S}_k}\Big(\prod_{i=2}^k x_i^{-1-1/(k-1)}\Big)\Big(\prod_{i=k+1}^d x_i^{-1}\Big)x_d\,\mathrm{d}x_2\cdots\mathrm{d}x_d\\ &\le C\int_{\mathcal{S}_k}\Big(\prod_{i=2}^k x_i^{-1-1/(k-1)}\Big)\Big(\prod_{i=k+1}^d x_i^{-1+1/(d-k)}\Big)\,\mathrm{d}x_2\cdots\mathrm{d}x_d\\ &\le Cn/n.\end{aligned}$$

If $x\notin\mathcal{S}_{\epsilon',d}$, then $|\sum_{j=1}^d\epsilon_jx_j|\ge 3x_d$. Since $|\sum_{j=1}^d\epsilon_jx_j|<d\cdot 20/n$ on $\mathcal{S}_{\epsilon'}$, Lemma 6.16 implies

$$\begin{aligned}\int_{\mathcal{S}_k\cap\mathcal{S}_{\epsilon'}\setminus\mathcal{S}_{\epsilon',d}}|K_{n,\epsilon'}^\infty(x)|\,\mathrm{d}x &\le C\sum_{\epsilon_d}\int_{\mathcal{S}_k\cap\mathcal{S}_{\epsilon'}\setminus\mathcal{S}_{\epsilon',d}}\Big(\prod_{i=1}^{d-1}x_i^{-1}\Big)x_d^{-\delta}\Big|\sum_{j=1}^d\epsilon_jx_j\Big|^{-1+\delta}\,\mathrm{d}x\\ &\le C\sum_{\epsilon_d}\int_{\mathcal{S}_k\cap\mathcal{S}_{\epsilon'}}\Big(\prod_{i=2}^k x_i^{-1-1/(k-1)}\Big)\\ &\qquad\Big(\prod_{i=k+1}^d x_i^{-1+(1-\delta)/(d-k)}\Big)\Big|\sum_{j=1}^d\epsilon_jx_j\Big|^{-1+\delta}\,\mathrm{d}x\\ &\le Cn\Big(\frac{1}{n}\Big)^{1-\delta}\Big(\frac{1}{n}\Big)^{\delta},\end{aligned}$$

whenever $0<\delta<1$ and $k=2,\ldots,d-1$. This proves that the second sum in (6.16) is finite.

Step 3. To estimate the fourth sum, let us use Lemma 6.16:

$$\int_{\mathcal{S}_k\setminus\mathcal{S}_{\epsilon'}}|K_{n,\epsilon'}^\infty(x)|\,\mathrm{d}x\le C\sum_{\epsilon_d}\Big(\int_{(\mathcal{S}_k\setminus\mathcal{S}_{\epsilon'})\cap\mathcal{S}_{\epsilon,1}}+\int_{(\mathcal{S}_k\setminus\mathcal{S}_{\epsilon'})\setminus\mathcal{S}_{\epsilon,1}}\Big)\Big(\prod_{i=1}^{d-1}x_i^{-1}\Big)\Big|\sum_{j=1}^d\epsilon_jx_j\Big|^{-1}\,\mathrm{d}x,$$

$k=2,\ldots,d-1$. Since $|\sum_{j=1}^d\epsilon_jx_j|\ge d\cdot 12/n$ on $\mathcal{S}_{\epsilon'}^c$, we have

$$\sum_{\epsilon_d}\int_{(\mathcal{S}_k\setminus\mathcal{S}_{\epsilon'})\cap\mathcal{S}_{\epsilon,1}}\Big(\prod_{i=1}^{d-1}x_i^{-1}\Big)\Big|\sum_{j=1}^d\epsilon_jx_j\Big|^{-1}\,\mathrm{d}x$$

$$
\begin{aligned}
&\le C\sum_{\epsilon_d}\int_{(\mathcal{S}_k\setminus\mathcal{S}_{\epsilon'})\cap\mathcal{S}_{\epsilon,1}} x_1^{\delta}\Big(\prod_{i=1}^{d} x_i^{-1}\Big)x_d\Big|\sum_{j=1}^{d}\epsilon_j x_j\Big|^{-1-\delta}\,\mathrm{d}x \\
&\le C\sum_{\epsilon_d}\int_{\mathcal{S}_k\setminus\mathcal{S}_{\epsilon'}}\Big(\prod_{i=2}^{k} x_i^{-1+(\delta-1)/(k-1)}\Big)\Big(\prod_{i=k+1}^{d} x_i^{1/(d-k)-1}\Big)\Big|\sum_{j=1}^{d}\epsilon_j x_j\Big|^{-1-\delta}\,\mathrm{d}x \\
&\le C\sum_{\epsilon_d}\Big(\frac{1}{n}\Big)^{\delta-1}\Big(\frac{1}{n}\Big)\Big(\frac{1}{n}\Big)^{-\delta} \\
&\le C,
\end{aligned}
$$

whenever $0<\delta\le 1$. Similarly,

$$
\begin{aligned}
&\sum_{\epsilon_d}\int_{(\mathcal{S}_k\setminus\mathcal{S}_{\epsilon'})\setminus\mathcal{S}_{\epsilon,1}}\Big(\prod_{i=1}^{d-1} x_i^{-1}\Big)\Big|\sum_{j=1}^{d}\epsilon_j x_j\Big|^{-1}\,\mathrm{d}x \\
&\quad\le C\sum_{\epsilon_d}\int_{\mathcal{S}_k} x_1^{-1}\Big(\prod_{i=1}^{d} x_i^{-1}\Big)x_d\,\mathrm{d}x \\
&\quad\le C\sum_{\epsilon_d}\int_{\mathcal{S}_k}\Big(\prod_{i=1}^{k} x_i^{-1-1/k}\Big)\Big(\prod_{i=k+1}^{d} x_i^{1/(d-k)-1}\Big)\,\mathrm{d}x \\
&\quad\le C.
\end{aligned}
$$

Step 4. In the third sum of (6.16) the inequality

$$\int_{\mathcal{S}_1\setminus\mathcal{S}'}|K_n^{\infty}(x)|\,\mathrm{d}x\le C$$

can be computed as in Step 3 with $\delta=1$. If $k=d$, then instead of Lemma 6.16, we use Lemma 6.15 to show that

$$\int_{\mathcal{S}_d\setminus\mathcal{S}'}|K_{n,\epsilon'}^{\infty}(x)|\,\mathrm{d}x\le C,$$

which finishes the proof. ■

6.2.4 Proof for $q=2$

Define

$$\theta(s):=\begin{cases}(1-|s|^2)^{\alpha} & \text{if } |s|\le 1\\ 0 & \text{if } |s|>1\end{cases}\qquad (s\in\mathbb{R})$$

and

$$\theta_0(x):=\theta(\|x\|_2)\qquad (x\in\mathbb{R}^d).$$

In this section, we use another method. We will express the Riesz means in terms of the Fourier transform of θ_0.

To this end, we need the concept of Bessel functions. First, we introduce the **gamma function**,

$$\Gamma(x) := \int_0^\infty t^{x-1} \mathrm{e}^{-t} \, \mathrm{d}t.$$

Integration by parts yields

$$\Gamma(x) = \Big[\frac{t^x \mathrm{e}^{-t}}{x}\Big]_0^\infty + \frac{1}{x} \int_0^\infty t^x \mathrm{e}^{-t} \, \mathrm{d}t = \frac{1}{x}\Gamma(x+1).$$

Since $\Gamma(1) = 1$, we have

$$\Gamma(x+1) = x\Gamma(x) \quad (x > 0) \qquad \text{and} \qquad \Gamma(n) = (n-1)!. \tag{6.17}$$

It is easy to see that

$$\Gamma\Big(\frac{1}{2}\Big) = \int_0^\infty t^{-1/2} \mathrm{e}^{-t} \, \mathrm{d}t = 2 \int_0^\infty \mathrm{e}^{-u^2} \, \mathrm{d}u = \sqrt{\pi}.$$

The **beta function** is defined by

$$B(x,y) := \int_0^1 s^{x-1}(1-s)^{y-1} \, \mathrm{d}s = \int_0^1 s^{y-1}(1-s)^{x-1} \, \mathrm{d}s.$$

The relationship between the beta and gamma function reads as follows:

$$\Gamma(x+y)B(x,y) = \Gamma(x)\Gamma(y). \tag{6.18}$$

Indeed, substituting $s = u/(1+u)$, we obtain

$$\begin{aligned} \Gamma(x+y)B(x,y) &= \Gamma(x+y) \int_0^1 s^{y-1}(1-s)^{x-1} \, \mathrm{d}s \\ &= \Gamma(x+y) \int_0^\infty u^{y-1} \Big(\frac{1}{1+u}\Big)^{x+y} \mathrm{d}u \\ &= \int_0^\infty \int_0^\infty u^{y-1} \Big(\frac{1}{1+u}\Big)^{x+y} v^{x+y-1} \mathrm{e}^{-v} \, \mathrm{d}v \mathrm{d}u. \end{aligned}$$

The substitution $v = t(1+u)$ in the inner integral yields

$$\begin{aligned} \Gamma(x+y)B(x,y) &= \int_0^\infty \int_0^\infty u^{y-1} t^{x+y-1} \mathrm{e}^{-t(1+u)} \, \mathrm{d}t \, \mathrm{d}u \\ &= \int_0^\infty t^x \mathrm{e}^{-t} \int_0^\infty (ut)^{y-1} \mathrm{e}^{-tu} \, \mathrm{d}u \, \mathrm{d}t \\ &= \int_0^\infty t^{x-1} \mathrm{e}^{-t} \Gamma(y) \, \mathrm{d}t \\ &= \Gamma(x)\Gamma(y), \end{aligned}$$

which shows (6.18).

For $k > -1/2$, the **Bessel function** is defined by

$$J_k(t) := \frac{(t/2)^k}{\Gamma(k+1/2)\Gamma(1/2)} \int_{-1}^{1} \mathrm{e}^{\mathrm{i}ts}(1-s^2)^{k-1/2}\,\mathrm{d}s \qquad (t \in \mathbb{R}).$$

Note that the Bessel functions are real valued. We prove some basic properties of the Bessel functions.

Lemma 6.18 *We have*

$$J_k'(t) = kt^{-1}J_k(t) - J_{k+1}(t) \qquad (t \neq 0).$$

Proof. By integrating by parts,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(t^{-k}J_k(t)) &= \frac{\mathrm{i}2^{-k}}{\Gamma(k+1/2)\Gamma(1/2)} \int_{-1}^{1} \mathrm{e}^{\mathrm{i}ts} s(1-s^2)^{k-1/2}\,\mathrm{d}s \\
&= \frac{\mathrm{i}2^{-k}}{\Gamma(k+1/2)\Gamma(1/2)} \int_{-1}^{1} \frac{\mathrm{i}t}{2k+1} \mathrm{e}^{\mathrm{i}ts}(1-s^2)^{k+1/2}\,\mathrm{d}s \\
&= \frac{-2^{-k-1}t}{(k+1/2)\Gamma(k+1/2)\Gamma(1/2)} \int_{-1}^{1} \mathrm{e}^{\mathrm{i}ts}(1-s^2)^{k+1/2}\,\mathrm{d}s \\
&= -t^{-k}J_{k+1}(t).
\end{aligned}$$

In the last step, we used (6.17). The lemma follows immediately. ∎

Lemma 6.19 *For $k > -1/2$ and $t > 0$,*

$$J_k(t) \leq C_k t^k \qquad \text{and} \qquad J_k(t) \leq C_k t^{-1/2},$$

where C is independent of t.

Proof. The first estimate trivially follows from the definition of J_k. The second one follows from the first one if $0 < t \leq 1$. Assume that $t > 1$ and integrate the complex valued function $\mathrm{e}^{\mathrm{i}tz}(1-z^2)^{k-1/2}$ $(z \in \mathbb{C})$ over the boundary of the rectangle whose lower side is $[-1,1]$ and whose height is $R > 0$. By Cauchy's theorem,

$$\begin{aligned}
0 &= \mathrm{i}\int_R^0 \mathrm{e}^{\mathrm{i}t(-1+\mathrm{i}s)}(s^2+2\mathrm{i}s)^{k-1/2}\,\mathrm{d}s + \int_{-1}^{1} \mathrm{e}^{\mathrm{i}ts}(1-s^2)^{k-1/2}\,\mathrm{d}s \\
&\quad + \mathrm{i}\int_0^R \mathrm{e}^{\mathrm{i}t(1+\mathrm{i}s)}(s^2-2\mathrm{i}s)^{k-1/2}\,\mathrm{d}s + \epsilon(R),
\end{aligned}$$

where $\epsilon(R) \to 0$ as $R \to \infty$. Hence

$$\begin{aligned}
\int_{-1}^{1} \mathrm{e}^{\mathrm{i}ts}(1-s^2)^{k-1/2}\,\mathrm{d}s &= \mathrm{i}\mathrm{e}^{-\mathrm{i}t}\int_0^\infty \mathrm{e}^{-ts}(s^2+2\mathrm{i}s)^{k-1/2}\,\mathrm{d}s \\
&\quad - \mathrm{i}\mathrm{e}^{\mathrm{i}t}\int_0^\infty \mathrm{e}^{-ts}(s^2-2\mathrm{i}s)^{k-1/2}\,\mathrm{d}s \\
&=: I_1 + I_2.
\end{aligned}$$

Observe that

$$(s^2+2\mathrm{i}s)^{k-1/2}=(2\mathrm{i}s)^{k-1/2}+\phi(s),$$

where $|\phi(s)|\leq Cs^{k+1/2}$ if $0<s\leq 1$ or $s>1$ and $k\leq 3/2$ and $|\phi(s)|\leq Cs^{2k-1}$ if $s>1$ and $k>3/2$. Indeed, by Lagrange's mean value theorem

$$|\phi(s)|=|(2\mathrm{i}s)^{k-1/2}||(\frac{s}{2\mathrm{i}}+1)^{k-1/2}-1|\leq C_k s^{k+1/2}|\frac{\xi}{2\mathrm{i}}+1|^{k-3/2},$$

where $0<\xi<s$. Thus $|s^2+2\mathrm{i}s|^{k-1/2}\leq C_k s^{k-1/2}+|\phi(s)|$ and

$$\begin{aligned}|I_1| &\leq \int_0^\infty \mathrm{e}^{-ts}(C_k s^{k-1/2}+|\phi(s)|)\,\mathrm{d}s\\ &= C_k t^{-1}\int_0^\infty \mathrm{e}^{-u}(u/t)^{k-1/2}\,\mathrm{d}u+\int_0^1 \mathrm{e}^{-ts}|\phi(s)|\,\mathrm{d}s+\int_1^\infty \mathrm{e}^{-ts}|\phi(s)|\,\mathrm{d}s.\end{aligned}$$

The first term is $C_k\Gamma(k+1/2)t^{-k-1/2}$, the second term can be estimated by

$$\Gamma(k+3/2)t^{-k-3/2}\leq C_k t^{-k-1/2}$$

and the third one can be estimated by $\Gamma(k+3/2)t^{-k-3/2}$ if $k\leq 3/2$ or by $C_k\mathrm{e}^{-t}$ if $k>3/2$, both are less than $C_k t^{-k-1/2}$. The integral I_2 can be estimated in the same way. ■

Lemma 6.20 *If $k>-1/2$, $l>-1$ and $t>0$, then*

$$J_{k+l+1}(t)=\frac{t^{l+1}}{2^l\Gamma(l+1)}\int_0^1 J_k(ts)s^{k+1}(1-s^2)^l\,\mathrm{d}s.$$

Proof. Observe that

$$\begin{aligned}J_k(t) &= \frac{2(t/2)^k}{\Gamma(k+1/2)\Gamma(1/2)}\int_0^1 \cos(ts)(1-s^2)^{k-1/2}\,\mathrm{d}s\\ &= \sum_{j=0}^\infty(-1)^j\frac{2(t/2)^k t^{2j}}{(2j)!\Gamma(k+1/2)\Gamma(1/2)}\int_0^1 s^{2j}(1-s^2)^{k-1/2}\,\mathrm{d}s\\ &= \sum_{j=0}^\infty(-1)^j\frac{(t/2)^k t^{2j}}{(2j)!\Gamma(k+1/2)\Gamma(1/2)}\int_0^1 u^{j-1/2}(1-u)^{k-1/2}\,\mathrm{d}u\\ &= \sum_{j=0}^\infty(-1)^j\frac{(t/2)^k t^{2j}}{(2j)!\Gamma(k+1/2)\Gamma(1/2)}B(j+1/2,k+1/2)\\ &= \frac{(t/2)^k}{\Gamma(1/2)}\sum_{j=0}^\infty(-1)^j\frac{\Gamma(j+1/2)}{\Gamma(j+k+1)}\frac{t^{2j}}{(2j)!}.\end{aligned}$$

Thus

$$\begin{aligned}
&\int_0^1 J_k(ts)s^{k+1}(1-s^2)^l\,\mathrm{d}s\\
&= \int_0^1 \Big(\frac{(ts/2)^k}{\Gamma(1/2)}\sum_{j=0}^{\infty}(-1)^j\frac{\Gamma(j+1/2)}{\Gamma(j+k+1)}\frac{(ts)^{2j}}{(2j)!}\Big)s^{k+1}(1-s^2)^l\,\mathrm{d}s\\
&= \frac{(t/2)^k}{\Gamma(1/2)}\sum_{j=0}^{\infty}(-1)^j\frac{\Gamma(j+1/2)}{\Gamma(j+k+1)}\frac{t^{2j}}{(2j)!}\int_0^1 s^{2k+2j+1}(1-s^2)^l\,\mathrm{d}s\\
&= \frac{(t/2)^k}{\Gamma(1/2)}\sum_{j=0}^{\infty}(-1)^j\frac{\Gamma(j+1/2)}{2\Gamma(j+k+1)}\frac{t^{2j}}{(2j)!}\int_0^1 u^{k+j}(1-u)^l\,\mathrm{d}u\\
&= \frac{(t/2)^k}{\Gamma(1/2)}\sum_{j=0}^{\infty}(-1)^j\frac{\Gamma(j+1/2)}{2\Gamma(j+k+1)}\frac{t^{2j}}{(2j)!}B(k+j+1,l+1)\\
&= \frac{2^l\Gamma(l+1)}{t^{l+1}}\frac{(t/2)^{k+l+1}}{\Gamma(1/2)}\sum_{j=0}^{\infty}(-1)^j\frac{\Gamma(j+1/2)}{\Gamma(k+l+j+2)}\frac{t^{2j}}{(2j)!}\\
&= \frac{2^l\Gamma(l+1)}{t^{l+1}}J_{k+l+1}(t),
\end{aligned}$$

which proves the lemma. ■

If θ_0 is radial as above, then its Fourier transform is also radial and can be computed with the help of the Bessel functions. Recall that the **Fourier transform** of $f \in L_1(\mathbb{R}^d)$ is defined by

$$\widehat{f}(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} f(t)\mathrm{e}^{-\mathrm{i}x\cdot t}\,\mathrm{d}t \qquad (x \in \mathbb{R}^d).$$

Theorem 6.21 *For $x \in \mathbb{R}^d$ and $r = \|x\|_2$,*

$$\widehat{\theta}_0(x) = (2\pi)^{-d/2}\,r^{-d/2+1}\int_0^\infty \theta(s)J_{d/2-1}(rs)s^{d/2}\,\mathrm{d}s.$$

Proof. Obviously, $\theta_0 \in L_1(\mathbb{R}^d)$ because $\int_0^\infty |\theta(r)|r^{d-1}\,\mathrm{d}r < \infty$. Let $r = \|x\|_2$, $x = rx'$, $s = \|u\|_2$ and $u = su'$. Then

$$\widehat{\theta}_0(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\theta_0(u)\mathrm{e}^{-\mathrm{i}x\cdot u}\,\mathrm{d}u = \frac{1}{(2\pi)^d}\int_0^\infty \theta(s)(\int_{\Sigma_{d-1}} \mathrm{e}^{-\mathrm{i}rsx'\cdot u'}\,\mathrm{d}u')s^{d-1}\,\mathrm{d}s,$$

where Σ_{d-1} is the sphere. In the inner integral, we integrate first over the parallel $P_\delta := \{u' \in \Sigma_{d-1} : x'\cdot u' = \cos\delta\}$ orthogonal to x' obtaining a function of $0 \le \delta \le \pi$, which we then integrate over $[0,\pi]$. If ω_{d-2} denotes the surface area of Σ_{d-2}, then the measure of P_δ is

$$\omega_{d-2}(\sin\delta)^{d-2} = \frac{2\pi^{(d-1)/2}}{\Gamma((d-1)/2)}(\sin\delta)^{d-2}.$$

Hence

$$\begin{aligned}
\int_{\Sigma_{d-1}} \mathrm{e}^{-\mathrm{i}rsx'\cdot u'}\,\mathrm{d}u' &= \int_0^\pi \mathrm{e}^{-\mathrm{i}rs\cos\delta}\omega_{d-2}(\sin\delta)^{d-2}\,\mathrm{d}\delta \\
&= \omega_{d-2}\int_{-1}^1 \mathrm{e}^{\mathrm{i}rs\xi}(1-\xi^2)^{(d-3)/2}\,\mathrm{d}\xi \\
&= \frac{2\pi^{(d-1)/2}}{\Gamma((d-1)/2)}\frac{\Gamma(d/2-1/2)\Gamma(1/2)}{(rs/2)^{d/2-1}}J_{d/2-1}(rs) \\
&= (2\pi)^{d/2}(rs)^{-d/2+1}J_{d/2-1}(rs),
\end{aligned}$$

which finishes the proof of the theorem. ■

Note that this theorem works for any radial function.

Corollary 6.22 *If* $\alpha > 0$, *then*

$$\widehat{\theta}_0(x) = (2\pi)^{-d/2}2^\alpha\Gamma(\alpha+1)\|x\|_2^{-d/2-\alpha}J_{d/2+\alpha}(\|x\|_2).$$

Proof. By Theorem 6.21,

$$\widehat{\theta}_0(x) = (2\pi)^{-d/2}\|x\|_2^{-d/2+1}\int_0^1 J_{d/2-1}(\|x\|_2 s)s^{d/2}(1-s^2)^\alpha\,\mathrm{d}s.$$

Applying Lemma 6.20 with $k = d/2-1$, $l = \alpha$, we see that

$$\widehat{\theta}_0(x) = (2\pi)^{-d/2}\|x\|_2^{-d/2+1}J_{d/2+\alpha}(\|x\|_2)\|x\|_2^{-\alpha-1}2^\alpha\Gamma(\alpha+1),$$

which shows the corollary. ■

Corollary 6.22 and Lemma 6.18 imply that $\widehat{\theta}_0(x)$ as well as all of its derivatives can be estimated by $\|x\|_2^{-d/2-\alpha-1/2}$.

Corollary 6.23 *For all* $i_1,\ldots,i_d \geq 0$ *and* $\alpha > 0$,

$$|\partial_1^{i_1}\cdots\partial_d^{i_d}\widehat{\theta}_0(x)| \leq C\|x\|_2^{-d/2-\alpha-1/2} \qquad (x \neq 0).$$

The same result holds for

$$\theta(s) := \begin{cases} (1-|s|^\gamma)^\alpha & \text{if } |s| \leq 1 \\ 0 & \text{if } |s| > 1 \end{cases} \qquad (s \in \mathbb{R})$$

and $\theta_0(x) := \theta(\|x\|_2)$ $(x \in \mathbb{R}^d)$, whenever $\gamma \in \mathbb{N}$ (see Lu [61, p. 132]). From now on, we assume that $\gamma \in \mathbb{N}$. Now we are ready to express the Riesz means using the Fourier transform of θ_0. Observe that $\widehat{\theta}_0 \in L_1(\mathbb{R}^d)$ if and only if $\alpha > (d-1)/2$.

Theorem 6.24 *If $n \in \mathbb{N}^d$, $f \in L_1(\mathbb{T}^d)$ and $\alpha > (d-1)/2$, then*

$$\sigma_n^{2,\alpha} f(x) = n^d \int_{\mathbb{R}^d} f(x-t)\widehat{\theta}_0(nt)\,\mathrm{d}t.$$

Proof. If $f(t) = \mathrm{e}^{\mathrm{i}k\cdot t}$ $(k \in \mathbb{Z}^d, t \in \mathbb{T}^d)$, then

$$n^d \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}k\cdot(x-t)}\widehat{\theta}_0(nt)\,\mathrm{d}t = \mathrm{e}^{\mathrm{i}k\cdot x}\int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}k\cdot t/n}\widehat{\theta}_0(t)\,\mathrm{d}t = \theta_0\Big(\frac{-k}{n}\Big)\mathrm{e}^{\mathrm{i}k\cdot x} = \sigma_n^{2,\alpha} f(x).$$

The theorem holds also for trigonometric polynomials. Let f be an arbitrary element from $L_1(\mathbb{T}^d)$ and (f_k) be a sequence of trigonometric polynomials such that $f_k \to f$ in the $L_1(\mathbb{T}^d)$-norm. It follows from the form (5.1) of $\sigma_n^{2,\alpha} f$ and from the fact that $K_n^{2,\alpha} \in L_1(\mathbb{T}^d)$ that $\sigma_n^{2,\alpha} f_k \to \sigma_n^{2,\alpha} f$ in the $L_1(\mathbb{T}^d)$ norm as $k \to \infty$.

On the other hand, since $\widehat{\theta}_0 \in L_1(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} f_k(x-t)\widehat{\theta}_0(nt)\,\mathrm{d}t \longrightarrow \int_{\mathbb{R}^d} f(x-t)\widehat{\theta}_0(nt)\,\mathrm{d}t$$

in the $L_1(\mathbb{T}^d)$-norm as $k \to \infty$. ∎

Proof of Theorem 6.1 for $q = 2$. Observe that Theorem 6.1 follows from Theorem 6.24. Indeed, since f is periodic,

$$\begin{aligned} \sigma_n^{2,\alpha} f(x) &= n^d \sum_{k\in\mathbb{Z}^d} \int_{2k\pi}^{2(k+1)\pi} f(x-t)\widehat{\theta}_0(nt)\,\mathrm{d}t \\ &= n^d \sum_{k\in\mathbb{Z}^d} \int_0^{2\pi} f(x-u)\widehat{\theta}_0(n(u+2k\pi))\,\mathrm{d}u. \end{aligned}$$

By (5.1),

$$K_n^{q,\alpha}(u) = (2\pi)^d n^d \sum_{k\in\mathbb{Z}^d} \widehat{\theta}_0(n(u+2k\pi)) \tag{6.19}$$

and

$$\int_0^{2\pi} |K_n^{q,\alpha}(u)|\,\mathrm{d}u \le (2\pi)^d n^d \sum_{k\in\mathbb{Z}^d} \int_0^{2\pi} |\widehat{\theta}_0(n(u+2k\pi))|\,\mathrm{d}u = (2\pi)^d \|\widehat{\theta}_0\|_1.$$

∎

7 $H_p^\square(\mathbb{T}^d)$ Hardy spaces

To prove almost everywhere convergence of the Riesz means, we will need the concept of Hardy spaces and their atomic decomposition. A distribution f is in the **Hardy space** $H_p^\square(\mathbb{T}^d)$ and in the **weak Hardy space** $H_{p,\infty}^\square(\mathbb{T}^d)$ $(0 < p \le \infty)$ if

$$\|f\|_{H_p^\square} := \|\sup_{0<t} |f * P_t^d|\|_p < \infty$$

and

$$\|f\|_{H^\square_{p,\infty}} := \| \sup_{0<t} |f * P_t^d| \|_{p,\infty} < \infty,$$

respectively, where

$$P_t^d(x) := \sum_{k\in\mathbb{Z}^d} \mathrm{e}^{-t\|k\|_2} \mathrm{e}^{\mathrm{i}k\cdot x} \qquad (x \in \mathbb{T}^d, t > 0)$$

is the d-dimensional periodic **Poisson kernel**. Since $P_t^d \in L_1(\mathbb{T}^d)$, the convolution in the definition of the norms are well defined. In the one-dimensional case, we get back the usual Poisson kernel

$$P_t(x) := P_t^1(x) = \sum_{k=-\infty}^{\infty} r^{|k|}\mathrm{e}^{\mathrm{i}kx} = \frac{1-r^2}{1+r^2-2r\cos x} \qquad (x \in \mathbb{T}),$$

where $r := \mathrm{e}^{-t}$. It is known (see e.g. Stein [78] or Weisz [94]) that

$$H_p^\square(\mathbb{T}^d) \sim L_p(\mathbb{T}^d) \qquad (1 < p \le \infty)$$

and $H_1^\square(\mathbb{T}^d) \subset L_1(\mathbb{T}^d) \subset H_{1,\infty}^\square(\mathbb{T}^d)$. Moreover,

$$\|f\|_{H^\square_{1,\infty}} = \sup_{\rho>0} \rho\, \lambda(\sup_{0<t} |f * P_t^d| > \rho) \le C\|f\|_1 \qquad (f \in L_1(\mathbb{T}^d)). \tag{7.1}$$

The *atomic decomposition* provides a useful characterization of Hardy spaces. A bounded function a is an $H_p^\square$**-atom** if there exists a cube $I \subset \mathbb{T}^d$ such that

(i) $\operatorname{supp} a \subset I$,

(ii) $\|a\|_\infty \le |I|^{-1/p}$,

(iii) $\int_I a(x)x^k \,\mathrm{d}x = 0$ for all multi-indices $k = (k_1, \ldots, k_d)$ with $|k| \le \lfloor d(1/p - 1)\rfloor$.

In the definition, the cubes can be replaced by balls and (ii) by

(ii') $\|a\|_q \le |I|^{1/q-1/p}$ $(1 < q \le \infty)$.

We could suppose that the integral in (iii) is zero for all multi-indices k for which $|k| \le N$, where $N \ge \lfloor d(1/p-1)\rfloor$. The best possible choice of such numbers N is $\lfloor d(1/p-1)\rfloor$. Each Hardy space has an atomic decomposition. In other words, every function from the Hardy space can be decomposed into the sum of atoms (see e.g. Latter [57], Lu [61], Coifman and Weiss [24], Wilson [105, 106], Stein [78] and Weisz [94]).

Theorem 7.1 *A function f is in $H_p^\square(\mathbb{T}^d)$ $(0 < p \le 1)$ if and only if there exist a sequence $(a^k, k \in \mathbb{N})$ of $H_p^\square$-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that*

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty \quad \textit{and} \quad \sum_{k=0}^{\infty} \mu_k a^k = f \quad \textit{in the sense of distributions.}$$

Moreover,

$$\|f\|_{H_p^\square} \sim \inf \Big(\sum_{k=0}^{\infty} |\mu_k|^p \Big)^{1/p}.$$

The "only if" part of the theorem holds also for $0 < p < \infty$. The following result gives a sufficient condition for an operator to be bounded from $H_p^\square(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$ (see e.g. Weisz [94]). Let I^r be the interval having the same center as the interval $I \subset \mathbb{T}$ and length $2^r|I|$ $(r \in \mathbb{N})$. For a rectangle

$$R = I_1 \times \cdots \times I_d \quad \text{let} \quad R^r = I_1^r \times \cdots \times I_d^r.$$

Theorem 7.2 *For each $n \in \mathbb{N}^d$, let $V_n : L_1(\mathbb{T}^d) \to L_1(\mathbb{T}^d)$ be a bounded linear operator and let*

$$V_* f := \sup_{n\in\mathbb{N}^d} |V_n f|.$$

Suppose that

$$\int_{\mathbb{T}^d\setminus I^r} |V_* a|^{p_0} \,\mathrm{d}\lambda \le C_{p_0}$$

for all $H_{p_0}^\square$-atoms a and for some fixed $r \in \mathbb{N}$ and $0 < p_0 \le 1$, where the cube I is the support of the atom. If V_* is bounded from $L_{p_1}(\mathbb{T}^d)$ to $L_{p_1}(\mathbb{T}^d)$ for some $1 < p_1 \le \infty$, then

$$\|V_* f\|_p \le C_p \|f\|_{H_p^\square} \qquad (f \in H_p^\square(\mathbb{T}^d) \cap L_1(\mathbb{T}^d)) \tag{7.2}$$

for all $p_0 \le p \le p_1$. If $\lim_{k\to\infty} f_k = f$ in the $H_p^\square$-norm implies that $\lim_{k\to\infty} V_n f_k = V_n f$ in the sense of distributions ($n \in \mathbb{N}^d$), then (7.2) holds for all $f \in H_p^\square(\mathbb{T}^d)$.

Proof. Observe that, under the conditions of Theorem 7.2, the L_{p_0}-norms of $V_* a$ are uniformly bounded for all $H_{p_0}^\square$-atoms a. Indeed,

$$\begin{aligned}
\int_{\mathbb{T}^d} |V_* a|^{p_0} \,\mathrm{d}\lambda &= \int_{I^r} |V_* a|^{p_0} \,\mathrm{d}\lambda + \int_{\mathbb{T}^d\setminus I^r} |V_* a|^{p_0} \,\mathrm{d}\lambda \\
&\le \Big(\int_{I^r} |V_* a|^{p_1} \,\mathrm{d}\lambda \Big)^{p_0/p_1} |I^r|^{1-p_0/p_1} + C_{p_0} \\
&\le C_{p_0} \Big(\int_{I^r} |a|^{p_1} \,\mathrm{d}\lambda \Big)^{p_0/p_1} |I|^{1-p_0/p_1} + C_{p_0} \\
&\le C_{p_0} \Big(|I|^{-p_1/p_0} |I^r| \Big)^{p_0/p_1} |I|^{1-p_0/p_1} + C_{p_0} \\
&= C_{p_0}.
\end{aligned}$$

There is an atomic decomposition such that

$$f = \sum_{k=0}^\infty \mu_k a_k \quad \text{in the } H_{p_0}^\square\text{-norm} \quad \text{and} \quad \Big(\sum_{k=0}^\infty |\mu_k|^{p_0} \Big)^{1/p_0} \le C_{p_0} \|f\|_{H_{p_0}^\square},$$

where the convergence holds also in the $H_1^\square$-norm and in the L_1-norm if $f \in H_1^\square(\mathbb{T}^d)$. Then

$$V_n f = \sum_{k=0}^\infty \mu_k V_n a_k$$

and

$$|V_* f| \leq \sum_{k=0}^{\infty} |\mu_k||V_* a_k|$$

for $f \in H_1^\square(\mathbb{T}^d)$. Thus

$$\|V_* f\|_{p_0}^{p_0} \leq \sum_{k=0}^{\infty} |\mu_k|^{p_0} \|V_* a_k\|_{p_0}^{p_0} \leq C_{p_0} \|f\|_{H_{p_0}^\square}^{p_0} \qquad (f \in H_1^\square(\mathbb{T}^d)). \tag{7.3}$$

Obviously, the same inequality holds for the operators V_n. This and interpolation proves the theorem if $p_0 = 1$. Assume that $p_0 < 1$. Since $H_1^\square(\mathbb{T}^d)$ is dense in $L_1(\mathbb{T}^d)$ as well as in $H_{p_0}^\square(\mathbb{T}^d)$, we can extend uniquely the operators V_n and V_* such that (7.3) holds for all $f \in H_{p_0}^\square(\mathbb{T}^d)$. Let us denote these extended operators by V_n' and V_*'. Then $V_n f = V_n' f$ and $V_* f = V_*' f$ for all $f \in H_1^\square(\mathbb{T}^d)$. It is enough to show that these equalities hold for all $f \in H_{p_0}^\square(\mathbb{T}^d) \cap L_1(\mathbb{T}^d)$. We get by interpolation from (7.3) that the operator

$$V_*' \quad \text{is bounded from} \quad H_{p,\infty}^\square(\mathbb{T}^d) \quad \text{to} \quad L_{p,\infty}(\mathbb{T}^d) \tag{7.4}$$

when $p_0 < p < p_1$. For the basic definitions and theorems on interpolation theory, see Bergh and Löfström [10], Bennett and Sharpley [6] or Weisz [94]. Since $p_0 < 1$, the boundedness in (7.4) holds especially for $p = 1$, and so (7.1) implies that V_*' is of weak type $(1,1)$:

$$\sup_{\rho>0} \rho\, \lambda(|V_*' f| > \rho) = \|V_*' f\|_{1,\infty} \leq C \|f\|_{H_{1,\infty}^\square} \leq C\|f\|_1 \qquad (f \in L_1(\mathbb{R}^d)). \tag{7.5}$$

Obviously, the same holds for V_n'. Since V_n is bounded on $L_1(\mathbb{T}^d)$ if $f_k \in H_1^\square(\mathbb{T}^d)$ such that $\lim_{k\to\infty} f_k = f$ in the L_1-norm, then $\lim_{k\to\infty} V_n f_k = V_n f$ in the L_1-norm. Inequality (7.5) implies that $\lim_{k\to\infty} V_n f_k = V_n' f$ in the $L_{1,\infty}$-norm, hence $V_n f = V_n' f$ for all $f \in L_1(\mathbb{T}^d)$. Similarly, for a fixed $N \in \mathbb{N}$, the operator

$$V_{N,*} f := \sup_{|n| \leq N} |V_n f|$$

satisfies (7.5) for all $f \in H_1^\square(\mathbb{T}^d)$ and its extension $V_{N,*}'$ for all $f \in L_1(\mathbb{T}^d)$. The inequality

$$\begin{aligned}\sup_{\rho>0} \rho\, \lambda(|V_{N,*}' f - V_{N,*} f| > \rho) \quad &\leq \quad \sup_{\rho>0} \rho\, \lambda(|V_{N,*}' f - V_{N,*}' f_k| > \rho/2) \\ &\quad + \sup_{\rho>0} \rho\, \lambda(|V_{N,*} f_k - V_{N,*} f| > \rho/2) \to 0\end{aligned}$$

as $k \to \infty$, shows that $V_{N,*}' f = V_{N,*} f$ for all $f \in L_1(\mathbb{T}^d)$. Moreover, for a fixed ρ,

$$\begin{aligned}&\lambda(|V_*' f - V_{N,*} f| > \rho) \\ &\quad \leq \quad \lambda(|V_*' f - V_*' f_k| > \rho/3) + \lambda(|V_* f_k - V_{N,*} f_k| > \rho/3) + \lambda(|V_{N,*} f_k - V_{N,*} f| > \rho/3) \\ &\quad \leq \quad \lambda(V_*'(f - f_k) > \rho/3) + \lambda(V_* f_k - V_{N,*} f_k > \rho/3) + \lambda(V_{N,*}(f_k - f) > \rho/3) \\ &\quad \leq \quad \frac{C}{\rho} \|f - f_k\|_1 + \lambda(V_* f_k - V_{N,*} f_k > \rho/3) \\ &\quad < \quad \epsilon\end{aligned}$$

if k and N are large enough. Hence $\lim_{N\to\infty} V_{N,*}f = V'_*f$ in measure for all $f \in L_1(\mathbb{T}^d)$. On the other hand, $\lim_{N\to\infty} V_{N,*}f = V_*f$ a.e., which implies that

$$V_*f = V'_*f \quad \text{for all} \quad f \in L_1(\mathbb{T}^d).$$

Consequently, (7.5) holds also for V_* and (7.3) for all $f \in H_{p_0}^\square(\mathbb{T}^d) \cap L_1(\mathbb{T}^d)$.

Assume that V_n is defined also for distributions and that $\lim_{k\to\infty} f_k = f$ in the $H_p^\square$-norm implies $\lim_{k\to\infty} V_n f_k = V_n f$ in the sense of distributions ($n \in \mathbb{N}^d$). Suppose that $p < 1$ and $f_k \in H_p^\square(\mathbb{T}^d) \cap L_1(\mathbb{T}^d)$ ($k \in \mathbb{N}$). Since by (7.2), $V_n f_k$ is convergent in the L_p-norm as $k \to \infty$, we can identify the distribution $V_n f$ with the L_p-limit $\lim_{k\to\infty} V_n f_k$. Hence the same holds for $V_{N,*}f$: $V_{N,*}f = \lim_{k\to\infty} V_{N,*}f_k$ in the L_p-norm. Moreover,

$$\begin{aligned}
\|V'_*f - V_{N,*}f\|_p &\leq \|V'_*f - V'_*f_k\|_p + \|V_*f_k - V_{N,*}f_k\|_p + \|V_{N,*}f_k - V_{N,*}f\|_p \\
&\leq C_p\|f - f_k\|_{H_p^\square} + \|V_*f_k - V_{N,*}f_k\|_p + \|V_{N,*}f_k - V_{N,*}f\|_p \\
&< \epsilon
\end{aligned}$$

if k and N are large enough. Thus $\lim_{N\to\infty} V_{N,*}f = V'_*f$ in the L_p-norm and, on the other hand, $\lim_{N\to\infty} V_{N,*}f = V_*f$ a.e., which implies that $V_*f = V'_*f$ for all $f \in H_p^\square(\mathbb{T}^d)$. Consequently, (7.2) holds for all $f \in H_p^\square(\mathbb{T}^d)$. ■

Unfortunately, for a linear operator V, the uniform boundedness of the L_{p_0}-norms of Va is not enough for the boundedness $V : H_{p_0}^\square(\mathbb{T}^d) \to L_{p_0}(\mathbb{T}^d)$ (see [13, 64, 65, 14, 70]). The next weak version of Theorem 7.2 can be proved similarly (see also the proof in Weisz [94]).

Theorem 7.3 *For each $n \in \mathbb{N}^d$, let $V_n : L_1(\mathbb{T}^d) \to L_1(\mathbb{T}^d)$ be a bounded linear operator and let*

$$V_*f := \sup_{n\in\mathbb{N}^d} |V_n f|.$$

Suppose that

$$\sup_{\rho>0} \rho^p \lambda\Big(\{|V_*a| > \rho\} \cap \{\mathbb{T}^d \setminus I^r\}\Big) \leq C_p$$

for all $H_p^\square$-atoms a and for some fixed $r \in \mathbb{N}$ and $0 < p < 1$. If V_* is bounded from $L_{p_1}(\mathbb{T}^d)$ to $L_{p_1}(\mathbb{T}^d)$ ($1 < p_1 \leq \infty$), then

$$\|V_*f\|_{p,\infty} \leq C_p\|f\|_{H_p^\square} \qquad (f \in H_p^\square(\mathbb{T}^d) \cap L_1(\mathbb{T}^d)). \tag{7.6}$$

If $\lim_{k\to\infty} f_k = f$ in the $H_p^\square$-norm implies that $\lim_{k\to\infty} V_n f_k = V_n f$ in the sense of distributions ($n \in \mathbb{N}^d$), then (7.6) holds for all $f \in H_p^\square(\mathbb{T}^d)$.

The next result follows from inequality (7.5).

Corollary 7.4 *If $p_0 < 1$ in Theorem 7.2, then for all $f \in L_1(\mathbb{T}^d)$,*

$$\sup_{\rho>0} \rho\, \lambda(|V_*f| > \rho) \leq C\|f\|_1.$$

Theorem 7.2 and Corollary 7.4 can be regarded also as an alternative tool to the Calderon-Zygmund decomposition lemma for proving weak type $(1,1)$ inequalities. In many cases, this method can be applied better and more simply than the Calderon-Zygmund decomposition lemma.

8 Almost everywhere convergence of the ℓ_q-summability means

We define the **maximal Riesz operator** by

$$\sigma_*^{q,\alpha} f := \sup_{n\in\mathbb{N}} |\sigma_n^{q,\alpha} f|.$$

If $\alpha = 1$, we obtain the **maximal Fejér operator** and write it simply as $\sigma_*^q f$.

Using Theorems 7.2 and 7.3 and some estimations of the kernel functions, we can show that the maximal Riesz operator is bounded from $H_p^\square(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$. This was proved by Stein, Taibleson and Weiss [79] and Lu [61] for $q = 2$, by Oswald [68] for Fourier transforms and for $q = \infty$, $\gamma = 2$, by Weisz [94, 96, 100, 99, 101] for $q = 1, 2, \infty$ and more general summability methods.

Theorem 8.1 *If $q = 1, \infty$, $\alpha \geq 1$ and $d/(d+1) < p \leq \infty$, then*

$$\|\sigma_*^{q,\alpha} f\|_p \leq C_p \|f\|_{H_p^\square} \qquad (f \in H_p^\square(\mathbb{T}^d)) \tag{8.1}$$

and for $f \in H_{d/(d+1)}^\square(\mathbb{T}^d)$,

$$\|\sigma_*^{q,\alpha} f\|_{d/(d+1),\infty} = \sup_{\rho>0} \rho \lambda(\sigma_*^{q,\alpha} f > \rho)^{(d+1)/d} \leq C \|f\|_{H_{d/(d+1)}^\square}. \tag{8.2}$$

If $q = 2$ and $\alpha > (d-1)/2$, then the same holds with the critical index $d/(d/2 + \alpha + 1/2)$ instead of $d/(d+1)$.

This theorem will be proved in Subsection 8.2. Recall that $H_p^\square(\mathbb{T}^d) \sim L_p(\mathbb{T}^d)$ for $1 < p \leq \infty$ and so (8.1) yields

$$\|\sigma_*^{q,\alpha} f\|_p \leq C_p \|f\|_p \qquad (f \in L_p(\mathbb{T}^d), 1 < p \leq \infty).$$

If p is smaller than or equal to the critical index, then this theorem is not true (Oswald [68], Stein, Taibleson and Weiss [79]).

Theorem 8.2 *If $q = \infty$ and $\alpha = 1$ (resp. $q = 2$ and $\alpha > (d-1)/2$), then the operator $\sigma_*^{q,\alpha}$ is not bounded from $H_p^\square(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$ if p is smaller than or equal to the critical index $d/(d+1)$ (resp. $d/(d/2 + \alpha + 1/2)$).*

Of course, (8.2) cannot be true for $p < d/(d+1)$, i.e., $\sigma_*^{q,\alpha}$ is not bounded from $H_p^\square(\mathbb{T}^d)$ to the weak $L_{p,\infty}(\mathbb{T}^d)$ space for $p < d/(d+1)$. If the operator was bounded, then by interpolation (8.1) would hold for $p = d/(d+1)$, which contradicts Theorem 8.2.

Marcinkiewicz [62] verified for two-dimensional Fourier series that the cubic (i.e., $q = \infty$) Fejér means of a function $f \in L \log L(\mathbb{T}^2)$ converge almost everywhere to f as $n \to \infty$. Later Zhizhiashvili [108, 109] extended this result to all $f \in L_1(\mathbb{T}^2)$ and to Cesàro means, Oswald [68] to Fourier transform and $\gamma = 2$ and the author [101] to higher dimensions. The same result for $q = 2$ can be found in Stein and Weiss [80], Lu [61] and Weisz [96], for $q = 1$ in Berens, Li and Xu [7] and Weisz [100, 99].

Corollary 8.3 *Suppose that $q = 1, \infty$ and $\alpha \geq 1$ or $q = 2$ and $\alpha > (d-1)/2$. If $f \in L_1(\mathbb{T}^d)$, then*

$$\sup_{\rho>0} \rho\, \lambda(\sigma_*^{q,\alpha} f > \rho) \leq C\|f\|_1.$$

The density argument of Marcinkiewicz and Zygmund implies

Corollary 8.4 *Suppose that $q = 1, \infty$ and $\alpha \geq 1$ or $q = 2$ and $\alpha > (d-1)/2$. If $f \in L_1(\mathbb{T}^d)$, then*

$$\lim_{n\to\infty} \sigma_n^{q,\alpha} f = f \qquad a.e.$$

Proof. Since the trigonometric polynomials are dense in $L_1(\mathbb{T}^2)$, the corollary follows from Theorem 3.2 and Corollary 8.3. ■

8.1 Further results for the Bochner-Riesz means

The boundedness of the operator $\sigma_*^{2,\alpha}$ is complicated and not completely solved if $q = 2$ and $\alpha \leq (d-1)/2$. All results of this section can be found in Grafakos [43, Chapter 10], so we omit the proofs. The following result is due to Tao [83].

Theorem 8.5 *Suppose that $d \geq 2$ and $q = \gamma = 2$. If $0 < \alpha \leq (d-1)/2$ and*

$$1 < p < \frac{2d-1}{d+2\alpha} \qquad or \qquad p > \frac{2d}{d-1-2\alpha},$$

then the maximal operator $\sigma_*^{2,\alpha}$ is not bounded from $L_p(\mathbb{T}^d)$ to $L_{p,\infty}(\mathbb{T}^d)$ (see Figure 20).

By Theorems 8.5 and 4.6, Figure 20 shows the region where $\sigma_*^{2,\alpha}$ is unbounded from $L_p(\mathbb{T}^d)$ to $L_{p,\infty}(\mathbb{T}^d)$. Obviously, the operator $\sigma_*^{2,\alpha}$ is unbounded from $L_p(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$ on the same region. Note that the exact region of the boundedness or unboundedness of $\sigma_*^{2,\alpha}$ is still unknown (see Figure 23).

Stein [80, p. 276] proved that Theorem 6.4 holds also for the maximal operator $\sigma_*^{2,\alpha}$.

Theorem 8.6 *Suppose that $d \geq 2$ and $q = \gamma = 2$. If $0 < \alpha \leq \frac{d-1}{2}$ and*

$$\frac{2(d-1)}{d-1+2\alpha} < p < \frac{2(d-1)}{d-1-2\alpha},$$

then the maximal Bochner-Riesz operator $\sigma_*^{2,\alpha}$ is bounded on $L_p(\mathbb{T}^d)$ (see Figure 15).

Carbery improved this theorem in [16] for $d = 2$.

Theorem 8.7 *Suppose that $d = 2$ and $q = \gamma = 2$. If $0 < \alpha \leq 1/2$ and*

$$\frac{2}{1+2\alpha} < p < \frac{4}{1-2\alpha},$$

then the maximal Bochner-Riesz operator $\sigma_*^{2,\alpha}$ is bounded on $L_p(\mathbb{T}^d)$ (see Figure 21).

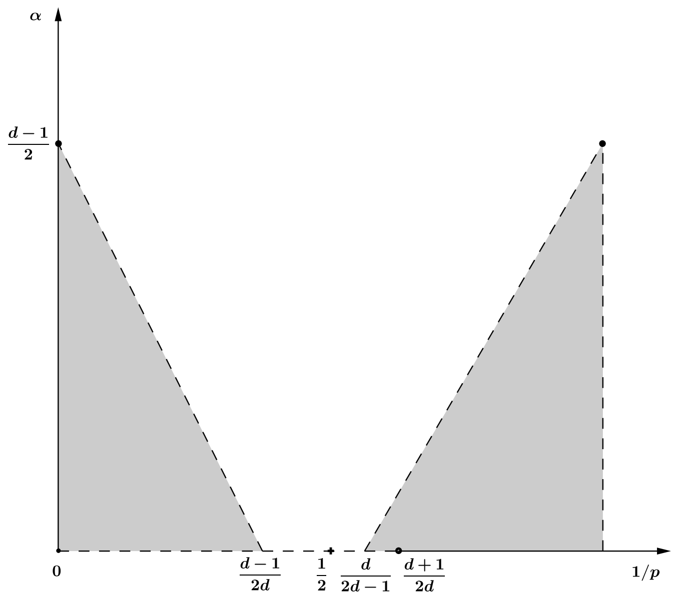

Figure 20: Unboundedness of $\sigma_*^{2,\alpha}$ from $L_p(\mathbb{T}^d)$ to $L_{p,\infty}(\mathbb{T}^d)$.

Christ [23] generalized this result to higher dimensions.

Theorem 8.8 *Suppose that $d \geq 3$ and $q = \gamma = 2$. If $\frac{d-1}{2(d+1)} \leq \alpha \leq \frac{d-1}{2}$ and*

$$\frac{2(d-1)}{d-1+2\alpha} < p < \frac{2d}{d-1-2\alpha},$$

then the maximal Bochner-Riesz operator $\sigma_*^{2,\alpha}$ is bounded on $L_p(\mathbb{T}^d)$ (see Figure 22).

The following result follows from analytic interpolation and from Theorems 8.6 and 8.8 (see e.g. Stein and Weiss [80, p. 276, p. 205]).

Theorem 8.9 *Suppose that $d \geq 3$ and $q = \gamma = 2$. If $0 < \alpha < \frac{d-1}{2(d+1)}$ and*

$$\frac{2(d-1)}{d-1+2\alpha} < p < \frac{2(d-1)}{d-1-4\alpha},$$

then the maximal Bochner-Riesz operator $\sigma_*^{2,\alpha}$ is bounded on $L_p(\mathbb{T}^d)$ (see Figure 22).

It is still an open question as to whether $\sigma_*^{2,\alpha}$ is bounded or unbounded in the region of Figure 23. If $d = 2$, then the question is open on the right hand side of the region of Figure 23 only, i.e., for $1/p \geq 1/2$.

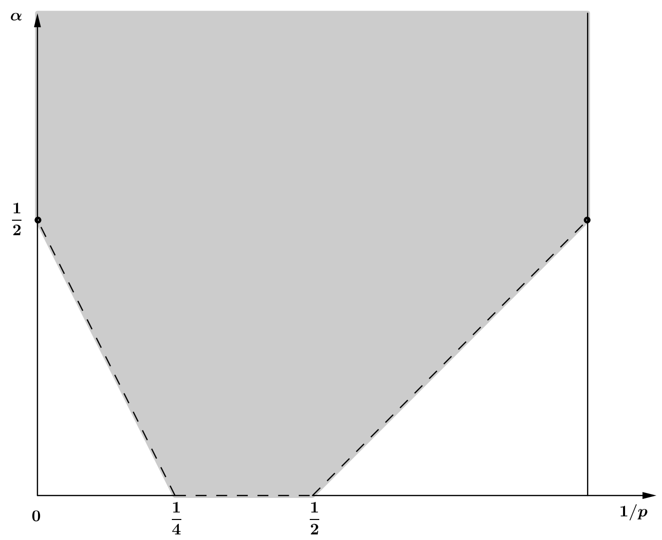

Figure 21: Boundedness of $\sigma_*^{2,\alpha}$ from $L_p(\mathbb{T}^d)$ to $L_{p,\infty}(\mathbb{T}^d)$ when $d = 2$.

Of course, in Theorems 8.6–8.9, $\sigma_*^{2,\alpha}$ is also bounded from $L_p(\mathbb{T}^d)$ to $L_{p,\infty}(\mathbb{T}^d)$. This implies that

$$\lim_{n\to\infty} \sigma_n^{q,\alpha} f = f \qquad \text{a.e.}$$

Figures 21 and 22 show the region where $\sigma_*^{2,\alpha}$ is bounded from $L_p(\mathbb{T}^d)$ to $L_{p,\infty}(\mathbb{T}^d)$ and almost everywhere convergence hold.

By Theorem 3.2, if $\sigma_*^{q,\alpha}$ is of **weak type** (p,p), i.e., it is bounded from $L_p(\mathbb{T}^d)$ to $L_{p,\infty}(\mathbb{T}^d)$, then $\lim_{n\to\infty} \sigma_n^{q,\alpha} f = f$ almost everywhere for all $f \in L_p(\mathbb{T}^d)$. The converse is also true when $1 \leq p \leq 2$: almost everywhere convergence implies that the corresponding maximal operator is of weak type (p,p). More exactly, if $X = L_p$, $1 \leq p \leq 2$ and T_n commutes with translation, then the converse of Theorem 3.2 holds (see Stein [76]). However, this result is no longer true for $p > 2$. The preceding theorems concerning the almost everywhere convergence were generalized by Carbery, Rubio de Francia and Vega [17] (see Figure 24).

Theorem 8.10 *Suppose that $d \geq 2$ and $q = \gamma = 2$. If $0 < \alpha \leq (d-1)/2$ and*

$$\frac{2(d-1)}{d-1+2\alpha} < p < \frac{2d}{d-1-2\alpha},$$

then for all $f \in L_p(\mathbb{T}^d)$

$$\lim_{n\to\infty} \sigma_n^{q,\alpha} f = f \qquad \text{a.e.}$$

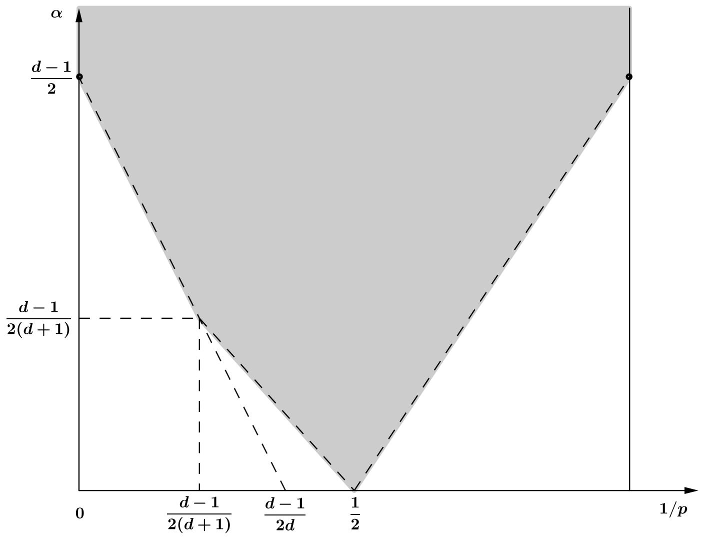

Figure 22: Boundedness of $\sigma_*^{2,\alpha}$ from $L_p(\mathbb{T}^d)$ to $L_{p,\infty}(\mathbb{T}^d)$ when $d \geq 3$.

(see Figure 24).

8.2 Proof of Theorem 8.1

In this section, we will prove Theorem 8.1 in four subsections. If $q = 1, \infty$, then we suppose here that $\alpha = \gamma = 1$. For other parameters, see Subsection 10.1.

8.2.1 Proof for $q = 1$ in the two-dimensional case

Proof of Theorem 8.1 for $q = 1$ and $d = 2$. First, we will show that

$$\int_{\mathbb{T}^2} |\sigma_*^1 a(x,y)|^p \, \mathrm{d}x \, \mathrm{d}y = \int_{\mathbb{T}^2} \sup_{n \geq 1} \left| \int_I a(u,v) K_n^1(x-u, y-v) \, \mathrm{d}u \, \mathrm{d}v \right|^p \mathrm{d}x \, \mathrm{d}y \leq C_p \tag{8.3}$$

for every $H_p^\square$-atom a, where $2/3 < p < 1$ and I is the support of the atom. By Theorems 6.2 and 7.2, this will imply (8.1). Without loss of generality, we can suppose that a is a $H_p^\square$-atom with support $I = I_1 \times I_2$ and

$$[-2^{-K-2}, 2^{-K-2}] \subset I_j \subset [-2^{-K-1}, 2^{-K-1}] \qquad (j = 1, 2)$$

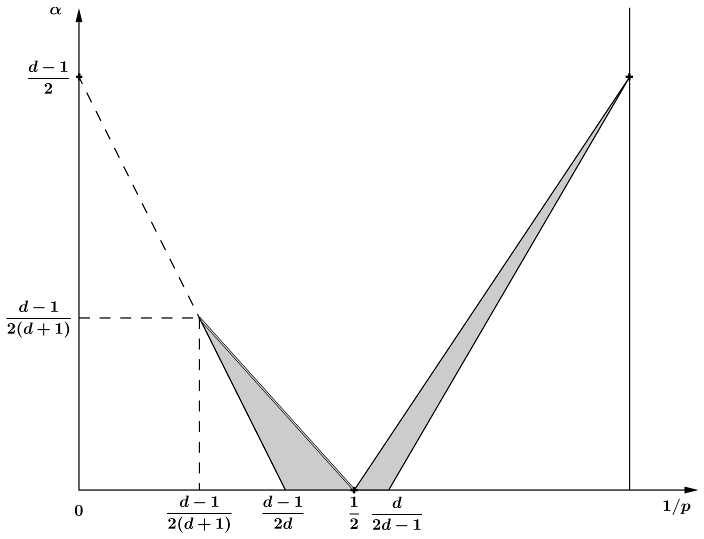

Figure 23: Open question of the boundedness of $\sigma_*^{2,\alpha}$ when $d \geq 3$.

for some $K \in \mathbb{N}$. By symmetry, we can assume that $\pi > x - u > y - v > 0$, and so, instead of (8.3), it is enough to show that

$$\int_{\mathbb{T}^2} \sup_{n \geq 1} \left| \int_I a(u,v) K_n^1(x-u, y-v) 1_{A_i}(x-u, y-v) \, du \, dv \right|^p dx \, dy \leq C_p$$

for all $i = 1, \dots, 10$, where

$$\begin{aligned}
A_1 &:= \{(x,y) : 0 < x \leq 2^{-K+5}, 0 < y < x < \pi, y \leq \pi/2\}, \\
A_2 &:= \{(x,y) : 2^{-K+5} < x < \pi, 0 < y \leq 2^{-K+2}, y \leq \pi/2\}, \\
A_3 &:= \{(x,y) : 2^{-K+5} < x < \pi, 2^{-K+2} < y \leq x/2, y \leq \pi/2\}, \\
A_4 &:= \{(x,y) : 2^{-K+5} < x < \pi, x/2 < y \leq x - 2^{-K+2}, y \leq \pi/2\}, \\
A_5 &:= \{(x,y) : 2^{-K+5} < x < \pi, x - 2^{-K+2} < y < x, y \leq \pi/2\} \\
A_6 &:= \{(x,y) : y > \pi/2, \pi - 2^{-K+5} \leq y < \pi, 0 < y < x < \pi\}, \\
A_7 &:= \{(x,y) : \pi/2 < y < \pi - 2^{-K+5}, \pi - 2^{-K+2} < x < \pi\}, \\
A_8 &:= \{(x,y) : \pi/2 < y < \pi - 2^{-K+5}, (\pi + y)/2 < x \leq \pi - 2^{-K+2}\}, \\
A_9 &:= \{(x,y) : \pi/2 < y < \pi - 2^{-K+5}, y + 2^{-K+2} < x \leq (\pi + y)/2\}, \\
A_{10} &:= \{(x,y) : \pi/2 < y < \pi - 2^{-K+5}, y < x \leq y + 2^{-K+2}\}.
\end{aligned}$$

These sets are similar to those in Theorem 6.1 (see Figure 19).

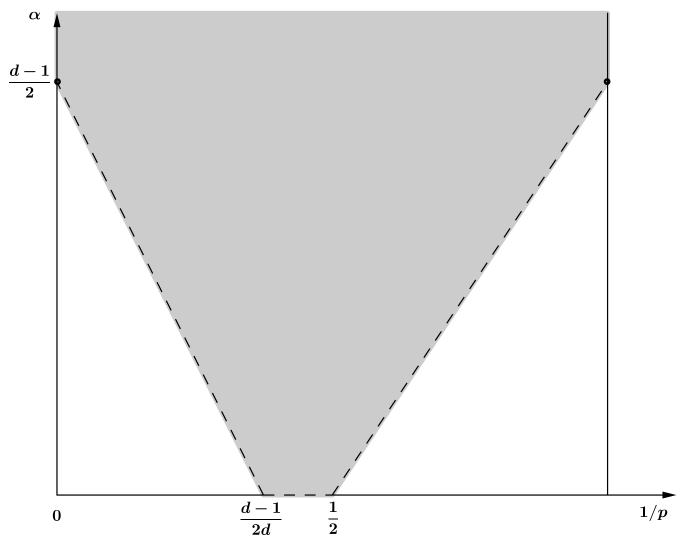

Figure 24: Almost everywhere convergence of $\sigma_n^{q,\alpha} f$, $f \in L_p(\mathbb{T}^d)$.

First of all, if $0 < x-u \leq 2^{-K+5}$, then $-2^{-K-1} < x \leq 2^{-K+6}$ and the same holds for y. If $\pi - 2^{-K+5} \leq y-v < \pi$, then $\pi - 2^{-K+6} < y \leq \pi + 2^{-K-1}$ and the same is true for x. By the definition of the atom and by Theorem 6.1,

$$\int_{\mathbb{T}^2} \sup_{n\geq 1} \Big| \int_I a(u,v) K_n^1(x-u,y-v) 1_{A_1 \cup A_6}(x-u,y-v) \, \mathrm{d}u \, \mathrm{d}v \Big|^p \, \mathrm{d}x \, \mathrm{d}y \leq C_p 2^{2K} 2^{-2K}.$$

Considering the set A_2, we have $2^{-K+5} < x-u < \pi$ and $0 < y-v \leq 2^{-K+2}$, thus $2^{-K+4} < x < \pi + 2^{-K-1}$ and $-2^{-K-1} < y \leq 2^{-K+3}$. Using (6.5), we conclude

$$\begin{aligned}
&\Big| \int_I a(u,v) K_n^1(x-u,y-v) 1_{A_2}(x-u,y-v) \, \mathrm{d}u \, \mathrm{d}v \Big| \\
&\leq C_p 2^{2K/p} \int_I (x-u-y+v)^{-3/2} (y-v)^{-1/2} 1_{A_2}(x-u,y-v) \, \mathrm{d}u \, \mathrm{d}v \\
&\leq C_p 2^{2K/p} 1_{\{2^{-K+4} < x < \pi + 2^{-K-1}\}} 1_{\{-2^{-K-1} < y \leq 2^{-K+3}\}} \\
&\qquad \int_I (x - 2^{-K+3})^{-3/2} (y-v)^{-1/2} \, \mathrm{d}u \, \mathrm{d}v \\
&\leq C_p 2^{2K/p - 3K/2} 1_{\{2^{-K+4} < x < \pi + 2^{-K-1}\}} 1_{\{-2^{-K-1} < y \leq 2^{-K+3}\}} (x - 2^{-K+3})^{-3/2}. \qquad (8.4)
\end{aligned}$$

Similarly, on A_7, $\pi/2 < y-v < \pi - 2^{-K+5}$ and $\pi - 2^{-K+2} < x-u < \pi$, thus $\pi/2 - 2^{-K-1} <$

$y < \pi - 2^{-K+4}$ and $\pi - 2^{-K+3} < x < \pi + 2^{-K-1}$. By (6.5),

$$\begin{aligned}
&\Big| \int_I a(u,v) K_n^1(x-u,y-v) 1_{A_7}(x-u,y-v)\,\mathrm{d}u\,\mathrm{d}v \Big| \\
&\leq C_p 2^{2K/p} \int_I (x-u-y+v)^{-3/2} (\pi - x + u)^{-1/2} 1_{A_7}(x-u,y-v)\,\mathrm{d}u\,\mathrm{d}v \\
&\leq C_p 2^{2K/p} 1_{\pi/2-2^{-K-1}<y<\pi-2^{-K+4}} 1_{\pi-2^{-K+3}<x<\pi+2^{-K-1}} \\
&\qquad \int_I (\pi - 2^{-K+3} - y)^{-3/2} (\pi - x + u)^{-1/2}\,\mathrm{d}u\,\mathrm{d}v \\
&\leq C_p 2^{2K/p-3K/2} 1_{\pi/2-2^{-K-1}<y<\pi-2^{-K+4}} 1_{\pi-2^{-K+3}<x<\pi+2^{-K-1}} (\pi - 2^{-K+3} - y)^{-3/2}.
\end{aligned}$$

If $p > 2/3$, then

$$\begin{aligned}
&\int_{\mathbb{T}^2} \sup_{n\geq 1} \Big| \int_I a(u,v) K_n^1(x-u,y-v) 1_{A_2\cup A_7}(x-u,y-v)\,\mathrm{d}u\,\mathrm{d}v \Big|^p \mathrm{d}x\,\mathrm{d}y \\
&\leq C_p 2^{2K-3Kp/2} \int_{2^{-K+4}}^{\pi+2^{-K-1}} \int_{-2^{-K-1}}^{2^{-K+3}} (x - 2^{-K+3})^{-3p/2}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + C_p 2^{2K-3Kp/2} \int_{\pi/2-2^{-K-1}}^{\pi-2^{-K+4}} \int_{\pi-2^{-K+3}}^{\pi+2^{-K-1}} (\pi - 2^{-K+3} - y)^{-3p/2}\,\mathrm{d}y\,\mathrm{d}x \\
&\leq C_p.
\end{aligned}$$

We may suppose that the center of I is zero, in other words $I := (-\nu,\nu)\times(-\nu,\nu)$. Let

$$A_1(u,v) := \int_{-\nu}^{u} a(t,v)\,\mathrm{d}t \qquad \text{and} \qquad A_2(u,v) := \int_{-\nu}^{v} A_1(u,t)\,\mathrm{d}t.$$

Observe that

$$|A_k(u,v)| \leq C_p 2^{K(2/p-k)}.$$

Integrating by parts, we can see that

$$\begin{aligned}
&\int_{I_1} a(u,v) K_n^1(x-u,y-v) 1_{A_3\cup A_8}(x-u,y-v)\,\mathrm{d}u \\
&= A_1(\nu,v) K_n^1(x-\nu,y-v) 1_{A_3\cup A_8}(x-\nu,y-v) \\
&\quad + \int_{-\nu}^{\nu} A_1(u,v) \partial_1 K_n^1(x-u,y-v) 1_{A_3\cup A_8}(x-u,y-v)\,\mathrm{d}u,
\end{aligned}$$

because $A_1(-\nu,v) = 0$. As $A_2(\nu,\nu) = \int_I a = 0$, integrating the first term again by parts, we obtain

$$\begin{aligned}
&\int_{I_1}\int_{I_2} a(u,v) K_n^1(x-u,y-v) 1_{A_3\cup A_8}(x-u,y-v)\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_{-\nu}^{\nu} A_2(\nu,v) \partial_2 K_n^1(x-\nu,y-v) 1_{A_3\cup A_8}(x-\nu,y-v)\,\mathrm{d}v \\
&\quad + \int_{I_1}\int_{I_2} A_1(u,v) \partial_1 K_n^1(x-u,y-v) 1_{A_3\cup A_8}(x-u,y-v)\,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

Since $x-u-y+v>(x-u)/2$ on the set A_3 and $x-u-y+v>(\pi-y+v)/2$ on the set A_8, we get from (6.9) that

$$\begin{aligned}
&\Big|\int_I a(u,v)K_n^1(x-u,y-v)1_{A_3}(x-u,y-v)\,\mathrm{d}u\,\mathrm{d}v\Big| \\
&\leq C_p 2^{2K/p-2K}\int_{I_2}(x-\nu)^{-1-\beta}(y-v)^{\beta-2}1_{A_3}(x-\nu,y-v)\,\mathrm{d}v \\
&\quad + C_p 2^{2K/p-K}\int_I (x-u)^{-1-\beta}(y-v)^{\beta-2}1_{A_3}(x-u,y-v)\,\mathrm{d}u\,\mathrm{d}v \\
&\leq C_p 2^{2K/p-3K}1_{\{2^{-K+4}<x<\pi+2^{-K-1}\}}1_{\{2^{-K+1}<y\leq x/2+2^{-K}\}} \\
&\quad (x-2^{-K-1})^{-1-\beta}(y-2^{-K-1})^{\beta-2} \qquad (8.5)
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\int_I a(u,v)K_n^1(x-u,y-v)1_{A_8}(x-u,y-v)\,\mathrm{d}u\,\mathrm{d}v\Big| \\
&\leq C_p 2^{2K/p-2K}\int_{I_2}(\pi-y+v)^{-1-\beta}(\pi-x+\nu)^{\beta-2}1_{A_8}(x-\nu,y-v)\,\mathrm{d}v \\
&\quad + C_p 2^{2K/p-K}\int_I (\pi-y+v)^{-1-\beta}(\pi-x+u)^{\beta-2}1_{A_8}(x-u,y-v)\,\mathrm{d}u\,\mathrm{d}v \\
&\leq C_p 2^{2K/p-3K}1_{\pi/2-2^{-K-1}<y<\pi-2^{-K+4}}1_{(\pi+y)/2-2^{-K}<x<\pi-2^{-K+1}} \\
&\quad (\pi-y-2^{-K-1})^{-1-\beta}(\pi-x-2^{-K-1})^{\beta-2}.
\end{aligned}$$

Choosing $\beta=1/2$, we conclude

$$\begin{aligned}
&\int_{\mathbb{T}^2}\sup_{n\geq 1}\Big|\int_I a(u,v)K_n^1(x-u,y-v)1_{A_3\cup A_8}(x-u,y-v)\,\mathrm{d}u\,\mathrm{d}v\Big|^p\,\mathrm{d}x\,\mathrm{d}y \\
&\leq C_p 2^{2K-3Kp}\Big(\int_{2^{-K+4}}^{\pi+2^{-K-1}}\int_{2^{-K+1}}^{x/2+2^{-K}}(x-2^{-K-1})^{-3p/2}(y-2^{-K-1})^{-3p/2}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\pi/2-2^{-K-1}}^{\pi-2^{-K+4}}\int_{(\pi+y)/2-2^{-K}}^{\pi-2^{-K+1}}(\pi-y-2^{-K-1})^{-3p/2}(\pi-x-2^{-K-1})^{-3p/2}\,\mathrm{d}x\,\mathrm{d}y\Big) \\
&\leq C_p 2^{2K-3Kp}2^{-K(1-3p/2)}2^{-K(1-3p/2)} \\
&\leq C_p.
\end{aligned}$$

Since $y-v>(x-u)/2$ on A_4 and $\pi-x+u>(\pi-y+v)/2$ on the set A_9, (6.9) implies

$$\begin{aligned}
&\Big|\int_I a(u,v)K_n^1(x-u,y-v)1_{A_4}(x-u,y-v)\,\mathrm{d}u\,\mathrm{d}v\Big| \\
&\leq C_p 2^{2K/p-2K}\int_{I_2}(x-\nu-y+v)^{-1-\beta}(x-u)^{\beta-2}1_{A_4}(x-\nu,y-v)\,\mathrm{d}v \\
&\quad + C_p 2^{2K/p-K}\int_I (x-u-y+v)^{-1-\beta}(x-u)^{\beta-2}1_{A_4}(x-u,y-v)\,\mathrm{d}u\,\mathrm{d}v
\end{aligned}$$

$$\leq C_p 2^{2K/p-3K} 1_{\{2^{-K+4}<x<\pi+2^{-K-1}\}} 1_{\{x/2-2^{-K}<y\leq x-2^{-K+1}\}} (x-y-2^{-K})^{-1-\beta}(x-2^{-K-1})^{\beta-2} \tag{8.6}$$

and

$$\begin{aligned}
&\Big|\int_I a(u,v)K_n^1(x-u,y-v)1_{A_9}(x-u,y-v)\,\mathrm{d}u\,\mathrm{d}v\Big| \\
&\leq C_p 2^{2K/p-2K}\int_{I_2}(x-\nu-y+v)^{-1-\beta}(\pi-y+v)^{\beta-2}1_{A_8}(x-\nu,y-v)\,\mathrm{d}v \\
&\quad + C_p 2^{2K/p-K}\int_I (x-u-y+v)^{-1-\beta}(\pi-y+v)^{\beta-2}1_{A_9}(x-u,y-v)\,\mathrm{d}u\,\mathrm{d}v \\
&\leq C_p 2^{2K/p-3K} 1_{\pi/2-2^{-K-1}<y<\pi-2^{-K+4}} 1_{y+2^{-K+1}<x<(\pi+y)/2+2^{-K}} \\
&\quad (x-y-2^{-K})^{-1-\beta}(\pi-y-2^{-K-1})^{\beta-2}
\end{aligned}$$

as before. Choosing again $\beta=1/2$, we obtain

$$\begin{aligned}
&\int_{\mathbb{T}^2}\sup_{n\geq 1}\Big|\int_I a(u,v)K_n^1(x-u,y-v)1_{A_4\cup A_9}(x-u,y-v)\,\mathrm{d}u\,\mathrm{d}v\Big|^p\,\mathrm{d}x\,\mathrm{d}y \\
&\leq C_p 2^{2K-3Kp}\Big(\int_{2^{-K+4}}^{\pi+2^{-K-1}}\int_{x/2-2^{-K}}^{x-2^{-K+1}}(x-y-2^{-K})^{-3p/2}(x-2^{-K-1})^{-3p/2}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\pi/2-2^{-K-1}}^{\pi-2^{-K+4}}\int_{y+2^{-K+1}}^{(\pi+y)/2+2^{-K}}(x-y-2^{-K})^{-3p/2}(\pi-y-2^{-K-1})^{-3p/2}\,\mathrm{d}x\,\mathrm{d}y\Big) \\
&\leq C_p 2^{2K-3Kp}2^{-K(1-3p/2)}2^{-K(1-3p/2)} \\
&\leq C_p.
\end{aligned}$$

Finally, inequality (6.7) implies

$$\begin{aligned}
&\Big|\int_I a(u,v)K_n^1(x-u,y-v)1_{A_5}(x-u,y-v)\,\mathrm{d}u\,\mathrm{d}v\Big| \\
&\leq C_p 2^{2K/p}\int_I (y-2^{-K-1})^{-2}1_{A_5}(x-u,y-v)\,\mathrm{d}u\,\mathrm{d}v \\
&\leq C_p 2^{2K/p-2K} 1_{\{2^{-K+4}<x<\pi+2^{-K-1}\}} 1_{\{x-2^{-K+3}<y\leq x+2^{-K}\}}(y-2^{-K-1})^{-2}
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\int_I a(u,v)K_n^1(x-u,y-v)1_{A_{10}}(x-u,y-v)\,\mathrm{d}u\,\mathrm{d}v\Big| \\
&\leq C_p 2^{2K/p}\int_I (\pi-x-2^{-K-1})^{-2}1_{A_{10}}(x-u,y-v)\,\mathrm{d}u\,\mathrm{d}v \\
&\leq C_p 2^{2K/p-2K} 1_{\pi/2-2^{-K-1}<y<\pi-2^{-K+4}} 1_{y-2^{-K}<x<y+2^{-K+3}}(\pi-x-2^{-K-1})^{-2},
\end{aligned}$$

hence

$$
\begin{aligned}
&\int_{\mathbb{T}^2} \sup_{n\geq 1} \Big| \int_I a(u,v) K_n^1(x-u,y-v) 1_{A_5\cup A_{10}}(x-u,y-v)\,\mathrm{d}u\,\mathrm{d}v \Big|^p \,\mathrm{d}x\,\mathrm{d}y \\
&\leq \; C_p 2^{2K-2Kp} \int_{2^{-K+4}}^{\pi+2^{-K-1}} \int_{x-2^{-K+3}}^{x+2^{-K}} (y-2^{-K-1})^{-2p}\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad + C_p 2^{2K-2Kp} \int_{\pi/2-2^{-K-1}}^{\pi-2^{-K+4}} \int_{y-2^{-K}}^{y+2^{-K+3}} (\pi-x-2^{-K-1})^{-2p}\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \; C_p 2^{2K-2Kp} \int_{2^{-K+3}}^{\pi+2^{-K+5}} \int_{y-2^{-K}}^{y+2^{-K+3}} (y-2^{-K-1})^{-2p}\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad + C_p 2^{2K-2Kp} \int_{\pi/2-2^{-K+1}}^{\pi-2^{-K+3}} \int_{x+2^{-K}}^{x-2^{-K+3}} (\pi-x-2^{-K-1})^{-2p}\,\mathrm{d}y\,\mathrm{d}x \\
&\leq \; C_p,
\end{aligned}
$$

which finishes the proof of (8.1).

Next, we will verify the weak inequality (8.2). To this end, we use Theorem 7.3 and prove that

$$\sup_{\rho>0} \rho^{2/3} \lambda(\sigma_*^1 a > \rho) \leq C$$

for all $H_{2/3}^{\square}$-atom a. Since

$$\rho^{2/3}\lambda(|g|>\rho) \leq \int |g|^{2/3},$$

taking into account the above inequalities, we have to show that

$$\rho^{2/3}\lambda\Big(\sup_{n\geq 1} \Big| \int_I a(u,v) K_n^1(x-u,y-v) 1_{A_i}(x-u,y-v)\,\mathrm{d}u\,\mathrm{d}v \Big| > \rho \Big) \leq C$$

for $i = 2,3,4,7,8,9$ and $\rho > 0$. We will prove the inequality for the first three sets. For the last three, the proof is similar.

If the expression in (8.4) with $p = 2/3$ is greater than ρ, then

$$1_{\{2^{-K+4}<x<\pi+2^{-K-1}\}}(x-2^{-K+3}) \leq C\rho^{-2/3} 2^{2K-K} 1_{\{-2^{-K-1}<y\leq 2^{-K+3}\}}$$

and

$$
\begin{aligned}
&\lambda\Big(\sup_{n\geq 1} \Big| \int_I a(u,v) K_n^1(x-u,y-v) 1_{A_2}(x-u,y-v)\,\mathrm{d}u\,\mathrm{d}v \Big| > \rho \Big) \\
&\qquad\qquad \leq \; C\rho^{-2/3} 2^K \int 1_{\{-2^{-K-1}<y\leq 2^{-K+3}\}}\,\mathrm{d}y \\
&\qquad\qquad = \; C\rho^{-2/3}.
\end{aligned}
$$

If (8.5) is greater than ρ, then

$$1_{\{2^{-K+1}<y\leq x/2+2^{-K}\}}(y-2^{-K-1}) \leq C\rho^{-\frac{1}{2-\beta}} 1_{\{2^{-K+4}<x<\pi+2^{-K-1}\}}(x-2^{-K-1})^{-\frac{1+\beta}{2-\beta}}.$$

Choosing β such that $-\frac{1+\beta}{2-\beta}+1<0$, i.e., $1/2<\beta\leq 1$, we obtain

$$\begin{aligned}
&\lambda\Big(\sup_{n\geq 1}\Big|\int_I a(u,v)K_n^1(x-u,y-v)1_{A_3}(x-u,y-v)\,\mathrm{d}u\,\mathrm{d}v\Big|>\rho\Big)\\
&\qquad\leq \int_{2^{-K+4}}^{\rho^{-1/3}+2^{-K-1}} x\,\mathrm{d}x + C\rho^{-\frac{1}{2-\beta}}\int_{\rho^{-1/3}+2^{-K-1}}^{2\pi}(x-2^{-K-1})^{-\frac{1+\beta}{2-\beta}}\,\mathrm{d}x\\
&\qquad\leq C\rho^{-2/3}+C\rho^{-\frac{1}{2-\beta}}\rho^{\frac{-1}{3}(-\frac{1+\beta}{2-\beta}+1)}\\
&\qquad= C\rho^{-2/3}.
\end{aligned}$$

For A_4, we get from (8.6) that

$$\begin{aligned}
&1_{\{x/2-2^{-K}<y\leq x-2^{-K+1}\}}(x-y-2^{-K})\\
&\qquad\qquad\leq C\rho^{-\frac{1}{1+\beta}}1_{\{2^{-K+4}<x<\pi+2^{-K-1}\}}(x-2^{-K-1})^{\frac{\beta-2}{1+\beta}}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\lambda\Big(\sup_{n\geq 1}\Big|\int_I a(u,v)K_n^1(x-u,y-v)1_{A_4}(x-u,y-v)\,\mathrm{d}u\,\mathrm{d}v\Big|>\rho\Big)\\
&\qquad\leq \int_{2^{-K+4}}^{\rho^{-1/3}+2^{-K-1}} x\,\mathrm{d}x + C\rho^{-\frac{1}{1+\beta}}\int_{\rho^{-1/3}+2^{-K-1}}^{2\pi}(x-2^{-K-1})^{\frac{\beta-2}{1+\beta}}\,\mathrm{d}x\\
&\qquad\leq C\rho^{-2/3}+C\rho^{-\frac{1}{1+\beta}}\rho^{\frac{-1}{3}(\frac{\beta-2}{1+\beta}+1)}\\
&\qquad= C\rho^{-2/3}.
\end{aligned}$$

Here, we have chosen β such that $\frac{\beta-2}{1+\beta}+1<0$, i.e., $0<\beta\leq 1$. The proof of the theorem is complete. ■

8.2.2 Proof for $q=1$ in higher dimensions ($d\geq 3$)

Proof of Theorem 8.1 for $q=1$ and $d\geq 3$. To prove (8.1), we will show that

$$\int_{\mathbb{T}^d}|\sigma_*^1 a(x)|^p\,\mathrm{d}x=\int_{\mathbb{T}^d}\sup_{n\geq 1}\Big|\int_I a(u)K_n^1(x-u)\,\mathrm{d}u\Big|^p\,\mathrm{d}x\leq C_p \tag{8.7}$$

for every $H_p^\square$-atom a, where $\frac{d}{d+1}<p<1$ and I is the support of the atom. We may suppose that a is a $H_p^\square$-atom with support $I=I_1\times\cdots\times I_d$ and

$$[-2^{-K-2},2^{-K-2}]\subset I_j\subset[-2^{-K-1},2^{-K-1}]\qquad (j=1,\ldots,d) \tag{8.8}$$

for some $K\in\mathbb{N}$. By symmetry, we can assume that

$$\pi>x_1-u_1>x_2-u_2>\cdots>x_d-u_d>0.$$

If $0 < x_1 - u_1 \leq 2^{-K+4}$, then $-2^{-K-1} < x_1 \leq 2^{-K+5}$ and if $\pi - 2^{-K+4} \leq x_d - u_d < \pi$, then $\pi - 2^{-K+5} \leq x_d < \pi + 2^{-K-1}$. By the definition of the atom and by Theorem 2.3,

$$\int_{\mathbb{T}^d} \sup_{n\geq 1} \Big| \int_I a(u) K_n^1(x-u) 1_{\{2^{-K+4}\geq x_1-u_1 > x_2-u_2>\cdots>x_d-u_d>0\}} \,\mathrm{d}u \Big|^p \,\mathrm{d}x \leq C_p 2^{Kd} 2^{-Kd}.$$

We get the same inequality if we integrate over the set

$$\{\pi > x_1 - u_1 > x_2 - u_2 > \cdots > x_d - u_d \geq \pi - 2^{-K+4}\}.$$

Hence, instead of (8.7), it is enough to prove that

$$\int_{\mathbb{T}^d} \sup_{n\geq 1} \Big| \int_I a(u) K_n^1(x-u) 1_{\mathcal{S}}(x-u) \,\mathrm{d}u \Big|^p \,\mathrm{d}x \leq C_p,$$

where

$$\mathcal{S} := \{x \in \mathbb{T}^d : \pi > x_1 > x_2 > \cdots > x_d > 0, x_1 > 2^{-K+4}, x_d < \pi - 2^{-K+4}\}.$$

Let

$$\mathcal{S}_{(i_l,j_l),k} := \begin{cases} x \in \mathcal{S} : x_{i_l} - x_{j_l} > 2^{-K+2}, l = 1, \ldots, k-1, x_{i_k} - x_{j_k} \leq 2^{-K+2}, & \text{if } k < d; \\ x \in \mathcal{S} : x_{i_l} - x_{j_l} > 2^{-K+2}, l = 1, \ldots, d-1, & \text{if } k = d \end{cases}$$

and

$$\mathcal{S}_{(i_l,j_l),k,1} := \begin{cases} x \in \mathcal{S}_{(i_l,j_l),k} : x_{j_k} > 2^{-K+2}, x_{j_{d-1}} \leq \pi/2, & \text{if } k < d; \\ x \in \mathcal{S}_{(i_l,j_l),k} : x_{j_{d-1}} > 2^{-K+2}, x_{j_{d-1}} \leq \pi/2, & \text{if } k = d, \end{cases}$$

$$\mathcal{S}_{(i_l,j_l),k,2} := \begin{cases} x \in \mathcal{S}_{(i_l,j_l),k} : x_{j_k} \leq 2^{-K+2}, x_{j_{d-1}} \leq \pi/2, & \text{if } k < d; \\ x \in \mathcal{S}_{(i_l,j_l),k} : x_{j_{d-1}} \leq 2^{-K+2}, x_{j_{d-1}} \leq \pi/2, & \text{if } k = d, \end{cases}$$

$$\mathcal{S}_{(i_l,j_l),k,3} := \begin{cases} x \in \mathcal{S}_{(i_l,j_l),k} : \pi - x_{i_k} > 2^{-K+2}, x_{j_{d-1}} > \pi/2, & \text{if } k < d; \\ x \in \mathcal{S}_{(i_l,j_l),k} : \pi - x_{i_{d-1}} > 2^{-K+2}, x_{j_{d-1}} > \pi/2, & \text{if } k = d, \end{cases}$$

$$\mathcal{S}_{(i_l,j_l),k,4} := \begin{cases} x \in \mathcal{S}_{(i_l,j_l),k} : \pi - x_{i_k} \leq 2^{-K+2}, x_{j_{d-1}} > \pi/2, & \text{if } k < d; \\ x \in \mathcal{S}_{(i_l,j_l),k} : \pi - x_{i_{d-1}} \leq 2^{-K+2}, x_{j_{d-1}} > \pi/2, & \text{if } k = d. \end{cases}$$

Then

$$\begin{aligned} &\int_{\mathbb{T}^d} \sup_{n\geq 1} \Big| \int_I a(u) K_n^1(x-u) 1_{\mathcal{S}}(x-u) \,\mathrm{d}u \Big|^p \,\mathrm{d}x \\ \leq\ & \sum_{(i_l,j_l)\in\mathcal{I}} \sum_{k=1}^{d} \sum_{m=1}^{4} \int_{\mathbb{T}^d} \sup_{n\geq 1} \Big| \int_I a(u) K^1_{n,(i_l,j_l)}(x-u) 1_{\mathcal{S}_{(i_l,j_l),k,m}}(x-u) \,\mathrm{d}u \Big|^p \,\mathrm{d}x. \end{aligned} \tag{8.9}$$

Step 1. In this step, we estimate the first $d-1$ summands in (8.9) on the set $\mathcal{S}_{(i_l,j_l),k,1}$ by

$$C_p 2^{Kd} \sum_{(i_l,j_l)\in\mathcal{I}} \sum_{k=1}^{d-1} \int_{\mathbb{T}^d} \sup_{n\geq 1} \Big(\int_I |K^1_{n,(i_l,j_l)}(x-u)| 1_{\mathcal{S}_{(i_l,j_l),k,1}}(x-u) \,\mathrm{d}u \Big)^p \,\mathrm{d}x.$$

Since $x_{i_{d-1}} - u_{i_{d-1}} - (x_{j_{d-1}} - u_{j_{d-1}}) \leq x_{i_l} - u_{i_l} - (x_{j_l} - u_{j_l})$, (6.12) implies

$$\begin{aligned}
&\int_I K^1_{n,(i_l,j_l)}(x-u) 1_{\mathcal{S}_{(i_l,j_l),k,1}}(x-u)\,\mathrm{d}u \\
&\leq C \int_I \prod_{l=1}^{d-2} (x_{i_l} - u_{i_l} - (x_{j_l} - u_{j_l}))^{-1-\beta} (x_{j_{d-1}} - u_{j_{d-1}})^{\beta(d-2)-2} 1_{\mathcal{S}_{(i_l,j_l),k,1}}(x-u)\,\mathrm{d}u \\
&\leq C \int_I \prod_{l=1}^{k-1} (x_{i_l} - u_{i_l} - (x_{j_l} - u_{j_l}))^{-1-\beta} \prod_{l=k}^{d-2} (x_{i_l} - u_{i_l} - (x_{j_l} - u_{j_l}))^{-1-\beta+1/(d-k)} \\
&\quad (x_{i_{d-1}} - u_{i_{d-1}} - (x_{j_{d-1}} - u_{j_{d-1}}))^{1/(d-k)-1} \\
&\quad (x_{j_{d-1}} - u_{j_{d-1}})^{\beta(d-2)-2} 1_{\mathcal{S}_{(i_l,j_l),k,1}}(x-u)\,\mathrm{d}u.
\end{aligned}$$

In the first product, we estimate the factors and in the second one, we integrate. More exactly,

$$x_{i_l} - u_{i_l} - (x_{j_l} - u_{j_l}) > x_{i_l} - x_{j_l} - 2^{-K}, \qquad l = 1, \ldots, k-1,$$

and

$$x_{j_{d-1}} - u_{j_{d-1}} > x_{j_{d-1}} - 2^{-K-1}.$$

For the integration, we first choose the index i_{d-1} $(= i'_{d-1})$ and then i_{d-2} if $i_{d-2} \neq i_{d-1}$ or j_{d-2} if $j_{d-2} \neq j_{d-1}$. Repeating this process, we get a sequence $(i'_l, l = k, \ldots, d-1)$. Note that $i'_l \neq i'_m$ if $l \neq m$, $l, m = k, \ldots, d-1$. We integrate the term

$$(x_{i_k} - u_{i_k} - (x_{j_k} - u_{j_k}))^{-1-\beta+1/(d-k)} \quad \text{in} \quad u_{i'_k},$$

the term

$$(x_{i_{k+1}} - u_{i_{k+1}} - (x_{j_{k+1}} - u_{j_{k+1}}))^{-1-\beta+1/(d-k)} \quad \text{in} \quad u_{i'_{k+1}},$$

..., and finally the term

$$(x_{i_{d-1}} - u_{i_{d-1}} - (x_{j_{d-1}} - u_{j_{d-1}}))^{-1-\beta+1/(d-k)} \quad \text{in} \quad u_{i'_{d-1}}.$$

Since $x_{i_l} - u_{i_l} - (x_{j_l} - u_{j_l}) \leq 2^{-K+2}$ $(l = k, \ldots, d-1)$ and we can choose β such that $\beta < 1/(d-1)$, we have

$$\int_{I_l} (x_{i_l} - u_{i_l} - (x_{j_l} - u_{j_l}))^{-1-\beta+1/(d-k)} 1_{\mathcal{S}_{(i_l,j_l),k,1}}(x-u)\,\mathrm{d}u_{i_l} (\text{or}\,\mathrm{d}u_{j_l}) \leq C 2^{-K(1/(d-k)-\beta)},$$

$(l = k, \ldots, d-1)$. If $x - u \in \mathcal{S}_{(i_l,j_l),k,1}$, then

$$x_{i_l} - x_{j_l} > 2^{-K+2} + u_{i_l} - u_{j_l} > 2^{-K+2} - 2^{-K} > 2^{-K+1}, \qquad l = 1, \ldots, k-1,$$

and

$$x_{j_{d-1}} > 2^{-K+2} + u_{j_{d-1}} > 2^{-K+2} - 2^{-K-1} > 2^{-K+1}.$$

Moreover,

$$x_{i_l} - x_{j_l} \leq 2^{-K+2} + u_{i_l} - u_{j_l} < 2^{-K+3}, \qquad l = k, \ldots, d-1,$$

and

$$x_{i_l} - x_{j_l} > u_{i_l} - u_{j_l} > -2^{-K}, \qquad l = k, \ldots, d-1.$$

Hence

$$\begin{aligned}
&\int_I K^1_{n,(i_l,j_l)}(x-u) 1_{\mathcal{S}_{(i_l,j_l),k,1}}(x-u)\,\mathrm{d}u \\
&\leq C 2^{-Kk} 2^{-K(1/(d-k)-\beta)(d-k-1)} 2^{-K/(d-k)} \prod_{l=1}^{k-1} (x_{i_l} - x_{j_l} - 2^{-K})^{-1-\beta} 1_{\{x_{i_l}-x_{j_l}>2^{-K+1}\}} \\
&\quad \prod_{l=k}^{d-1} 1_{\{-2^{-K}<x_{i_l}-x_{j_l}<2^{-K+3}\}} (x_{j_{d-1}} - 2^{-K-1})^{\beta(d-2)-2} 1_{\{x_{j_{d-1}}>2^{-K+1}\}}
\end{aligned}$$

and

$$\begin{aligned}
&C_p 2^{Kd} \sum_{(i_l,j_l)\in\mathcal{I}} \sum_{k=1}^{d-1} \int_{\mathbb{T}^d} \Big(\int_I K^1_{n,(i_l,j_l)}(x-u) 1_{\mathcal{S}_{(i_l,j_l),k,1}}(x-u)\,\mathrm{d}u \Big)^p \,\mathrm{d}x \\
&\leq C_p 2^{Kd} 2^{-Kkp} 2^{-K(1-\beta(d-k-1))p} \sum_{(i_l,j_l)\in\mathcal{I}} \sum_{k=1}^{d-1} \\
&\quad \int_{\mathbb{T}^d} \prod_{l=1}^{k-1} (x_{i_l} - x_{j_l} - 2^{-K})^{-(1+\beta)p} 1_{\{x_{i_l}-x_{j_l}>2^{-K+1}\}} \\
&\quad \prod_{l=k}^{d-1} 1_{\{-2^{-K}<x_{i_l}-x_{j_l}<2^{-K+3}\}} (x_{j_{d-1}} - 2^{-K-1})^{(\beta(d-2)-2)p} 1_{\{x_{j_d}>2^{-K+1}\}}\,\mathrm{d}x \\
&\leq C_p 2^{Kd} 2^{-Kkp} 2^{-K(1-\beta(d-k-1))p} \\
&\quad \sum_{(i_l,j_l)\in\mathcal{I}} \sum_{k=1}^{d-1} 2^{-K(1-(1+\beta)p)(k-1)} 2^{-K(d-k)} 2^{-K(1-(2-\beta(d-2))p)} \\
&\leq C_p,
\end{aligned}$$

whenever $1-(1+\beta)p < 0$, $1-(2-\beta(d-2))p < 0$ and $\beta < 1/(d-1)$. Since β can be arbitrarily near to $1/(d-1)$, we obtain $p > \frac{d-1}{d}$.

Step 2. For $k = d$, we have $x_{i_l} - x_{j_l} > 2^{-K+1}$ for all $l = 1, \ldots, d-1$ and $x_{j_{d-1}} > 2^{-K+1}$. Suppose again that the center of I is zero, in other words $I := \prod_{j=1}^d (-\nu,\nu)$. If we introduce

$$A_1(u) := \int_{-\nu}^{u_1} a(t_1, u_2, \ldots, u_d)\,\mathrm{d}t_1$$

and

$$A_k(u) := \int_{-\nu}^{u_k} A_{k-1}(u_1, \ldots, u_{k-1}, t_k, u_{k+1}, \ldots, u_d)\,\mathrm{d}t_k \qquad (2 \leq k \leq d), \tag{8.10}$$

then

$$|A_k(u)| \leq C_p 2^{K(d/p-k)}.$$

Integrating by parts, we can see that

$$\begin{aligned}
&\int_{I_1} a(u) K^1_{n,(i_l,j_l)}(x-u) 1_{\mathcal{S}_{(i_l,j_l),d,1}}(x-u)\,\mathrm{d}u_1 \\
&\quad = \; A_1(\nu,u_2,\ldots,u_d)(K^1_{n,(i_l,j_l)} 1_{\mathcal{S}_{(i_l,j_l),d,1}})(x_1-\nu,x_2-u_2,\ldots,x_d-u_d) \\
&\qquad + \int_{-\nu}^{\nu} A_1(u)\partial_1 K^1_{n,(i_l,j_l)}(x-u) 1_{\mathcal{S}_{(i_l,j_l),d,1}}(x-u)\,\mathrm{d}u_1,
\end{aligned}$$

because $A_1(-\nu,u_2,\ldots,u_d)=0$. Integrating the first term again by parts, we obtain

$$\begin{aligned}
&\int_{I_1}\int_{I_2} a(u) K^1_{n,(i_l,j_l)}(x-u) 1_{\mathcal{S}_{(i_l,j_l),d,1}}(x-u)\,\mathrm{d}u_1 du_2 \\
&= \; A_2(\nu,\nu,u_3,\ldots,u_d)(K^1_{n,(i_l,j_l)} 1_{\mathcal{S}_{(i_l,j_l),d,1}})(x_1-\nu,x_2-\nu,x_3-u_3,\ldots,x_d-u_d) \\
&\quad + \int_{-\nu}^{\nu} A_2(\nu,u_2,\ldots,u_d)(\partial_2 K^1_{n,(i_l,j_l)} 1_{\mathcal{S}_{(i_l,j_l),d,1}})(x_1-\nu,x_2-u_2,\ldots,x_d-u_d)\,\mathrm{d}u_2 \\
&\quad + \int_{I_1}\int_{I_2} A_1(u)(\partial_1 K^1_{n,(i_l,j_l)} 1_{\mathcal{S}_{(i_l,j_l),d,1}})(x-u)\,\mathrm{d}u_1 du_2.
\end{aligned}$$

Since $A_d(\nu,\ldots,\nu)=\int_I a=0$, repeating this process, we get that

$$\begin{aligned}
&\int_I a(u) K^1_{n,(i_l,j_l)}(x-u) 1_{\mathcal{S}_{(i_l,j_l),d,1}}(x-u)\,\mathrm{d}u \qquad (8.11)\\
&= \; \sum_{k=1}^{d} \int_{I_k}\cdots\int_{I_d} A_k(\nu,\ldots\nu,u_k,\ldots,u_d) \\
&\quad (\partial_k K^1_{n,(i_l,j_l)} 1_{\mathcal{S}_{(i_l,j_l),d,1}})(x_1-\nu,\ldots,x_1-\nu,x_k-u_k,\ldots,x_d-u_d)\,\mathrm{d}u_k\cdots du_d.
\end{aligned}$$

Inequality (6.13) implies

$$\begin{aligned}
&\Big|\int_I a(u) K^1_{n,(i_l,j_l)}(x-u) 1_{\mathcal{S}_{(i_l,j_l),d,1}}(x-u)\,\mathrm{d}u\Big| \\
&\quad \le \; C_p \sum_{k=1}^{d} 2^{K(d/p-k)} \int_{I_k}\cdots\int_{I_d} \prod_{l=1}^{k-1}(x_{i_l}-\nu-(x_{j_l}-\nu))^{-1-\beta} \\
&\qquad \prod_{l=k}^{d-1}(x_{i_l}-u_{i_l}-(x_{j_l}-u_{j_l}))^{-1-\beta}(x_{j_{d-1}}-u_{j_{d-1}})^{\beta(d-1)-2} \\
&\qquad 1_{\mathcal{S}_{(i_l,j_l),d,1}}(x-u)\,\mathrm{d}u_k\cdots du_d \qquad (8.12)\\
&\quad \le \; C_p 2^{K(d/p-k)} 2^{-K(d-k+1)} \prod_{l=1}^{d-1}(x_{i_l}-x_{j_l}-2^{-K})^{-1-\beta} 1_{\{x_{i_l}-x_{j_l}>2^{-K+1}\}} \\
&\qquad (x_{j_{d-1}}-2^{-K-1})^{\beta(d-1)-2} 1_{\{x_{j_{d-1}}>2^{-K+1}\}}
\end{aligned}$$

and

$$\sum_{(i_l,j_l)\in\mathcal{I}} \int_{\mathbb{T}^d} \sup_{n\geq 1} \Big| \int_I a(u) K^1_{n,(i_l,j_l)}(x-u) 1_{\mathcal{S}_{(i_l,j_l),d,1}}(x-u)\,\mathrm{d}u\Big|^p \,\mathrm{d}x$$

$$\begin{aligned}
&\leq C_p 2^{Kd} 2^{-Kdp-Kp} \sum_{(i_l,j_l)\in\mathcal{I}} \int_{\mathbb{T}^d} \prod_{l=1}^{d-1} (x_{i_l}-x_{j_l}-2^{-K})^{-(1+\beta)p} 1_{\{x_{i_l}-x_{j_l}>2^{-K+1}\}} \\
&\qquad (x_{j_{d-1}}-2^{-K-1})^{(\beta(d-1)-2)p} 1_{\{x_{j_{d-1}}>2^{-K+1}\}}\,\mathrm{d}x \\
&\leq C_p 2^{Kd} 2^{-Kdp-Kp} \sum_{(i_l,j_l)\in\mathcal{I}} 2^{-K(1-(1+\beta)p)(d-1)} 2^{-K(1-(2-\beta(d-1))p)} \\
&\leq C_p,
\end{aligned}$$

whenever $1-(1+\beta)p<0$, $1-(2-\beta(d-1))p<0$ and $\beta<2/(d-1)$. In other words

$$p > \frac{d}{d+1}.$$

Step 3. Now we investigate the first $d-1$ summands and the set $\mathcal{S}_{(i_l,j_l),k,2}$ in (8.9). In this case,

$$x_{i_k}-u_{i_k}-(x_{j_k}-u_{j_k}) \leq 2^{-K+2}$$

and $x_{j_{d-1}} < x_{i_k} < 2^{-K+4}$. Note that $x_{j_{d-1}} \leq 2^{-K+2}$ for $k=d$. Observe that $k=1$ can be excluded, because $i_1=1$, $j_1=d$ and this contradicts the definition of $\mathcal{S}$, where $x_1>2^{-K+4}$. Using the method of Step 1, we get that

$$\int_I K^1_{n,(i_l,j_l)}(x-u) 1_{\mathcal{S}_{(i_l,j_l),k,2}}(x-u)\,\mathrm{d}u$$

$$\begin{aligned}
&\leq C \int_I \prod_{l=1}^{d-2} (x_{i_l}-u_{i_l}-(x_{j_l}-u_{j_l}))^{-1-\beta} (x_{j_{d-1}}-u_{j_{d-1}})^{\beta(d-2)-2} 1_{\mathcal{S}_{(i_l,j_l),k,1}}(x-u)\,\mathrm{d}u \\
&\leq C \int_I \prod_{l=1}^{k-1} (x_{i_l}-u_{i_l}-(x_{j_l}-u_{j_l}))^{-1-\beta} \prod_{l=k}^{d-2} (x_{i_l}-u_{i_l}-(x_{j_l}-u_{j_l}))^{-1-\beta+(1-\epsilon)/(d-k-1)} \\
&\qquad (x_{i_{d-1}}-u_{i_{d-1}}-(x_{j_{d-1}}-u_{j_{d-1}}))^{\epsilon-1} (x_{j_{d-1}}-u_{j_{d-1}})^{\beta(d-2)-2} 1_{\mathcal{S}_{(i_l,j_l),k,2}}(x-u)\,\mathrm{d}u \\
&\leq C 2^{-K(k-1)} 2^{-K((1-\epsilon)/(d-k-1)-\beta)(d-k-1)} 2^{-K\epsilon} \\
&\qquad \prod_{l=1}^{k-1} (x_{i_l}-x_{j_l}-2^{-K})^{-1-\beta} 1_{\{x_{i_l}-x_{j_l}>2^{-K+1}\}} \\
&\qquad \prod_{l=k}^{d-1} 1_{\{-2^{-K}<x_{i_l}-x_{j_l}<2^{-K+3}\}} 2^{-K(\beta(d-2)-1)} 1_{\{x_{j_{d-1}}\leq 2^{-K+4}\}}
\end{aligned}$$

and so

$$\sum_{(i_l,j_l)\in\mathcal{I}} \sum_{k=1}^{d-1} \int_{\mathbb{T}^d} \sup_{n\geq 1} \Big| \int_I a(u) K^1_{n,(i_l,j_l)}(x-u) 1_{\mathcal{S}_{(i_l,j_l),k,2}}(x-u)\,\mathrm{d}u\Big|^p \,\mathrm{d}x$$

$$
\begin{aligned}
&\le C_p 2^{Kd} 2^{-K(k-1)p} 2^{-K(1-\beta(d-k-1))p} 2^{-K(\beta(d-2)-1)p} \\
&\qquad \sum_{(i_l,j_l)\in\mathcal{I}} \sum_{k=2}^{d-1} \int_{\mathbb{T}^d} \prod_{l=1}^{k-1} (x_{i_l}-x_{j_l}-2^{-K})^{-(1+\beta)p} 1_{\{x_{i_l}-x_{j_l}>2^{-K+1}\}} \\
&\qquad \prod_{l=k}^{d-1} 1_{\{-2^{-K}<x_{i_l}-x_{j_l}<2^{-K+3}\}} 1_{\{x_{j_d}\le 2^{-K+4}\}} \,\mathrm{d}x \\
&\le C_p 2^{Kd} 2^{-K(k-1)p} 2^{-K(1-\beta(d-k-1))p} 2^{-K(\beta(d-2)-1)p} \\
&\qquad \sum_{(i_l,j_l)\in\mathcal{I}} \sum_{k=2}^{d-1} 2^{-K(1-(1+\beta)p)(k-1)} 2^{-K(d-k+1)} \\
&\le C_p.
\end{aligned}
$$

We have used that $0<\epsilon<1$, $1/(d-2)<\beta<(1-\epsilon)/(d-3)$ and $1-(1+\beta)p<0$. (Since the term for $k=1$ is zero, we can suppose here that $d>3$.) This implies that $\epsilon<1/(d-2)$. Since β can be chosen arbitrarily near to $(1-\epsilon)/(d-3)$ and ϵ to 0, we obtain

$$
p > \frac{d-3}{d-2}.
$$

Step 4. Here, we consider the dth summand and the set $\mathcal{S}_{(i_l,j_l),d,2}$ in (8.9). Similarly to (8.12),

$$
\begin{aligned}
&\Big| \int_I a(u) K^1_{n,(i_l,j_l)}(x-u) 1_{\mathcal{S}_{(i_l,j_l),d,2}}(x-u)\,\mathrm{d}u \Big| \\
&\le C_p \sum_{k=1}^{d} 2^{K(d/p-k)} \int_{I_k} \cdots \int_{I_d} \prod_{l=1}^{k-1} (x_{i_l}-\nu-(x_{j_l}-\nu))^{-1-\beta} \\
&\qquad \prod_{l=k}^{d-1} (x_{i_l}-u_{i_l}-(x_{j_l}-u_{j_l}))^{-1-\beta} \\
&\qquad (x_{j_{d-1}}-u_{j_{d-1}})^{\beta(d-1)-2} 1_{\mathcal{S}_{(i_l,j_l),d,2}}(x-u)\,\mathrm{d}u_k \cdots \mathrm{d}u_d \\
&\le C_p 2^{K(d/p-k)} 2^{-K(d-k)} \prod_{l=1}^{d-1} (x_{i_l}-x_{j_l}-2^{-K})^{-1-\beta} 1_{\{x_{i_l}-x_{j_l}>2^{-K+1}\}} \\
&\qquad 2^{-K(\beta(d-1)-1)} 1_{\{x_{j_{d-1}}\le 2^{-K+4}\}},
\end{aligned} \tag{8.13}
$$

thus

$$
\begin{aligned}
&\sum_{(i_l,j_l)\in\mathcal{I}} \int_{\mathbb{T}^d} \sup_{n\ge 1} \Big| \int_I a(u) K^1_{n,(i_l,j_l)}(x-u) 1_{\mathcal{S}_{(i_l,j_l),d,2}}(x-u)\,\mathrm{d}u \Big|^p \,\mathrm{d}x \\
&\le C_p 2^{Kd} 2^{-Kdp} \sum_{(i_l,j_l)\in\mathcal{I}} \int_{\mathbb{T}^d} \prod_{l=1}^{d-1} (x_{i_l}-x_{j_l}-2^{-K})^{-(1+\beta)p} 1_{\{x_{i_l}-x_{j_l}>2^{-K+1}\}}
\end{aligned}
$$

$$
\begin{aligned}
& 2^{-K(\beta(d-1)-1)p} 1_{\{x_{j_{d-1}} \leq 2^{-K+4}\}} \, \mathrm{d}x \\
\leq \; & C_p 2^{Kd} 2^{-Kdp} \sum_{(i_l,j_l)\in\mathcal{I}} 2^{-K(1-(1+\beta)p)(d-1)} 2^{-K(\beta(d-1)-1)p} 2^{-K} \\
\leq \; & C_p,
\end{aligned}
$$

whenever $p > \frac{d}{d+1}$. The corresponding inequalities for the sets $\mathcal{S}_{(i_l,j_l),k,3}$ and $\mathcal{S}_{(i_l,j_l),k,4}$ can be proved similarly. This proves (8.7) and (8.1).

By Theorem 7.3, to prove the weak inequality (8.2), it is enough to show that

$$
\sup_{\rho>0} \rho^{d/(d+1)} \lambda(\sigma_*^1 a > \rho) \leq C
$$

for all $H_{d/(d+1)}^\square$-atoms a. Observe that

$$
\rho^{d/(d+1)} \lambda(|g| > \rho) \leq \int |g|^{d/(d+1)} \tag{8.14}
$$

implies that we have to show only that

$$
\rho^{d/(d+1)} \lambda\Big(\sup_{n\geq 1} \Big| \int_I a(u) K_{n,(i_l,j_l)}^1(x-u) 1_{\mathcal{S}_{(i_l,j_l),k,l}}(x-u) \, \mathrm{d}u \Big| > \rho\Big) \leq C \tag{8.15}
$$

for $k=d$, $l=1$ or $k=d$, $l=2$ and all $\rho>0$.

Step 5. Suppose that $k=d$ and $l=1$. We can see as in Lemma 6.14 that

$$
\begin{aligned}
|\partial_q K_{n,(i_l,j_l)}^1(x)| \;\leq\; & C \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1-\beta_l} x_{j_{d-1}}^{d-3+\sum_{l=1}^{d-1}(\beta_l-1)} 1_{\{x_{j_{d-1}} \leq \pi/2\}} \\
& + C \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1-\beta_l} \\
& \quad (\pi - x_{i_{d-1}})^{d-3+\sum_{l=1}^{d-1}(\beta_l-1)} 1_{\{x_{j_{d-1}} > \pi/2\}},
\end{aligned} \tag{8.16}
$$

whenever $q=1,\ldots,d$, $0<\beta_l<1$, $d-3+\sum_{l=1}^{d-1}(\beta_l-1)<0$ and $\beta_l + \frac{\beta_{d-1}}{d-2} < 1$ for all $l=1,\ldots,d-2$. We get similarly to (8.12) in Step 2 with $p=d/(d+1)$ that

$$
\begin{aligned}
& \Big| \int_I a(u) K_{n,(i_l,j_l)}^1(x-u) 1_{\mathcal{S}_{(i_l,j_l),d,1}}(x-u) \, \mathrm{d}u \Big| \\
\leq \; & C \sum_{k=1}^{d} 2^{K(d+1-k)} \int_{I_k} \cdots \int_{I_d} \prod_{l=1}^{k-1} (x_{i_l} - \nu - (x_{j_l} - \nu))^{-1-\beta_l} \\
& \prod_{l=k}^{d-1} (x_{i_l} - u_{i_l} - (x_{j_l} - u_{j_l}))^{-1-\beta_l} (x_{j_{d-1}} - u_{j_{d-1}})^{d-3+\sum_{l=1}^{d-1}(\beta_l-1)} \\
& 1_{\mathcal{S}_{(i_l,j_l),d,1}}(x-u) \, \mathrm{d}u_k \cdots \mathrm{d}u_d \\
\leq \; & C 2^{K(d+1-k)} 2^{-K(d-k+1)} \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l} - 2^{-K})^{-1-\beta_l} 1_{\{x_{i_l} - x_{j_l} > 2^{-K+1}\}} \\
& (x_{j_{d-1}} - 2^{-K-1})^{d-3+\sum_{l=1}^{d-1}(\beta_l-1)} 1_{\{x_{j_{d-1}} > 2^{-K+1}\}}.
\end{aligned}
$$

If this is greater than ρ, then

$$\begin{aligned}
&(x_{i_1}-x_{j_1}-2^{-K})1_{\{x_{i_1}-x_{j_1}>2^{-K+1}\}} \\
&\quad\leq C\rho^{\frac{-1}{1+\beta_1}}\prod_{l=2}^{d-1}(x_{i_l}-x_{j_l}-2^{-K})^{\frac{-1-\beta_l}{1+\beta_1}}1_{\{x_{i_l}-x_{j_l}>2^{-K+1}\}} \\
&\qquad(x_{j_{d-1}}-2^{-K-1})^{\frac{d-3+\sum_{l=1}^{d-1}(\beta_l-1)}{1+\beta_1}}1_{\{x_{j_{d-1}}>2^{-K+1}\}}.
\end{aligned}$$

We know that either $i_1>i_2$ or $j_1<j_2$. Suppose that the first inequality is satisfied and let $x':=(x_1,\ldots,x_{i_1-1},x_{i_1+1},\ldots,x_d)$. Let

$$\mathcal{R}_k:=\begin{cases} x':x_{i_l}-x_{j_l}-2^{-K}>\rho^{\frac{-1}{d+1}}, l=2,\ldots,k-1; \\ x':x_{i_l}-x_{j_l}-2^{-K}\leq\rho^{\frac{-1}{d+1}}, l=k,\ldots,d-1\end{cases}$$

and

$$\begin{aligned}
\mathcal{R}_{k,1} &:= \{x'\in\mathcal{R}_k : x_{j_{d-1}}-2^{-K+1}>\rho^{\frac{-1}{d+1}}\}, \\
\mathcal{R}_{k,2} &:= \{x'\in\mathcal{R}_k : x_{j_{d-1}}-2^{-K+1}\leq\rho^{\frac{-1}{d+1}}\}
\end{aligned}$$

for some $k=2,\ldots,d-1$. We may assume that $x'\in\mathcal{R}_{k,1}$ or $x'\in\mathcal{R}_{k,2}$. In both cases, let $\beta_2=\beta_3=\cdots=\beta_{k-1}$ and $\beta_k=\beta_{k+1}=\cdots=\beta_{d-1}$. Then

$$\begin{aligned}
&\lambda\Big(\sup_{n\geq1}\Big|\int_I a(u)K^1_{n,(i_l,j_l)}(x-u)1_{\mathcal{S}_{(i_l,j_l),d,1}}(x-u)\,\mathrm{d}u\Big|1_{\mathcal{R}_{k,1}}(x')>\rho\Big) \\
&\quad\leq C\rho^{\frac{-1}{1+\beta_1}}\int\prod_{l=2}^{k-1}(x_{i_l}-x_{j_l}-2^{-K})^{\frac{-1-\beta_2}{1+\beta_1}}1_{\{x_{i_l}-x_{j_l}>2^{-K+1}\}} \\
&\qquad\prod_{l=k}^{d-1}(x_{i_l}-x_{j_l}-2^{-K})^{\frac{-1-\beta_k}{1+\beta_1}}1_{\{x_{i_l}-x_{j_l}>2^{-K+1}\}} \\
&\qquad(x_{j_{d-1}}-2^{-K-1})^{\frac{d-3+(\beta_1-1)+(\beta_2-1)(k-2)+(\beta_k-1)(d-k)}{1+\beta_1}}1_{\{x_{j_{d-1}}>2^{-K+1}\}}1_{\mathcal{R}_{k,1}}(x')\,\mathrm{d}x' \\
&\quad\leq C\rho^{\frac{-1}{1+\beta_1}}\rho^{\frac{-1}{d+1}(\frac{-1-\beta_2}{1+\beta_1}+1)(k-2)}\rho^{\frac{-1}{d+1}(\frac{-1-\beta_k}{1+\beta_1}+1)(d-k)}\rho^{\frac{-1}{d+1}(\frac{-2+\beta_1+\beta_2(k-2)+\beta_k(d-k)}{1+\beta_1}+1)} \\
&\quad\leq C\rho^{\frac{-d}{d+1}}, \qquad (8.17)
\end{aligned}$$

whenever

$$1+\beta_1<1+\beta_2,\ 1+\beta_1>1+\beta_k,\ 1+\beta_1<2-\beta_1-\beta_2(k-2)-\beta_k(d-k).$$

Substituting $\beta_i=1/d+\mu_i$ with small μ_i in these inequalities, we obtain

$$\mu_1<\mu_2,\ \mu_1>\mu_k,\ 0<-2\mu_1-\mu_2(k-2)-\mu_k(d-k).$$

If $\mu_1<0$, $\mu_2>0$ and $\mu_k<0$ are small enough, then these inequalities are satisfied for a fixed k.

For the set $\mathcal{R}_{k,2}$, we get the same inequality as in (8.17) if

$$1+\beta_1<1+\beta_2,\ 1+\beta_1>1+\beta_k,\ 0<2-\beta_1-\beta_2(k-2)-\beta_k(d-k)<1+\beta_1,$$

or, after the substitution,

$$\mu_1<\mu_2,\ \mu_1>\mu_k,\ -\frac{1}{d}-1-\mu_1<-2\mu_1-\mu_2(k-2)-\mu_k(d-k)<0.$$

These inequalities are again satisfied if $\mu_1>0$, $\mu_2>0$ and $\mu_k<0$ are small enough, which shows (8.15) for $k=d$, $l=1$.

Step 6. Let $k=d$ and $l=2$. Setting $\beta_1=1/d+\epsilon$, $\epsilon>0$ and $\beta_l=\beta$, $l=2,\ldots,d-1$ in (8.16), we obtain

$$\begin{aligned}
&|\partial_q K^1_{n,(i_l,j_l)}(x)| \\
\leq\ & C(x_{i_1}-x_{j_1})^{-\frac{d+1}{d}-\epsilon}\prod_{l=2}^{d-1}(x_{i_l}-x_{j_l})^{-1-\beta}x_{j_{d-1}}^{\beta(d-2)+\frac{1}{d}+\epsilon-2}1_{\{x_{j_{d-1}}\leq\pi/2\}} \\
&+C(x_{i_1}-x_{j_1})^{-\frac{d+1}{d}-\epsilon}\prod_{l=2}^{d-1}(x_{i_l}-x_{j_l})^{-1-\beta}(\pi-x_{i_{d-1}})^{\beta(d-2)+\frac{1}{d}+\epsilon-2}1_{\{x_{j_{d-1}}>\pi/2\}} \\
\leq\ & C(x_{i_1}-x_{j_1})^{-\frac{d+1}{d}}\prod_{l=2}^{d-1}(x_{i_l}-x_{j_l})^{-1-\beta-\frac{\epsilon}{d-2}}x_{j_{d-1}}^{\beta(d-2)+\frac{1}{d}+\epsilon-2}1_{\{x_{j_{d-1}}\leq\pi/2\}} \\
&+C(x_{i_1}-x_{j_1})^{-\frac{d+1}{d}}\prod_{l=2}^{d-1}(x_{i_l}-x_{j_l})^{-1-\beta-\frac{\epsilon}{d-2}}(\pi-x_{i_{d-1}})^{\beta(d-2)+\frac{1}{d}+\epsilon-2}1_{\{x_{j_{d-1}}>\pi/2\}},
\end{aligned}$$

if $\beta(d-2)+\frac{1}{d}+\epsilon-2<0$. Similarly to (8.13),

$$\begin{aligned}
&\Big|\int_I a(u)K^1_{n,(i_l,j_l)}(x-u)1_{\mathcal{S}_{(i_l,j_l),d,2}}(x-u)\,\mathrm{d}u\Big| \\
\leq\ & C\sum_{k=1}^{d}2^{K(d+1-k)}\int_{I_k}\cdots\int_{I_d}(x_{i_1}-u_{i_1}-(x_{j_1}-u_{j_1}))^{-\frac{d+1}{d}} \\
&\prod_{l=2}^{k-1}(x_{i_l}-\nu-(x_{j_l}-\nu))^{-1-\beta-\frac{\epsilon}{d-2}}\prod_{l=k}^{d-1}(x_{i_l}-u_{i_l}-(x_{j_l}-u_{j_l}))^{-1-\beta-\frac{\epsilon}{d-2}} \\
&(x_{j_{d-1}}-u_{j_{d-1}})^{\beta(d-2)+\frac{1}{d}+\epsilon-2}1_{\mathcal{S}_{(i_l,j_l),d,2}}(x-u)\,\mathrm{d}u_k\cdots du_d \\
\leq\ & C2^{K(d+1-k)}2^{-K(d-k)}(x_{i_1}-x_{j_1}-2^{-K})^{-\frac{d+1}{d}}1_{\{x_{i_1}-x_{j_1}>2^{-K+1}\}} \\
&\prod_{l=2}^{d-1}(x_{i_l}-x_{j_l}-2^{-K})^{-1-\beta-\frac{\epsilon}{d-2}}1_{\{x_{i_l}-x_{j_l}>2^{-K+1}\}}2^{-K(\beta(d-2)+\frac{1}{d}+\epsilon-1)}1_{\{x_{j_{d-1}}\leq 2^{-K+4}\}}.
\end{aligned}$$

Here $\beta(d-2)+\frac{1}{d}+\epsilon-1>0$ and $u_{i_1}=u_{j_1}=\nu$ if $k\geq 2$. If the last expression is greater than ρ, then

$$(x_{i_1}-x_{j_1}-2^{-K})1_{\{x_{i_1}-x_{j_1}>2^{-K+1}\}}\ \leq\ C\rho^{-\frac{d}{d+1}}2^{-\frac{Kd}{d+1}(\beta(d-2)+\frac{1}{d}+\epsilon-2)}$$

$$\prod_{l=2}^{d-1}(x_{i_l}-x_{j_l}-2^{-K})^{(-1-\beta-\frac{\epsilon}{d-2})\frac{d}{d+1}} 1_{\{x_{i_l}-x_{j_l}>2^{-K+1}\}}1_{\{x_{j_{d-1}}\leq 2^{-K+4}\}}$$

and

$$\begin{aligned}
\lambda\Big(\sup_{n\geq 1}\Big|\int_I a(u)K^1_{n,(i_l,j_l)}(x-u)1_{\mathcal{S}_{(i_l,j_l),d,2}}(x-u)\,\mathrm{d}u\Big|>\rho\Big) \\
\leq\ & C\rho^{-\frac{d}{d+1}}2^{-\frac{Kd}{d+1}(\beta(d-2)+\frac{1}{d}+\epsilon-2)}\int \\
& \prod_{l=2}^{d-1}(x_{i_l}-x_{j_l}-2^{-K})^{(-1-\beta-\frac{\epsilon}{d-2})\frac{d}{d+1}}1_{\{x_{i_l}-x_{j_l}>2^{-K+1}\}}1_{\{x_{j_{d-1}}\leq 2^{-K+4}\}}\,\mathrm{d}x' \\
\leq\ & C\rho^{-\frac{d}{d+1}}2^{-\frac{Kd}{d+1}(\beta(d-2)+\frac{1}{d}+\epsilon-2)}2^{-K((-1-\beta-\frac{\epsilon}{d-2})\frac{d}{d+1}+1)(d-2)}2^{-K} \\
\leq\ & C\rho^{\frac{-d}{d+1}},
\end{aligned}$$

whenever $(-1-\beta-\frac{\epsilon}{d-2})\frac{d}{d+1}+1<0$. Recall that $x':=(x_1,\ldots,x_{i_1-1},x_{i_1+1},\ldots,x_d)$. It is easy to see that we can choose a β that satisfies all conditions mentioned above. This implies inequality (8.15) for $k=d$, $l=2$. The cases $k=d$, $l=3$ and $k=d$, $l=4$ can be handled similarly. The proof of the theorem is complete. ∎

8.2.3 Proof for $q=\infty$

Proof of Theorem 8.1 for $q=\infty$. We will show again that

$$\int_{\mathbb{T}^d\setminus 2^7 I}|\sigma_*^\infty a(x)|^p\,\mathrm{d}x=\int_{\mathbb{T}^d\setminus 2^7 I}\sup_{n\in\mathbb{N}}\Big|\int_I a(u)K_n^\infty(x-u)\,\mathrm{d}u\Big|^p\,\mathrm{d}x\leq C_p \tag{8.18}$$

for every $H_p^\square$-atom a, where $\frac{d}{d+1}<p<1$. Assume again that I, the support of a, satisfies (8.8).

Here, we modify slightly the definition of the sets $\mathcal{S}$, $\mathcal{S}_{\epsilon'}$, $\mathcal{S}'$ and $\mathcal{S}_k$. Let

$$\mathcal{S}:=\{x\in\mathbb{T}^d:x_1>x_2>\cdots>x_d>0,x_1>2^{-K+5}\},$$

$$\mathcal{S}_{\epsilon'}:=\{x\in\mathbb{T}^d:\Big|\sum_{j=1}^{d-1}\epsilon_j x_j\Big|<d2^{-K+4}\},$$

$$\mathcal{S}':=\{x\in\mathbb{T}^d:\exists\epsilon,\Big|\sum_{j=1}^{d}\epsilon_j x_j\Big|<d2^{-K+4}\},$$

$$\mathcal{S}_k:=\{x\in\mathcal{S}:x_1>x_2>\cdots>x_k\geq 2^{-K+2}>x_{k+1}>\cdots>x_d>0\},$$

$k=1,\ldots,d$. The sets $\mathcal{S}_{\epsilon,1}$ and $\mathcal{S}_{\epsilon',d}$ are defined as before in Subsection 6.2.3.

$$\mathcal{S}_{\epsilon,1}:=\{x\in\mathbb{T}^d:|\sum_{j=1}^{d}\epsilon_j x_j|<4x_1\},$$

$$\mathcal{S}_{\epsilon',d} := \{x \in \mathbb{T}^d : \Big|\sum_{j=1}^{d-1} \epsilon_j x_j\Big| < 4x_d\}.$$

We may suppose again that $|\sum_{j=1}^{d} \epsilon_j x_j| \leq \pi$ and $|\sum_{j=1}^{d-1} \epsilon_j x_j| \leq \pi$. It is easy to see that the Lemmas 6.15, 6.16 and 6.17 also hold for these sets.

Instead of (8.18) it is enough to prove by symmetry that

$$\begin{aligned}
&\int_{\mathbb{T}^d\setminus 2^7 I} \sup_{n\in\mathbb{N}} \Big| \int_I a(u) K_n^\infty(x-u) 1_{\mathcal{S}}(x-u)\,\mathrm{d}u \Big|^p \,\mathrm{d}x \\
\leq\ & \sum_{k=1,d} \int_{\mathbb{T}^d\setminus 2^7 I} \sup_{n\in\mathbb{N}} \Big| \int_I a(u) K_n^\infty(x-u) 1_{\mathcal{S}_k\cap\mathcal{S}'}(x-u)\,\mathrm{d}u \Big|^p \,\mathrm{d}x \\
&+ \sum_{\epsilon'} \sum_{k=2}^{d-1} \int_{\mathbb{T}^d\setminus 2^7 I} \sup_{n\in\mathbb{N}} \Big| \int_I a(u) K_{n,\epsilon'}^\infty(x-u) 1_{\mathcal{S}_k\cap\mathcal{S}_{\epsilon'}}(x-u)\,\mathrm{d}u \Big|^p \,\mathrm{d}x \\
&+ \sum_{k=1,d} \int_{\mathbb{T}^d\setminus 2^7 I} \sup_{n\in\mathbb{N}} \Big| \int_I a(u) K_n^\infty(x-u) 1_{\mathcal{S}_k\setminus\mathcal{S}'}(x-u)\,\mathrm{d}u \Big|^p \,\mathrm{d}x \\
&+ \sum_{\epsilon'} \sum_{k=2}^{d-1} \int_{\mathbb{T}^d\setminus 2^7 I} \sup_{n\in\mathbb{N}} \Big| \int_I a(u) K_{n,\epsilon'}^\infty(x-u) 1_{\mathcal{S}_k\setminus\mathcal{S}_{\epsilon'}}(x-u)\,\mathrm{d}u \Big|^p \,\mathrm{d}x \\
\leq\ & C_p. \qquad (8.19)
\end{aligned}$$

Step 1. Let us consider the first sum of (8.19). Since $u \in I$, $x-u \in \mathcal{S}_{\epsilon'}$ or $x-u \in \mathcal{S}'$ implies that x_1 must be in an interval of length $C2^{-K}$. If $x-u \in \mathcal{S}_k$ and $u \in I$, then $x_i - u_i \geq 2^{-K+2}$ and so $x_i \geq 2^{-K+1}$ $(i = 1,\ldots,k)$. Moreover, $x_i - u_i < 2^{-K+2}$ and so $x_i < 2^{-K+3}$ $(i = k+1,\ldots,d)$. By Theorem 6.1, the integral of K_n^∞ can be estimated by a constant, thus

$$\int_{\mathbb{T}^d\setminus 2^7 I} \sup_{n\in\mathbb{N}} \Big| \int_I a(u) K_n^\infty(x-u) 1_{\mathcal{S}_1\cap\mathcal{S}'}(x-u)\,\mathrm{d}u \Big|^p \,\mathrm{d}x \leq C_p 2^{Kd} 2^{-Kd}.$$

If $k = d$, then (6.15) implies

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \Big| \int_I a(u) K_{n,\epsilon'}^\infty(x-u) 1_{\mathcal{S}_d\cap\mathcal{S}'}(x-u)\,\mathrm{d}u \Big| \\
&\leq\ C 2^{Kd/p} \int_I \prod_{i=1}^{d} (x_i - u_i)^{-1} 1_{\mathcal{S}_d\cap\mathcal{S}'}(x-u)\,\mathrm{d}u \\
&\leq\ C 2^{Kd/p} \int_I \prod_{i=2}^{d} (x_i - 2^{-K-1})^{-1-1/(d-1)} 1_{\mathcal{S}_d\cap\mathcal{S}'}(x-u)\,\mathrm{d}u \\
&\leq\ C 2^{Kd/p-Kd} 1_{I_1'}(x_1) \Big(\prod_{i=2}^{d} (x_i - 2^{-K-1})^{-d/(d-1)} 1_{\{x_i \geq 2^{-K+1}\}} \Big),
\end{aligned}$$

where the length of I_1' is $c2^{-K}$. Then

$$\begin{aligned}
&\int_{\mathbb{T}^d\setminus 2^7 I} \sup_{n\in\mathbb{N}} \Big| \int_I a(u) K_{n,\epsilon'}^{\infty}(x-u) 1_{\mathcal{S}_d\cap\mathcal{S}'}(x-u)\,\mathrm{d}u\Big|^p \,\mathrm{d}x \\
&\leq C_p 2^{Kd-Kdp} 2^{-K} \int_{\mathbb{T}^{d-1}} \Big(\prod_{i=2}^d (x_i - 2^{-K-1})^{-dp/(d-1)} 1_{\{x_i\geq 2^{-K+1}\}} \Big) \,\mathrm{d}x_2\cdots \mathrm{d}x_d \\
&\leq C_p,
\end{aligned}$$

when $p>(d-1)/d$.

Step 2. In the second sum let us investigate first the term multiplied by $1_{\mathcal{S}_{\epsilon',d}}(x-u)$ in the integrand for all $k=2,\ldots,d-1$. If $x-u\in\mathcal{S}_{\epsilon',d}$, then u_1 is in an interval of length $8(x_d-u_d)$. By (6.15),

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \Big| \int_I a(u) K_{n,\epsilon'}^{\infty}(x-u) 1_{\mathcal{S}_k\cap\mathcal{S}_{\epsilon'}\cap\mathcal{S}_{\epsilon',d}}(x-u)\,\mathrm{d}u\Big| \\
&\leq C 2^{Kd/p} \int_I \prod_{i=1}^d (x_i-u_i)^{-1} 1_{\mathcal{S}_k\cap\mathcal{S}_{\epsilon'}\cap\mathcal{S}_{\epsilon',d}}(x-u)\,\mathrm{d}u \\
&\leq C 2^{Kd/p} 1_{I_1'}(x_1) \int_{I_2\times\cdots\times I_d} \Big(\prod_{i=2}^k (x_i-u_i)^{-1-1/(k-1)} \Big) \\
&\qquad \Big(\prod_{i=k+1}^d (x_i-u_i)^{-1} \Big) (x_d-u_d) 1_{\mathcal{S}_k}(x-u)\,\mathrm{d}u_2\cdots du_d \\
&\leq C 2^{Kd/p} 1_{I_1'}(x_1) \int_{I_2\times\cdots\times I_d} \Big(\prod_{i=2}^k (x_i-2^{-K-1})^{-1-1/(k-1)} \Big) \\
&\qquad \Big(\prod_{i=k+1}^d (x_i-u_i)^{-1+1/(d-k)} \Big) 1_{\mathcal{S}_k}(x-u)\,\mathrm{d}u_2\cdots du_d \\
&\leq C 2^{Kd/p-K(k-1)-K} 1_{I_1'}(x_1) \\
&\qquad \Big(\prod_{i=2}^k (x_i-2^{-K-1})^{-k/(k-1)} 1_{\{x_i\geq 2^{-K+1}\}} \Big) \Big(\prod_{i=k+1}^d 1_{\{x_i<2^{-K+3}\}} \Big),
\end{aligned}$$

where the length of I_1' is $c2^{-K}$. Hence

$$\begin{aligned}
\sum_{k=2}^{d-1} \int_{\mathbb{T}^d\setminus 2^7 I} \sup_{n\in\mathbb{N}} \Big| \int_I a(u) K_{n,\epsilon'}^{\infty}(x-u) 1_{\mathcal{S}_k\cap\mathcal{S}_{\epsilon'}\cap\mathcal{S}_{\epsilon',d}}(x-u)\,\mathrm{d}u\Big|^p \,\mathrm{d}x& \\
\leq C_p 2^{Kd-Kkp} 2^{-K} 2^{-K(-kp/(k-1)+1)(k-1)} 2^{-K(d-k)}& \\
\leq C_p,&
\end{aligned}$$

whenever $p>(d-1)/d$.

If $x-u \notin \mathcal{S}_{\epsilon',d}$, then $|\sum_{j=1}^d \epsilon_j(x_j-u_j)| \geq 3(x_d-u_d)$. Applying this and Lemma 6.16, we can see that

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \Big| \int_I a(u) K_{n,\epsilon'}^{\infty}(x-u) 1_{\mathcal{S}_k\cap\mathcal{S}_{\epsilon'}\setminus\mathcal{S}_{\epsilon',d}}(x-u)\,\mathrm{d}u \Big| \\
&\leq C \sum_{\epsilon_d} 2^{Kd/p} \int_I \Big(\prod_{i=1}^{d-1} (x_i-u_i)^{-1} \Big) (x_d-u_d)^{-\delta} \\
&\quad \Big| \sum_{j=1}^d \epsilon_j(x_j-u_j) \Big|^{-1+\delta} 1_{\mathcal{S}_k\cap\mathcal{S}_{\epsilon'}}(x-u)\,\mathrm{d}u \\
&\leq C \sum_{\epsilon_d} 2^{Kd/p} \int_I \Big(\prod_{i=2}^{k} (x_i-2^{-K-1})^{-1-1/(k-1)} \Big) \Big(\prod_{i=k+1}^{d} (x_i-u_i)^{-1+(1-\delta)/(d-k)} \Big) \\
&\quad \Big| \sum_{j=1}^d \epsilon_j(x_j-u_j) \Big|^{-1+\delta} 1_{\mathcal{S}_k\cap\mathcal{S}_{\epsilon'}}(x-u)\,\mathrm{d}u \\
&\leq C 2^{Kd/p-K(k-1)-K(1-\delta)-K(-1+\delta+1)} 1_{I_1'}(x_1) \\
&\quad \Big(\prod_{i=2}^{k} (x_i-2^{-K-1})^{-k/(k-1)} 1_{\{x_i\geq 2^{-K+1}\}} \Big) \Big(\prod_{i=k+1}^{d} 1_{\{x_i<2^{-K+3}\}} \Big)
\end{aligned}$$

and

$$\sum_{k=2}^{d-1} \int_{\mathbb{T}^d\setminus 2^7 I} \sup_{n\in\mathbb{N}} \Big| \int_I a(u) K_{n,\epsilon'}^{\infty}(x-u) 1_{\mathcal{S}_k\cap\mathcal{S}_{\epsilon'}\setminus\mathcal{S}_{\epsilon',d}}(x-u)\,\mathrm{d}u \Big|^p \mathrm{d}x \leq C_p,$$

as before, whenever $0<\delta<1$ and $p>(d-1)/d$. This proves that the second sum in (8.19) can be estimated by a constant.

Step 3. Now let us consider the fourth sum of (8.19):

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \Big| \int_I a(u) K_{n,\epsilon'}^{\infty}(x-u) 1_{\mathcal{S}_k\setminus\mathcal{S}_{\epsilon'}}(x-u)\,\mathrm{d}u \Big| \\
&\leq C \sum_{\epsilon_d} 2^{Kd/p} \int_I \Big(\prod_{i=1}^{d-1} (x_i-u_i)^{-1} \Big) \\
&\quad \Big| \sum_{j=1}^d \epsilon_j(x_j-u_j) \Big|^{-1} (1_{(\mathcal{S}_k\setminus\mathcal{S}_{\epsilon'})\cap\mathcal{S}_{\epsilon,1}}(x-u) + 1_{(\mathcal{S}_k\setminus\mathcal{S}_{\epsilon'})\setminus\mathcal{S}_{\epsilon,1}}(x-u))\,\mathrm{d}u,
\end{aligned}$$

$k=2,\ldots,d-1$. For the first sum, we get that

$$\begin{aligned}
&\sum_{\epsilon_d} 2^{Kd/p} \int_I \Big(\prod_{i=1}^{d-1} (x_i-u_i)^{-1} \Big) \Big| \sum_{j=1}^d \epsilon_j(x_j-u_j) \Big|^{-1} 1_{(\mathcal{S}_k\setminus\mathcal{S}_{\epsilon'})\cap\mathcal{S}_{\epsilon,1}}(x-u)\,\mathrm{d}u \\
&\leq \sum_{\epsilon_d} 2^{Kd/p} \int_I \Big(\prod_{i=1}^{d} (x_i-u_i)^{-1} \Big) (x_d-u_d)
\end{aligned}$$

$$
\begin{aligned}
& \Big|\sum_{j=1}^d \epsilon_j(x_j-u_j)\Big|^{-1-\delta+\delta} 1_{(\mathcal{S}_k\setminus\mathcal{S}_{\epsilon'})\cap\mathcal{S}_{\epsilon,1}}(x-u)\,\mathrm{d}u \\
\leq \ & C\sum_{\epsilon_d} 2^{Kd/p}\int_I \Big(\prod_{i=2}^k (x_i-u_i)^{-1+(\delta-1)/(k-1)}\Big)\Big(\prod_{i=k+1}^d (x_i-u_i)^{1/(d-k)-1}\Big) \\
& \Big|\sum_{j=1}^d \epsilon_j(x_j-u_j)\Big|^{-1-\delta} 1_{\mathcal{S}_k\setminus\mathcal{S}_{\epsilon'}}(x-u)\,\mathrm{d}u \\
\leq \ & C\sum_{\epsilon_d} 2^{Kd/p-Kk-K}\Big(\prod_{i=2}^k (x_i-2^{-K-1})^{-(k-\delta)/(k-1)} 1_{\{x_i\geq 2^{-K+1}\}}\Big) \\
& \Big(\prod_{i=k+1}^d 1_{\{x_i<2^{-K+3}\}}\Big)\Big|\sum_{j=1}^d \epsilon_j(x_j-2^{-K-1})\Big|^{-1-\delta} 1_{\{|\sum_{j=1}^d \epsilon_j(x_j-2^{-K-1})|\geq 2^{-K+3}\}},
\end{aligned}
$$

because

$$
|\sum_{j=1}^d \epsilon_j(x_j-2^{-K-1})| \geq |\sum_{j=1}^d \epsilon_j(x_j-u_j)| - |\sum_{j=1}^d \epsilon_j(u_j-2^{-K-1})| \geq d\cdot 2^{-K+2}
$$

and so

$$
\begin{aligned}
|\sum_{j=1}^d \epsilon_j(x_j-u_j)| \ & \geq \ |\sum_{j=1}^d \epsilon_j(x_j-2^{-K-1})| - |\sum_{j=1}^d \epsilon_j(u_j-2^{-K-1})| \\
& \geq \ \frac{1}{2}|\sum_{j=1}^d \epsilon_j(x_j-2^{-K-1})|.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sum_{\epsilon_d} 2^{Kd}\int_{\mathbb{T}^d\setminus 2^7 I} \Big|\int_I \Big(\prod_{i=1}^{d-1}(x_i-u_i)^{-1}\Big)\Big|\sum_{j=1}^d \epsilon_j(x_j-u_j)\Big|^{-1} 1_{(\mathcal{S}_k\setminus\mathcal{S}_{\epsilon'})\cap\mathcal{S}_{\epsilon,1}}(x-u)\,\mathrm{d}u\Big|^p\,\mathrm{d}x & \\
\leq \ C_p 2^{Kd-Kkp-Kp} 2^{-K(-(k-\delta)p/(k-1)+1)(k-1)} 2^{-K(d-k)} 2^{-K((-1-\delta)p+1)} & \\
\leq \ C_p, & \qquad (8.20)
\end{aligned}
$$

whenever $\delta=1/k$ and $p>k/(k+1)$, thus $p>(d-1)/d$.

Similarly,

$$
\begin{aligned}
& \sum_{\epsilon_d} 2^{Kd/p}\int_I \Big(\prod_{i=1}^{d-1}(x_i-u_i)^{-1}\Big)\Big|\sum_{j=1}^d \epsilon_j(x_j-u_j)\Big|^{-1} 1_{(\mathcal{S}_k\setminus\mathcal{S}_{\epsilon'})\setminus\mathcal{S}_{\epsilon,1}}(x-u)\,\mathrm{d}u \\
\leq \ & \sum_{\epsilon_d} 2^{Kd/p}\int_I (x_i-u_i)^{-1}\Big(\prod_{i=1}^d (x_i-u_i)^{-1}\Big)(x_d-u_d) 1_{\mathcal{S}_k}(x-u)\,\mathrm{d}u
\end{aligned}
$$

$$
\begin{aligned}
&\le C\sum_{\epsilon_d} 2^{Kd/p}\int_I \Big(\prod_{i=1}^k (x_i-u_i)^{-1-1/k}\Big)\Big(\prod_{i=k+1}^d (x_i-u_i)^{1/(d-k)-1}\Big)1_{\mathcal{S}_k}(x-u)\,\mathrm{d}u \\
&\le C\sum_{\epsilon_d} 2^{Kd/p-Kk-K}\Big(\prod_{i=1}^k (x_i-2^{-K-1})^{-(k+1)/k}1_{\{x_i\ge 2^{-K+1}\}}\Big)\Big(\prod_{i=k+1}^d 1_{\{x_i<2^{-K+3}\}}\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{\epsilon_d} 2^{Kd}\int_{\mathbb{T}^d\setminus 2^7I}\Big|\int_I \Big(\prod_{i=1}^{d-1}(x_i-u_i)^{-1}\Big)\Big|\sum_{j=1}^d \epsilon_j(x_j-u_j)\Big|^{-1} 1_{(\mathcal{S}_k\setminus\mathcal{S}_{\epsilon'})\setminus\mathcal{S}_{\epsilon,1}}(x-u)\,\mathrm{d}u\Big|^p\,\mathrm{d}x& \\
\le C_p 2^{Kd-Kkp-Kp}2^{-K(-(k+1)p/k+1)k}2^{-K(d-k)}& \\
\le C_p,& \qquad (8.21)
\end{aligned}
$$

if $p>(d-1)/d$. This yields that

$$
\sum_{k=2}^{d-1}\int_{\mathbb{T}^d\setminus 2^7I}\sup_{n\in\mathbb{N}}\Big|\int_I a(u)K_{n,\epsilon'}^\infty(x-u)1_{\mathcal{S}_k\setminus\mathcal{S}_{\epsilon'}}(x-u)\,\mathrm{d}u\Big|^p\,\mathrm{d}x\le C_p.
$$

Step 4. The inequality

$$
\int_{\mathbb{T}^d\setminus 2^7I}\sup_{n\in\mathbb{N}}\Big|\int_I a(u)K_n^\infty(x-u)1_{\mathcal{S}_1\setminus\mathcal{S}'}(x-u)\,\mathrm{d}u\Big|^p\,\mathrm{d}x\le C_p
$$

can be proved in the same way. Now let $\delta=1$. If $k=d$, then using the notation in (8.10), we get by integrations by parts, that

$$
\begin{aligned}
&\int_I a(u)K_{n,\epsilon'}^\infty(x-u)1_{\mathcal{S}_d\setminus\mathcal{S}'}(x-u)\,\mathrm{d}u \\
&\quad= \sum_{l=1}^d\int_{I_l}\cdots\int_{I_d} A_l(u^{(l)})(\partial_l K_{n,\epsilon'}^\infty 1_{\mathcal{S}_d\setminus\mathcal{S}'})(x-u^{(l)})\,\mathrm{d}u_l\cdots\mathrm{d}u_d
\end{aligned}
$$

as in (8.11), where $u^{(l)}:=(\nu,\dots,\nu,u_l,\dots,u_d)$. We remark that

$$
A_d(\nu,\dots,\nu)=\int_I a=0.
$$

By Lemma 6.17,

$$
\begin{aligned}
&\sup_{n\in\mathbb{N}}\Big|\int_I a(u)K_{n,\epsilon'}^\infty(x-u)1_{\mathcal{S}_d\setminus\mathcal{S}'}(x-u)\,\mathrm{d}u\Big| \\
&\quad\le C\sum_{\epsilon_d}\sum_{l=1}^d 2^{Kd/p-Kl}\int_{I_l}\cdots\int_{I_d}\Big(\prod_{i=1}^d (x_i-u_i^{(l)})^{-1}\Big)
\end{aligned}
$$

$$\begin{aligned}
& \Big|\sum_{j=1}^{d} \epsilon_j(x_j-u_j^{(l)})\Big|^{-1} 1_{\mathcal{S}_d\setminus\mathcal{S}'}(x-u^{(l)})\,\mathrm{d}u_l\ldots\,\mathrm{d}u_d \\
& + C\sum_{l=1}^{d} 2^{Kd/p-Kl}\int_{I_l}\cdots\int_{I_d}\Big(\prod_{i=1}^{d}(x_i-u_i^{(l)})^{-1}\Big) \\
& \qquad (x_d-u_d^{(l)})^{-1} 1_{(\mathcal{S}_d\setminus\mathcal{S}')\cap\mathcal{S}_{\epsilon',d}}(x-u^{(l)})\,\mathrm{d}u_l\cdots\,\mathrm{d}u_d \\
=: \; & A(x)+B(x). \qquad (8.22)
\end{aligned}$$

For the first sum, we obtain

$$\begin{aligned}
A(x) &= C\sum_{\epsilon_d}\sum_{l=1}^{d} 2^{Kd/p-Kl}\int_{I_l}\cdots\int_{I_d}\Big(\prod_{i=1}^{d}(x_i-u_i^{(l)})^{-1}\Big)\Big|\sum_{j=1}^{d}\epsilon_j(x_j-u_j^{(l)})\Big|^{-1} \\
& \quad (1_{(\mathcal{S}_d\setminus\mathcal{S}')\cap\mathcal{S}_{\epsilon,1}}(x-u^{(l)})+1_{(\mathcal{S}_d\setminus\mathcal{S}')\setminus\mathcal{S}_{\epsilon,1}}(x-u^{(l)}))\,\mathrm{d}u_l\cdots\,\mathrm{d}u_d \\
&=: A_1(x)+A_2(x).
\end{aligned}$$

Then

$$\begin{aligned}
A_1(x) &= C\sum_{\epsilon_d} 2^{Kd/p-Kl}\int_{I_l}\cdots\int_{I_d}\Big(\prod_{i=1}^{d}(x_i-u_i^{(l)})^{-1}\Big) \\
& \quad \Big|\sum_{j=1}^{d}\epsilon_j(x_j-u_j^{(l)})\Big|^{-1} 1_{(\mathcal{S}_d\setminus\mathcal{S}')\cap\mathcal{S}_{\epsilon,1}}(x-u^{(l)})\,\mathrm{d}u_l\cdots\,\mathrm{d}u_d \\
&\leq C\sum_{\epsilon_d} 2^{Kd/p-Kl}\int_{I_l}\cdots\int_{I_d}\Big(\prod_{i=2}^{d}(x_i-u_i^{(l)})^{-1+(\delta-1)/(d-1)}\Big) \\
& \quad \Big|\sum_{j=1}^{d}\epsilon_j(x_j-u_j^{(l)})\Big|^{-1-\delta} 1_{\mathcal{S}_d\setminus\mathcal{S}'}(x-u^{(l)})\,\mathrm{d}u_l\cdots\,\mathrm{d}u_d. \qquad (8.23)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
A_2(x) &= C\sum_{\epsilon_d} 2^{Kd/p-Kl}\int_{I_l}\cdots\int_{I_d}\Big(\prod_{i=1}^{d}(x_i-u_i^{(l)})^{-1}\Big) \\
& \quad \Big|\sum_{j=1}^{d}\epsilon_j(x_j-u_j^{(l)})\Big|^{-1} 1_{(\mathcal{S}_d\setminus\mathcal{S}')\setminus\mathcal{S}_{\epsilon,1}}(x-u^{(l)})\,\mathrm{d}u_l\cdots\,\mathrm{d}u_d \qquad (8.24) \\
&\leq C\sum_{\epsilon_d} 2^{Kd/p-Kl}\int_{I_l}\cdots\int_{I_d}\Big(\prod_{i=1}^{d}(x_i-u_i^{(l)})^{-1-1/d}\Big) 1_{\mathcal{S}_d}(x-u^{(l)})\,\mathrm{d}u_l\cdots\,\mathrm{d}u_d
\end{aligned}$$

and the inequality

$$\int_{\mathbb{T}^d\setminus 2^7 I} |A(x)|^p\,\mathrm{d}x \leq C_p$$

can be seen as in (8.20) and (8.21) for $p > d/(d+1)$.

Next, we investigate the second sum of (8.22). Since $u \in I$, $x - u \in \mathcal{S}_d \cap \mathcal{S}_{\epsilon',d}$ implies that $x_i \geq 2^{-K+1}$ $(i = 1, \ldots, d)$ and x_1 is in an interval I_1' with length $4x_d + C2^{-K-1} \leq Cx_d$. Then

$$\begin{aligned}
B(x) &= \sum_{l=1}^{d} 2^{Kd/p-Kl} \int_{I_l} \cdots \int_{I_d} \Big(\prod_{i=1}^{d} (x_i - u_i^{(l)})^{-1} \Big) \\
&\qquad (x_d - u_d^{(l)})^{-1} 1_{(\mathcal{S}_d \setminus \mathcal{S}') \cap \mathcal{S}_{\epsilon',d}}(x - u^{(l)}) \, \mathrm{d}u_l \cdots \mathrm{d}u_d \\
&\leq \sum_{l=1}^{d} 2^{Kd/p-Kl} \int_{I_l} \cdots \int_{I_d} \Big(\prod_{i=1}^{d} (x_i - 2^{-K-1})^{-1} \Big) \\
&\qquad (x_d - 2^{-K-1})^{-1} 1_{\mathcal{S}_d \cap \mathcal{S}_{\epsilon',d}}(x - u^{(l)}) \, \mathrm{d}u_l \cdots \mathrm{d}u_d \\
&\leq C 2^{Kd/p-Kd-K} \Big(\prod_{i=2}^{d-1} (x_i - 2^{-K-1})^{-1-1/d} 1_{\{x_i \geq 2^{-K+1}\}} \Big) \\
&\qquad (x_d - 2^{-K-1})^{-2-2/d} 1_{\{x_d \geq 2^{-K+1}\}} 1_{I_1'}(x_1),
\end{aligned} \tag{8.25}$$

consequently, integrating first in x_1,

$$\begin{aligned}
\int_{\mathbb{T}^d \setminus 2^7 I} |B(x)|^p \, \mathrm{d}x &\leq C_p \int_{\mathbb{T}^{d-1}} 2^{Kd-Kdp-Kp} \Big(\prod_{i=2}^{d-1} (x_i - 2^{-K-1})^{-(d+1)p/d} 1_{\{x_i \geq 2^{-K+1}\}} \Big) \\
&\qquad (x_d - 2^{-K-1})^{-(2d+2)p/d+1} 1_{\{x_d \geq 2^{-K+1}\}} \, \mathrm{d}x_2 \ldots \mathrm{d}x_d \\
&\leq C_p 2^{Kd-Kdp-Kp} 2^{-K(-(d+1)p/d+1)(d-2)} 2^{-K(-(2d+2)p/d+2)} \\
&\leq C_p,
\end{aligned}$$

whenever $p > \frac{d}{d+1}$. This finishes the proof of (8.18) as well as (8.1).

Step 5. For the weak inequality (8.2), it is enough to show that

$$\sup_{\rho > 0} \rho^{d/(d+1)} \lambda \left(\{ \sigma_*^\infty a > \rho \} \cap (\mathbb{T}^d \setminus 2^7 I) \right) \leq C$$

for all $H_{d/(d+1)}^\square$-atoms a. Observe that (8.14) implies that we have to show only that

$$\rho^{d/(d+1)} \lambda \Big(\sup_{n \in \mathbb{N}} \Big| \int_I a(u) K_n^\infty (x - u) 1_{\mathcal{S}_d \setminus \mathcal{S}'}(x - u) \, \mathrm{d}u \Big| > \rho, \mathbb{T}^d \setminus 2^7 I \Big) \leq C.$$

Similarly to (8.24) with $p = d/(d+1)$,

$$\begin{aligned}
A_2(x) &\leq C \sum_{\epsilon_d} 2^{K(d+1)-Kl} \int_{I_l} \cdots \int_{I_d} (x_1 - u_1^{(l)})^{-1} \Big(\prod_{i=1}^{d} (x_i - u_i^{(l)})^{-1} \Big) \\
&\qquad 1_{(\mathcal{S}_d \setminus \mathcal{S}') \setminus \mathcal{S}_{\epsilon,1}}(x - u^{(l)}) \, \mathrm{d}u_l \cdots \mathrm{d}u_d \\
&\leq C (x_1 - 2^{-K-1})^{-2} 1_{\{x_1 \geq 2^{-K+1}\}} \Big(\prod_{i=2}^{d} (x_i - 2^{-K-1})^{-1} 1_{\{x_i \geq 2^{-K+1}\}} \Big).
\end{aligned}$$

If this is greater than $C\rho$, then by translation, we may suppose that

$$x_d \le \rho^{-1} x_1^{-2} \Big(\prod_{i=2}^{d-1} x_i^{-1} \Big) \tag{8.26}$$

and each x_i is positive. We assume that

$$0 \le x_d < \cdots < x_{k+1} < \rho^{-1/(d+1)} < x_k < \cdots < x_1 \tag{8.27}$$

for some $k = 0, 1, \ldots, d$. The case $k = d$ contradicts (8.26). For another k and for some $0 \le \epsilon \le 1$,

$$x_d = x_d^{\epsilon} x_d^{1-\epsilon} \le \Big(\prod_{i=k+1}^{d-1} x_i^{\epsilon/(d-k-1)} \Big) \Big(\rho^{-1} x_1^{-2} \prod_{i=2}^{d-1} x_i^{-1} \Big)^{1-\epsilon}.$$

Then

$$\begin{aligned}
&\int 1_{\{x_1^{-2} \prod_{i=2}^{d} x_i^{-1} \ge \rho\}} \, \mathrm{d}x \\
&\le \int \rho^{\epsilon-1} \Big(\prod_{i=k+1}^{d-1} x_i^{\epsilon/(d-k-1)+\epsilon-1} \Big) \Big(\prod_{i=1}^{k} x_i^{-1-1/k} \Big)^{1-\epsilon} \mathrm{d}x_1 \cdots \mathrm{d}x_{d-1} \\
&\le \rho^{\epsilon-1} \rho^{-\frac{1}{d+1}(\frac{\epsilon}{d-1-k}+\epsilon)(d-1-k)} \rho^{-\frac{1}{d+1}(-\frac{k+1}{k}(1-\epsilon)+1)k} \\
&= \rho^{-\frac{d}{d+1}},
\end{aligned}$$

whenever we choose ϵ such that $-\frac{k+1}{k}(1-\epsilon)+1 < 0$. If $k = 0$, then let $\epsilon = 1$ and if $k = d-1$, then $\epsilon = 0$.

On the other hand, by (8.23),

$$\begin{aligned}
A_1(x) &\le C \sum_{\epsilon_d} 2^{K(d+1)-Kl} \int_{I_l} \cdots \int_{I_d} (x_1 - u_1^{(l)})^{\delta} \Big(\prod_{i=1}^{d} (x_i - u_i^{(l)})^{-1} \Big) \\
&\qquad \Big| \sum_{j=1}^{d} \epsilon_j (x_j - u_j^{(l)}) \Big|^{-1-\delta} 1_{(\mathcal{S}_d \setminus \mathcal{S}') \cap \mathcal{S}_{\epsilon,1}} (x - u^{(l)}) \, \mathrm{d}u_l \cdots \mathrm{d}u_d \\
&\le C \sum_{\epsilon_d} (x_1 - 2^{-K-1})^{-1+\delta} 1_{\{x_1 \ge 2^{-K+1}\}} \Big(\prod_{i=2}^{d} (x_i - 2^{-K-1})^{-1} 1_{\{x_i \ge 2^{-K+1}\}} \Big) \\
&\qquad \Big| \sum_{j=1}^{d} \epsilon_j (x_j - 2^{-K-1}) \Big|^{-1-\delta} 1_{\{|\sum_{j=1}^{d} \epsilon_j (x_j - 2^{-K-1})|^{-1-\delta} \ge 2^{-K+3}\}}.
\end{aligned}$$

We may suppose again that

$$x_1^{-1+\delta} \Big(\prod_{i=2}^{d} x_i^{-1} \Big) \Big| \sum_{j=1}^{d} \epsilon_j x_j \Big|^{-1-\delta} \ge \rho$$

and that (8.27) holds. Then

$$\Big(\prod_{i=2}^{k} x_i^{-1+(\delta-1)/(k-1)}\Big)\Big(\prod_{i=k+1}^{d} x_i^{-1}\Big)\Big|\sum_{j=1}^{d}\epsilon_j x_j\Big|^{-1-\delta} \geq \rho.$$

By a transformation,

$$\begin{aligned}
&\int 1_{\{x_1^{-1+\delta}(\prod_{i=2}^d x_i^{-1})|\sum_{j=1}^d \epsilon_j x_j|^{-1-\delta}\geq\rho\}}\,\mathrm{d}x \\
&\quad\leq \int 1_{\{(\prod_{i=2}^k t_i^{-1+(\delta-1)/(k-1)})(\prod_{i=k+1}^d t_i^{-1})|t_1|^{-1-\delta}\geq\rho\}}\,\mathrm{d}t.
\end{aligned}$$

Assume that $\rho^{-1/(d+1)} < |t_1|$. The case $k=d$ is again impossible. In other cases,

$$\begin{aligned}
&\int 1_{\{(\prod_{i=2}^k t_i^{-1+(\delta-1)/(k-1)})(\prod_{i=k+1}^d t_i^{-1})|t_1|^{-1-\delta}\geq\rho\}}\,\mathrm{d}t \\
&\leq \int \rho^{\epsilon-1}\Big(\prod_{i=k+1}^{d-1} t_i^{\epsilon/(d-k-1)+\epsilon-1}\Big)\Big(\prod_{i=2}^{k} t_i^{-1+(\delta-1)/(k-1)}\Big)^{1-\epsilon}|t_1|^{(-1-\delta)(1-\epsilon)}\,\mathrm{d}t_1\cdots\mathrm{d}t_{d-1} \\
&\leq \rho^{\epsilon-1}\rho^{-\frac{1}{d+1}(\frac{\epsilon}{d-1-k}+\epsilon)(d-1-k)}\rho^{-\frac{1}{d+1}((-\frac{k-\delta}{k-1}(1-\epsilon)+1)(k-1)+(-1-\delta)(1-\epsilon)+1)} \\
&= \rho^{-\frac{d}{d+1}},
\end{aligned}$$

if we choose ϵ and δ such that $-\frac{k-\delta}{k-1}(1-\epsilon)+1<0$ and $(-1-\delta)(1-\epsilon)+1<0$. The cases $k=0$ ($\epsilon=1$), $k=1$ ($\delta=1$) and $k=d-1$ ($\epsilon=0$) are included again.

If $x_d < |t_1| \leq \rho^{-1/(d+1)}$, then $k<d$ and

$$\begin{aligned}
&\int 1_{\{(\prod_{i=2}^k t_i^{-1+(\delta-1)/(k-1)})(\prod_{i=k+1}^d t_i^{-1})|t_1|^{-1-\delta}\geq\rho\}}\,\mathrm{d}t \\
&\leq \int \rho^{\epsilon-1}\Big(\prod_{i=k+1}^{d-1} t_i^{\epsilon/(d-k)+\epsilon-1}\Big)|t_1|^{\epsilon/(d-k)-(1+\delta)(1-\epsilon)}\Big(\prod_{i=2}^{k} t_i^{-1+(\delta-1)/(k-1)}\Big)^{1-\epsilon}\,\mathrm{d}t_1\cdots\mathrm{d}t_{d-1} \\
&\leq \rho^{\epsilon-1}\rho^{-\frac{1}{d+1}((\frac{\epsilon}{d-k}+\epsilon)(d-1-k)+\frac{\epsilon}{d-k}-(1+\delta)(1-\epsilon)+1)}\rho^{-\frac{1}{d+1}(-\frac{k-\delta}{k-1}(1-\epsilon)+1)(k-1)} \\
&= \rho^{-\frac{d}{d+1}},
\end{aligned}$$

assuming that $\frac{\epsilon}{d-k}-(1+\delta)(1-\epsilon)+1>0$ and $-\frac{k-\delta}{k-1}(1-\epsilon)+1<0$.

If $|t_1| < x_d$ and $|t_1| \leq \rho^{-1/(d+1)}$, then

$$\begin{aligned}
&\int 1_{\{(\prod_{i=2}^k t_i^{-1+(\delta-1)/(k-1)})(\prod_{i=k+1}^d t_i^{-1})|t_1|^{-1-\delta}\geq\rho\}}\,\mathrm{d}t \\
&\leq \int \rho^{-\frac{1-\epsilon}{1+\delta}}\Big(\prod_{i=k+1}^{d} t_i^{\frac{\epsilon}{d-k}-\frac{1-\epsilon}{1+\delta}}\Big)\Big(\prod_{i=2}^{k} t_i^{-1+\frac{\delta-1}{k-1}}\Big)^{\frac{1-\epsilon}{1+\delta}}\,\mathrm{d}t_2\cdots\mathrm{d}t_d \\
&\leq \rho^{-\frac{1-\epsilon}{1+\delta}}\rho^{-\frac{1}{d+1}(\frac{\epsilon}{d-k}-\frac{1-\epsilon}{1+\delta}+1)(d-k)}\rho^{-\frac{1}{d+1}(-\frac{k-\delta}{k-1}\frac{1-\epsilon}{1+\delta}+1)(k-1)} \\
&= \rho^{-\frac{d}{d+1}},
\end{aligned}$$

if $\frac{\epsilon}{d-k}-\frac{1-\epsilon}{1+\delta}+1>0$ and $-\frac{k-\delta}{k-1}\frac{1-\epsilon}{1+\delta}+1<0$.

Finally, we have to investigate $B(x)$. Similarly to (8.25), if

$$\Big(\prod_{i=1}^{d-1} x_i^{-1}\Big)x_d^{-2}1_{I_1'}(x_1)\geq \rho,$$

then

$$x_d \leq \rho^{-1/2}1_{I_1'}(x_1)\Big(\prod_{i=2}^{k} x_i^{-1-1(k-1)}\Big)^{1/2}\Big(\prod_{i=k+1}^{d-1} x_i^{-1}\Big)^{1/2}.$$

Then

$$\begin{aligned}\int 1_{\{(\prod_{i=1}^{d-1}x_i^{-1})x_d^{-2}1_{I_1'}(x_1)\geq\rho\}}\,\mathrm{d}x &\leq \int x_d 1_{\{(\prod_{i=1}^{d-1}x_i^{-1})x_d^{-2}\geq\rho\}}\,\mathrm{d}x_2\cdots\mathrm{d}x_d\\ &\leq \int \rho^{\epsilon-1}\Big(\prod_{i=k+1}^{d-1} x_i^{\epsilon/(d-k-1)+\epsilon/2-1/2}\Big)^2\\ &\qquad \Big(\prod_{i=2}^{k} x_i^{-1/2-1/2(k-1)}\Big)^{2(1-\epsilon)}\,\mathrm{d}x_2\ldots\mathrm{d}x_{d-1}\\ &\leq \rho^{\epsilon-1}\rho^{-\frac{1}{d+1}(\frac{2\epsilon}{d-1-k}+\epsilon)(d-1-k)+(-\frac{k}{k-1}(1-\epsilon)+1)(k-1)}\\ &= \rho^{-\frac{d}{d+1}},\end{aligned}$$

whenever we choose ϵ such that $-\frac{k+1}{k}(1-\epsilon)+1<0$. This finishes the proof of (8.2). ∎

8.2.4 Proof for $q=2$

Proof of Theorem 8.1 for $q=2$. Assume that $\alpha>(d-1)/2$, $\gamma\in\mathbb{N}$ and let us choose $N\in\mathbb{N}$ such that $N<\alpha-(d-1)/2\leq N+1$. As we mentioned in Section 7, we may suppose that the support of an atom a is a ball B with radius β, $2^{-K-1}<\beta\leq 2^{-K}$ ($K\in\mathbb{N}$). Moreover, we may suppose that the center of B is zero, i.e., $B=B(0,\beta)$. Obviously,

$$\begin{aligned}&\int_{\mathbb{T}^d\setminus(sB)}|\sigma_*^{2,\alpha}a(x)|^p\,\mathrm{d}x\\ &\leq \sum_{i=4\lfloor d^{1/2}\rfloor-1}^{\lfloor d^{1/2}2^K\pi\rfloor-1}\int_{B(0,(i+2)2^{-K})\setminus B(0,(i+1)2^{-K})\cap\mathbb{T}^d}\sup_{n\geq d^{1/2}2^{K+1}}|\sigma_n^{2,\alpha}a(x)|^p\,\mathrm{d}x\\ &\quad+\sum_{i=4\lfloor d^{1/2}\rfloor-1}^{\lfloor d^{1/2}2^K\pi\rfloor-1}\int_{B(0,(i+2)2^{-K})\setminus B(0,(i+1)2^{-K})\cap\mathbb{T}^d}\sup_{n<d^{1/2}2^{K+1}}|\sigma_n^{2,\alpha}a(x)|^p\,\mathrm{d}x\\ &=: (A)+(B),\end{aligned}$$

where $s=8d^{1/2}$. Note that if $K\leq 3$, then the integral is equal to 0.

Using Theorem 6.24, the definition of the atom and Taylor's formulae, we obtain

$$\begin{aligned}\sigma_n^{2,\alpha}a(x) &= \sum_{k\in\mathbb{Z}^d} n^d \int_{B+2k\pi} a(t)\widehat{\theta}_0(n(x-t))\,\mathrm{d}t \\ &= \sum_{k\in\mathbb{Z}^d} n^d \sum_{i_1+\cdots+i_d=N} (-1)^N \int_{B+2k\pi} a(t) \\ & \qquad \Big(\partial_1^{i_1}\cdots\partial_d^{i_d}\widehat{\theta}_0(n(x-2k\pi)-nv(t-2k\pi))\Big) n^N \prod_{j=1}^d \frac{(t_j-k_j)^{i_j}}{i_j!}\,\mathrm{d}t,\end{aligned}$$

where $0<v<1$. Then, by Corollary 6.23,

$$\begin{aligned}|\sigma_n^{2,\alpha}a(x)| &\leq C_p \sum_{k\in\mathbb{Z}^d} n^{(d-1)/2+N-\alpha}2^{Kd/p}2^{-KN} \\ & \qquad \int_{B+2k\pi} \|x-2k\pi-v(t-2k\pi)\|_2^{-d/2-\alpha-1/2}\,\mathrm{d}t. \end{aligned} \tag{8.28}$$

Moreover,

$$\begin{aligned}\sup_{n\geq d^{1/2}2^{K+1}} |\sigma_n^{2,\alpha}a(x)| &\leq C_p \sum_{k\in\mathbb{Z}^d} 2^{K((d-1)/2-\alpha)}2^{Kd/p} \\ & \qquad \int_{B+2k\pi} \|x-2k\pi-v(t-2k\pi)\|_2^{-d/2-\alpha-1/2}\,\mathrm{d}t \\ &=: A_1(x)+A_2(x),\end{aligned}$$

where A_1 denotes the term $k=0$ and A_2 the remaining sum. If $k=0$, $u\in B$ and $x\in B(0,(i+2)2^{-K})\setminus B(0,(i+1)2^{-K})\cap\mathbb{T}^d$ for some $i=4\lfloor d^{1/2}\rfloor-1,\ldots,\lfloor d^{1/2}2^K\pi\rfloor-1$, then

$$\|x-u\|_2 \geq \|x\|_2-\|u\|_2 \geq i2^{-K}.$$

In the case $k\neq 0$, $u\in B+2k\pi$ and $x\in B(0,(i+2)2^{-K})\setminus B(0,(i+1)2^{-K})\cap\mathbb{T}^d$, one can see that $\|x-u\|_2\geq\|k\|_2/4$. Then

$$A_1(x) \leq C_p 2^{K((d-1)/2-\alpha)}2^{Kd/p}\int_B \|i2^{-K}\|_2^{-d/2-\alpha-1/2}\,\mathrm{d}t \leq C_p 2^{Kd/p} i^{-d/2-\alpha-1/2}$$

and

$$\begin{aligned}A_2(x) &\leq C_p \sum_{k\in\mathbb{Z}^d,k\neq 0} 2^{K((d-1)/2-\alpha)}2^{Kd/p}\int_{B+2k\pi}\|k\|_2^{-d/2-\alpha-1/2}\,\mathrm{d}t \\ &\leq C_p \sum_{k\in\mathbb{Z}^d,k\neq 0} 2^{K(-d/2-1/2-\alpha)}2^{Kd/p}\|k\|_2^{-d/2-1/2-\alpha} \\ &\leq C_p \sum_{j=1}^\infty 2^{K(-d/2-1/2-\alpha)}2^{Kd/p} j^{(-d/2-1/2-\alpha)} j^{d-1} \\ &\leq C_p,\end{aligned}$$

whenever $x \in B(0,(i+2)2^{-K}) \setminus B(0,(i+1)2^{-K}) \cap \mathbb{T}^d$, $p \geq d/(d/2+\alpha+1/2)$ and $\alpha > (d-1)/2$. Hence,

$$(A) \leq C_p \sum_{i=4\lfloor d^{1/2} \rfloor -1}^{\lfloor d^{1/2} 2^K \pi \rfloor -1} 2^{-Kd} i^{d-1} 2^{Kd} i^{p(-d/2-\alpha-1/2)} + C_p \leq C_p,$$

if $p > d/(d/2+\alpha+1/2)$.

Similarly to (8.28), we obtain

$$\begin{aligned} |\sigma_n^{2,\alpha} a(x)| &\leq C_p \sum_{k \in \mathbb{Z}^d} n^{(d-1)/2+(N+1)-\alpha} 2^{Kd/p} 2^{-K(N+1)} \\ &\qquad \int_{B+2k\pi} \|x - 2k\pi - v(t-2k\pi)\|_2^{-d/2-\alpha-1/2} \, \mathrm{d}t. \end{aligned}$$

It is easy to see that $\sup_{n<d^{1/2}2^{K+1}} |\sigma_n^{2,\alpha} a(x)|$ can be estimated in the same way as $\sup_{n \geq d^{1/2}2^{K+1}} |\sigma_n^{2,\alpha} a(x)|$ above, which proves

$$\int_{\mathbb{T}^d \setminus (sB)} |\sigma_*^{2,\alpha} a(x)|^p \, \mathrm{d}x \leq C_p$$

as well as (8.1).

Let us introduce the set

$$E_\rho := \left\{ i \geq 4\lfloor d^{1/2} \rfloor - 1 : i^{-d/2-\alpha-1/2} > C^{-1} \rho 2^{-Kd/p} \right\},$$

where $p = d/(d/2+\alpha+1/2)$. To prove (8.2), observe that

$$\rho^p \, \lambda\Big(\{A_1 > \rho\} \cap \{\mathbb{T}^d \setminus (sB)\} \Big) \leq C\rho^p \sum_{i \in E_\rho} i^{d-1} 2^{-Kd}.$$

If k is the largest integer for which $k^{-d/2-\alpha-1/2} > C^{-1}\rho 2^{-Kd/p}$, then

$$\rho^p \, \lambda\Big(\{A_1 > \rho\} \cap \{\mathbb{T}^d \setminus (sB)\} \Big) \leq \rho^p 2^{-Kd} k^d \leq C.$$

The same inequality for (A_2) is trivial. We can estimate $\sup_{n<d^{1/2}2^{K+1}} |\sigma_n^{2,\alpha} a(x)|$ similarly, which shows (8.2). ■

9 Conjugate functions

For a distribution

$$f \sim \sum_{n \in \mathbb{Z}^d} \widehat{f}(n) \mathrm{e}^{\mathrm{i} n \cdot x},$$

the **conjugate distributions** or **Riesz transforms** are defined by

$$\tilde{f}^{(i)} \sim \sum_{n \in \mathbb{Z}^d} -\mathrm{i} \, \frac{n_i}{\|n\|_2} \widehat{f}(n) \mathrm{e}^{\mathrm{i} n \cdot x} \qquad (i = 1, \ldots, d)$$

(see e.g. Stein [78] or Weisz [94]). In the one-dimensional case,

$$\tilde{f} := \tilde{f}^{(1)} \sim \sum_{n=-\infty}^{\infty} (-\mathrm{i} \operatorname{sign} n) \widehat{f}(n) \mathrm{e}^{\mathrm{i}nx}$$

is called the **Hilbert transform**. As is well known, if f is an integrable function, then

$$\tilde{f}(x) = \text{p.v. } \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(x-t)}{2\tan(t/2)} \, \mathrm{d}t := \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{\epsilon < |t| < \pi} \frac{f(x-t)}{2\tan(t/2)} \, \mathrm{d}t \qquad \text{a.e.}$$

Note that p.v. is the abbreviation of the principal value. More generally, in the higher dimensional case, if $f \in L_1(\mathbb{T}^d)$, then there also exist a principal value form of the conjugate functions $\tilde{f}^{(i)}$. They do exist almost everywhere, but they are not integrable in general (see e.g. Shapiro [75], Stein and Weiss [77, 78, 80] or Weisz [94]). The following inequalities can be found in Stein [78] or Weisz [94].

Theorem 9.1 *For all $f \in H_p^\Box(\mathbb{T}^d)$,*

$$\|\tilde{f}^{(i)}\|_{H_p^\Box} \leq C_p \|f\|_{H_p^\Box} \qquad (0 < p < \infty, i = 0, \ldots, d) \tag{9.1}$$

and

$$\|f\|_{H_p^\Box} \sim \sum_{i=0}^{d} \|\tilde{f}^{(i)}\|_p \qquad ((d-1)/d < p < \infty). \tag{9.2}$$

Since $H_p^\Box(\mathbb{T}^d) \sim L_p(\mathbb{T}^d)$ for $1 < p < \infty$, Theorem 9.1 holds also for $L_p(\mathbb{T}^d)$ spaces $(1 < p < \infty)$.

The **conjugate Fejér and Riesz means** and **conjugate maximal operators** of a distribution f are introduced by

$$\tilde{\sigma}_n^{(i);q,\alpha} f(x) := \tilde{f}^{(i)} * K_n^{q,\alpha} \qquad (i = 1, \ldots, d)$$

and

$$\tilde{\sigma}_*^{(i);q,\alpha} f := \sup_{n \geq 1} |\tilde{\sigma}_n^{(i);q,\alpha} f|,$$

respectively. We use the notations

$$\tilde{f}^{(0)} = f, \quad \tilde{\sigma}_n^{(0);q,\alpha} f = \sigma_n^{q,\alpha} f \quad \text{and} \quad \tilde{\sigma}_*^{(0);q,\alpha} f = \sigma_*^{q,\alpha} f.$$

By (9.1) and (9.2), Theorem 8.1 can be generalized for the conjugate means.

Theorem 9.2 *If $q = 1, \infty$, $\alpha \geq 1$ and $d/(d+1) < p < \infty$, then for all $i = 0, 1, \ldots, d$,*

$$\|\tilde{\sigma}_*^{(i);q,\alpha} f\|_p \leq C_p \|f\|_{H_p^\Box} \qquad (f \in H_p^\Box(\mathbb{T}^d)). \tag{9.3}$$

If $q = 2$ and $\alpha > (d-1)/2$, then the same holds with $p > d/(d/2 + \alpha + 1/2)$. In particular, if $f \in L_1(\mathbb{T}^d)$, then

$$\sup_{\rho > 0} \rho \, \lambda(\tilde{\sigma}_*^{(i);q,\alpha} f > \rho) \leq C \|f\|_1. \tag{9.4}$$

If, in addition to the conditions just mentioned, we assume that $p > (d-1)/d$, then

$$\|\tilde{\sigma}_n^{(i);q,\alpha} f\|_{H_p^\Box} \leq C_p \|f\|_{H_p^\Box} \qquad (f \in H_p^\Box(\mathbb{T}^d)).$$

Proof. Theorem 8.1 and (9.1) imply

$$\|\tilde{\sigma}_*^{(i);q,\alpha} f\|_p = \|\sigma_*^{q,\alpha} \tilde{f}^{(i)}\|_p \le C_p \|\tilde{f}^{(i)}\|_{H_p^\square} \le C_p \|f\|_{H_p^\square} \qquad (f \in H_p^\square(\mathbb{T}^d)),$$

which is exactly (9.3). Inequality (9.4) follows by interpolation as above. Since

$$(\sigma_n^{q,\alpha} f)^{\sim(i)} = \tilde{\sigma}_n^{(i);q,\alpha} f,$$

we have by (9.3) that

$$\|(\sigma_n^{q,\alpha} f)^{\sim(i)}\|_p \le C_p \|f\|_{H_p^\square} \qquad (f \in H_p^\square(\mathbb{T}^d)).$$

Inequality (9.2) implies that

$$\|\tilde{\sigma}_n^{(i);q,\alpha} f\|_{H_p^\square} \le C_p \|f\|_{H_p^\square} \qquad (n \ge 1, f \in H_p^\square(\mathbb{T}^d))$$

if $p > (d-1)/d$. ■

The following result, which is a consequence of the density theorem (Theorem 3.2) due to Marcinkiewicz and Zygmund, is really a generalization of Corollary 8.4 because the conjugate function $\tilde{f}^{(i)}$ is not necessarily integrable.

Corollary 9.3 *Suppose that $q = 1, \infty$ and $\alpha \ge 1$ or $q = 2$ and $\alpha > (d-1)/2$. If $i = 0, 1, \ldots, d$ and $f \in L_1(\mathbb{T}^d)$, then*

$$\lim_{n\to\infty} \tilde{\sigma}_n^{(i);q,\alpha} f = \tilde{f}^{(i)} \qquad \text{a.e.}$$

Moreover, if $f \in H_p^\square(\mathbb{T}^d)$ with $d/(d+1) < p < \infty$, then this convergence also holds in the $H_p^\square(\mathbb{T}^d)$-norm. If $q = 2$ and $\alpha > (d-1)/2$, then the same holds with $p > d/(d/2+\alpha+1/2)$ and $p > (d-1)/d$.

10 ℓ_q-summation defined by θ

10.1 Summation generated by a function

Now, we introduce a general summability method, the so-called θ-summability generated by a single function. A natural question arises, under which conditions on θ can we prove the preceding results for the θ-means. *θ-summation* was considered in many papers and books, such as Butzer and Nessel [15], Trigub and Belinsky [86], Natanson and Zuk [66], Bokor, Schipp, Szili and Vértesi [73, 12, 74, 82, 81], and Feichtinger and Weisz [36, 37, 94, 100, 99, 101].

We suppose that $\theta : \mathbb{R} \to \mathbb{R}$ and

$$\begin{cases} \text{the support of } \theta \text{ is } [-c, c] \ (0 < c \le \infty), \\ \theta \text{ is even and continuous, } \theta(0) = 1, \\ \sum_{k=0}^\infty k^d \left|\Delta_1 \theta(\frac{k}{n})\right| < \infty, \\ \lim_{t\to\infty} t^d \theta(t) = 0, \end{cases} \tag{10.1}$$

where

$$\Delta_1\theta\Big(\frac{k}{n}\Big) := \theta\Big(\frac{k}{n}\Big) - \theta\Big(\frac{k+1}{n}\Big)$$

is the **first difference**. For $q=2$, we suppose furthermore that θ is non-increasing on $(0,\infty)$ or it has compact support. For $q=1,\infty$, Abel rearrangement implies that

$$\sum_{j\in\mathbb{Z}^d}\Big|\theta\Big(\frac{\|j\|_q}{n}\Big)\Big| \le C\sum_{k=0}^{\infty}k^{d-1}\Big|\theta\Big(\frac{k}{n}\Big)\Big| \le C\sum_{k=0}^{\infty}k^{d}\Big|\Delta_1\theta\Big(\frac{k}{n}\Big)\Big| < \infty.$$

The same holds for $q=2$ when θ is non-increasing on $(0,\infty)$. If θ has compact support, then obviously $\sum_{j\in\mathbb{Z}^d}\Big|\theta\Big(\frac{\|j\|_q}{n}\Big)\Big| < \infty$. The ℓ_q**-θ-means** of $f\in L_1(\mathbb{T}^d)$ are given by

$$\sigma_n^{q,\theta}f(x) := \sum_{j\in\mathbb{Z}^d}\theta\Big(\frac{\|j\|_q}{n}\Big)\widehat{f}(j)e^{\imath j\cdot x} = \int_{\mathbb{T}^d} f(x-u)K_n^{q,\theta}(u)\,du,$$

where

$$K_n^{q,\theta}(u) := \sum_{j\in\mathbb{Z}^d}\theta\Big(\frac{\|j\|_q}{n}\Big)e^{\imath j\cdot u}.$$

Let

$$\sigma_*^{q,\theta}f := \sup_{n\ge1}|\sigma_n^{q,\theta}f|$$

be the **maximal θ-operator**. If $q=1,\infty$, we have

$$K_n^{q,\theta}(u) = \sum_{j\in\mathbb{Z}^d}\sum_{k\ge\|j\|_q}\Delta_1\theta\Big(\frac{k}{n}\Big)e^{\imath j\cdot u} = \sum_{k=0}^{\infty}\Delta_1\theta\Big(\frac{k}{n}\Big)D_k^q(u)$$

and

$$\sigma_n^{q,\theta}f(x) = \sum_{k=0}^{\infty}\Delta_1\theta\Big(\frac{k}{n}\Big)s_k^q f(x).$$

If $\theta(t)=\max((1-|t|^\gamma)^\alpha,0)$, then we get back the Riesz (or in special case $\alpha=\gamma=1$, the Fejér) means.

Let first $q=1$ or ∞ and suppose in addition that

$$\begin{cases} \theta \text{ is twice continuously differentiable on } (0,c),\\ \theta''\neq 0 \text{ except at finitely many points and finitely many intervals,}\\ \lim_{t\to0+0} t\theta'(t) \text{ is finite,}\\ \lim_{t\to c-0} t\theta'(t) \text{ is finite,}\\ \lim_{t\to\infty} t\theta'(t)=0. \end{cases} \tag{10.2}$$

Let $\mathbf{X}$ and $\mathbf{Y}$ be two complete quasi-normed spaces of measurable functions, $L_\infty(\mathbb{T}^d)$ be continuously embedded into $\mathbf{X}$, and $L_\infty(\mathbb{T}^d)$ be dense in $\mathbf{X}$. Suppose that if $0\le f\le g$, $f,g\in\mathbf{Y}$, then $\|f\|_{\mathbf{Y}}\le\|g\|_{\mathbf{Y}}$. If $f_n,f\in\mathbf{Y}$, $f_n\ge0$ $(n\in\mathbb{N})$ and $f_n\nearrow f$ a.e. as $n\to\infty$, then assume that $\|f-f_n\|_{\mathbf{Y}}\to0$. Note that the spaces $L_p(\mathbb{T}^d)$ and $L_{p,\infty}(\mathbb{T}^d)$ $(0<p\le\infty)$ satisfy these properties. Recall that σ_*^q denotes the maximal Fejér operator.

Theorem 10.1 *Assume that $q = 1, \infty$ and (10.1) and (10.2) are satisfied. If $\sigma_*^q : \mathbf{X} \to \mathbf{Y}$ is bounded, i.e.,*

$$\|\sigma_*^q f\|_{\mathbf{Y}} \leq C\|f\|_{\mathbf{X}} \qquad (f \in \mathbf{X} \cap L_\infty(\mathbb{T}^d)),$$

then $\sigma_*^{q,\theta}$ is also bounded,

$$\|\sigma_*^{q,\theta} f\|_{\mathbf{Y}} \leq C\|f\|_{\mathbf{X}} \qquad (f \in \mathbf{X}).$$

Proof. By Abel rearrangement,

$$\sum_{k=0}^{m} \Delta_1\theta\Big(\frac{k}{n}\Big) D_k^q(x) = \sum_{k=0}^{m-1} \Delta_2\theta\Big(\frac{k}{n}\Big) kK_k^q(x) + \Delta_1\theta\Big(\frac{m}{n}\Big) mK_m^q(x),$$

where

$$\Delta_2\theta\Big(\frac{k}{n}\Big) := \Delta_1\theta\Big(\frac{k}{n}\Big) - \Delta_1\theta\Big(\frac{k+1}{n}\Big)$$

is the **second difference**. Observe that, for a fixed x, we have that $K_m^q(x)$ is uniformly bounded in m. By Lagrange's mean value theorem, there exists $m < \xi(m) < m+1$, such that

$$m\Delta_1\theta\Big(\frac{m}{n}\Big) = -\frac{m}{n}\theta'\Big(\frac{\xi(m)}{n}\Big)$$

and this converges to zero if $m \to \infty$. Thus,

$$K_n^{q,\theta}(x) = \sum_{k=0}^{\infty} k\, \Delta_2\theta\Big(\frac{k}{n}\Big) K_k^q(x).$$

Now we prove that

$$\sup_{n\geq 1} \sum_{k=0}^{\infty} k \left|\Delta_2\theta\Big(\frac{k}{n}\Big)\right| \leq C < \infty. \tag{10.3}$$

If $\theta'' \geq 0$ on the interval $(i/n, (j+2)/n)$, then θ is convex on this interval and this yields that

$$\Delta_2\theta\Big(\frac{k}{n}\Big) \geq 0 \quad \text{for} \quad i \leq k \leq j.$$

Hence

$$\begin{aligned}\sum_{k=i}^{j} k \left|\Delta_2\theta\Big(\frac{k}{n}\Big)\right| &= \sum_{k=i}^{j} k\, \Delta_2\theta\Big(\frac{k}{n}\Big) \\ &= \theta\Big(\frac{i}{n}\Big) + (i-1)\,\Delta_1\theta\Big(\frac{i}{n}\Big) - j\,\Delta_1\theta\Big(\frac{j+1}{n}\Big) - \theta\Big(\frac{j+1}{n}\Big).\end{aligned}$$

Applying again Lagrange's mean value theorem, we have

$$(i-1)\left|\Delta_1\theta\Big(\frac{i}{n}\Big)\right| = \frac{i-1}{n}\left|\theta'\Big(\frac{\xi(i)}{n}\Big)\right| = \frac{i-1}{\xi(i)}\left|\frac{\xi(i)}{n}\theta'\Big(\frac{\xi(i)}{n}\Big)\right| \leq C,$$

where $i < \xi(i) < i+1$. Here, we used the fact that the function $x \mapsto |x\theta'(x)|$ is bounded, which follows from (10.2). If $\theta'' = 0$ at an isolated point u or if θ'' is not twice continuously differentiable at u, $u \in (k/n, (k+1)/n)$, then the boundedness of $k\left|\Delta_2\theta\left(\frac{k}{n}\right)\right|$ can be seen in the same way. Since there are only finitely many intervals and isolated points satisfying the above properties, we have shown (10.3).

Hence

$$\sigma_n^{q,\theta} f(x) = \int_{\mathbb{T}^d} f(t) K_n^{q,\theta}(x-t)\,\mathrm{d}t = \sum_{k=0}^{\infty} \int_{\mathbb{T}^d} k\,\Delta_2\theta\Big(\frac{k}{n}\Big) f(t) K_k^q(x-t)\,\mathrm{d}t$$

for all $f \in L_\infty(\mathbb{T}^d)$. Thus

$$\sigma_*^{q,\theta} f \le C\sigma_*^q f \qquad (f \in L_\infty(\mathbb{T}^d))$$

and so

$$\|\sigma_*^{q,\theta} f\|_{\mathbf{Y}} \le C\|f\|_{\mathbf{X}} \qquad (f \in \mathbf{X} \cap L_\infty(\mathbb{T}^d)).$$

By a usual density argument, we finish the proof of the theorem. ■

This theorem implies that the analogues of Theorems 6.1, 6.2, 8.1, 9.2 and Corollary 8.3, 8.4 and 9.3 hold.

Now we give some examples for the θ-summation. It is easy to see that all the next examples satisfy (10.1) and (10.2).

Example 10.2 (Fejér and Riesz summation) Let

$$\theta(t) = \begin{cases} (1-|t|^\gamma)^\alpha & \text{if } |t| \le 1 \\ 0 & \text{if } |t| > 1 \end{cases}$$

for some $1 \le \alpha, \gamma < \infty$.

Example 10.3 (de La Vallée-Poussin summation) Let

$$\theta(t) = \begin{cases} 1 & \text{if } |t| \le 1/2 \\ -2|t|+2 & \text{if } 1/2 < |t| \le 1 \\ 0 & \text{if } |t| > 1. \end{cases}$$

Example 10.4 (Jackson-de La Vallée-Poussin summation) Let

$$\theta(t) = \begin{cases} 1 - 3t^2/2 + 3|t|^3/4 & \text{if } |t| \le 1 \\ (2-|t|)^3/4 & \text{if } 1 < |t| \le 2 \\ 0 & \text{if } |t| > 2. \end{cases}$$

Example 10.5 Let $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m$ and $\beta_0, \ldots, \beta_m$ ($m \in \mathbb{N}$) be real numbers, $\beta_0 = 1$, $\beta_m = 0$. Suppose that θ is even, $\theta(\alpha_j) = \beta_j$ ($j = 0, 1, \ldots, m$), $\theta(t) = 0$ for $t \ge \alpha_m$, θ is a polynomial on the interval $[\alpha_{j-1}, \alpha_j]$ ($j = 1, \ldots, m$).

Example 10.6 (Rogosinski summation) Let

$$\theta(t) = \begin{cases} \cos \pi t/2 & \text{if } |t| \leq 1+2j \\ 0 & \text{if } |t| > 1+2j \end{cases} \qquad \text{for some } j \in \mathbb{N}.$$

Example 10.7 (Weierstrass summation) Let

$$\theta(t) = \mathrm{e}^{-|t|^\gamma} \quad \text{for some} \quad 1 \leq \gamma < \infty.$$

Note that if $\gamma = 1$, then we obtain the Abel means.

Example 10.8 Let

$$\theta(t) = \mathrm{e}^{-(1+|t|^q)^\gamma} \quad \text{for some} \quad 1 \leq q < \infty, 0 < \gamma < \infty.$$

Example 10.9 (Picard and Bessel summations) Let

$$\theta(t) = (1+|t|^\gamma)^{-\alpha} \quad \text{for some} \quad 0 < \alpha < \infty, 1 \leq \gamma < \infty, \alpha\gamma > d.$$

If $q = 2$, then we have to assume other additional conditions instead of (10.2). Let $\theta_0(x) := \theta(\|x\|_2)$ satisfy

$$\theta_0 \in L_1(\mathbb{R}^d) \qquad \text{and} \qquad \widehat{\theta}_0 \in L_1(\mathbb{R}^d). \tag{10.4}$$

Assume that $\widehat{\theta}_0$ is $N+1$-times differentiable ($N \geq 0$) and there exists $d+N-1 < \beta \leq d+N$ such that

$$|\partial_1^{i_1} \cdots \partial_d^{i_d} \widehat{\theta}_0(x)| \leq C\|x\|_2^{-\beta-1} \qquad (x \neq 0), \tag{10.5}$$

whenever $i_1 + \cdots + i_d = N$ or $i_1 + \cdots + i_d = N+1$. If $\beta = d+N$, then it is enough to suppose (10.5) for $i_1 + \cdots + i_d = N+1$. Under the conditions (10.1), (10.4) and (10.5), the analogues of Theorems 6.1, 6.2, 8.1, 9.2 and Corollary 8.3, 8.4 and 9.3 hold with the critical index $d/(\beta+1)$. One can show ([96, 88]) that Example 10.2 with $\alpha > (d-1)/2$, $\gamma \in \mathbb{N}$ and $\beta = (d-1)/2 + \alpha$, Example 10.7 with $0 < \gamma < \infty$ and $\beta = d+N$, Example 10.8 with $0 < \gamma, q < \infty$ and $\beta = d+N$ and Example 10.9 with $\beta = d+N$ satisfy (10.1), (10.4) and (10.5).

10.2 Cesàro summability

The well known *Cesàro summation* is not generated by a function. For $k \in \mathbb{N}, \alpha \neq -1, -2, \ldots$, let

$$A_k^\alpha := \binom{k+\alpha}{k} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+k)}{k!}.$$

It is known (see Zygmund [110, p. 77]) that

$$A_k^\alpha = \sum_{i=0}^{k} A_{k-i}^{\alpha-1}, \qquad A_k^\alpha - A_{k-1}^\alpha = A_k^{\alpha-1}$$

and

$$A_k^\alpha = O(k^\alpha) \qquad (k \in \mathbb{N}). \tag{10.6}$$

Here, we assume that $q = 1$ or $q = \infty$. For $n \geq 1$, the **Cesàro (or (C, α))-means** of a function $f \in L_1(\mathbb{T}^d)$ are defined by

$$\begin{aligned} \sigma_n^{q,(c,\alpha)} f(x) &:= \frac{1}{A_{n-1}^\alpha} \sum_{k \in \mathbb{Z}^d, \, \|k\|_q \leq n} A_{n-1-\|k\|_q}^\alpha \widehat{f}(k) \mathrm{e}^{\mathrm{i}k \cdot x} \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-u) K_n^{q,(c,\alpha)}(u) \, \mathrm{d}u, \end{aligned}$$

where the **Cesàro kernel** is given by

$$\begin{aligned} K_n^{q,(c,\alpha)}(u) &:= \frac{1}{A_{n-1}^\alpha} \sum_{k \in \mathbb{Z}^d, \, \|k\|_q \leq n} A_{n-1-\|k\|_q}^\alpha \mathrm{e}^{\mathrm{i}k \cdot u} \\ &= \frac{1}{A_{n-1}^\alpha} \sum_{\|k\|_q \leq n} \sum_{j=\|k\|_q}^{n-1} A_{n-1-j}^{\alpha-1} \mathrm{e}^{\mathrm{i}k \cdot u} \\ &= \frac{1}{A_{n-1}^\alpha} \sum_{j=0}^{n-1} A_{n-1-j}^{\alpha-1} D_j^q(u). \end{aligned}$$

Hence

$$\sigma_n^{q,(c,\alpha)} f(x) = \frac{1}{A_{n-1}^\alpha} \sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} s_k^q f(x)$$

and if $\alpha = 1$, we get back the Fejér means.

The **conjugate Cesàro means** and **conjugate maximal operators** of a distribution f are introduced by

$$\tilde{\sigma}_n^{(i);q,(c,\alpha)} f(x) := \tilde{f}^{(i)} * K_n^{q,(c,\alpha)}, \qquad \tilde{\sigma}_*^{(i);q,(c,\alpha)} f := \sup_{n \geq 1} |\tilde{\sigma}_n^{(i);q,(c,\alpha)} f|,$$

where $i = 0, 1, \ldots, d$. We proved in [102, 101] that Theorems 6.1, 6.2, 8.1, 9.2 and Corollary 8.3, 8.4 and 9.3 hold for the Cesàro summability with the critical index $\frac{d}{d+\alpha \wedge 1}$. We use the notations

$$a \vee b := \max\{a, b\} \quad \text{and} \quad a \wedge b := \min\{a, b\}$$

for two real numbers a and b. Here, we give only some hints for the proofs.

Instead of the inequalities (6.8), we will use the inequality

$$\Big| \sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} \mathrm{soc}\,((k+1/2)u) \Big| \leq \frac{C}{(\sin(u/2))^\alpha} + \frac{C n^{\alpha-1}}{\sin(u/2)} \tag{10.7}$$

for $0 < \alpha \leq 1$. Indeed,

$$\sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} \mathrm{e}^{\mathrm{i}(k+1/2)u} = \mathrm{e}^{\mathrm{i}(n-1/2)u} \sum_{k=0}^{n-1} A_k^{\alpha-1} \mathrm{e}^{-\mathrm{i}ku}.$$

Since
$$\sum_{k=0}^{\infty} A_k^{\alpha-1} z^k = (1-z)^{-\alpha} \qquad (|z|<1),$$
we have
$$|\sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} \mathrm{e}^{\mathrm{i}(k+1/2)u}| = |(1-\mathrm{e}^{-\mathrm{i}u})^{-\alpha} - \sum_{k=n}^{\infty} A_k^{\alpha-1} \mathrm{e}^{-\mathrm{i}ku}|.$$
As $(A_k^{\alpha-1})_{k\in\mathbb{N}}$ decreases monotonically to 0, the last series converges and the absolute value of its sum can be estimated by $2A_n^{\alpha-1}(1-\mathrm{e}^{-\mathrm{i}u})^{-1}$. Thus
$$|\sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} \mathrm{e}^{\mathrm{i}(k+1/2)u}| \le C|(1-\mathrm{e}^{-\mathrm{i}u})^{-\alpha}| + CA_n^{\alpha-1}|(1-\mathrm{e}^{-\mathrm{i}u})^{-1}|,$$
which together with (10.6) proves (10.7).

For $q=1$, we get similarly to Lemma 6.11 that
$$\begin{aligned} K_n^{1,(c,\alpha)}(x) &= \sum_{(i_l,j_l)\in\mathcal{I}} \frac{(-1)^{i_{d-1}-1}}{A_{n-1}^{\alpha}} \prod_{l=1}^{d-1} (\cos x_{i_l} - \cos x_{j_l})^{-1} \\ &\quad \sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} (G_k(\cos x_{i_{d-1}}) - G_k(\cos x_{j_{d-1}})) \\ &=: \sum_{(i_l,j_l)\in\mathcal{I}} K_{n,(i_l,j_l)}^{1,(c,\alpha)}(x). \end{aligned}$$

Then the following three lemmas can be proved as are Lemmas 6.12, 6.13 and 6.14 (for details, see Weisz [103]).

Lemma 10.10 *For all* $0<\alpha\le 1$ *and* $0<\beta<\frac{\alpha+1}{d-1}$,
$$\begin{aligned} |K_{n,(i_l,j_l)}^{1,(c,\alpha)}(x)| &\le \frac{C}{n^{\alpha}} \prod_{l=1}^{d-1} (x_{i_l}-x_{j_l})^{-1-\beta} x_{j_{d-1}}^{\beta(d-1)-\alpha-1} 1_{\{x_{j_{d-1}}\le\pi/2\}} \\ &\quad + \frac{C}{n} \prod_{l=1}^{d-1} (x_{i_l}-x_{j_l})^{-1-\beta} x_{j_{d-1}}^{\beta(d-1)-2} 1_{\{x_{j_{d-1}}\le\pi/2\}} \\ &\quad + \frac{C}{n^{\alpha}} \prod_{l=1}^{d-1} (x_{i_l}-x_{j_l})^{-1-\beta} (\pi - x_{i_{d-1}})^{\beta(d-1)-\alpha-1} 1_{\{x_{j_{d-1}}>\pi/2\}} \\ &\quad + \frac{C}{n} \prod_{l=1}^{d-1} (x_{i_l}-x_{j_l})^{-1-\beta} (\pi - x_{i_{d-1}})^{\beta(d-1)-2} 1_{\{x_{j_{d-1}}>\pi/2\}}. \end{aligned}$$

Lemma 10.11 *For all* $0 < \alpha \le 1$ *and* $0 < \beta < \frac{\alpha+1}{d-2}$,

$$
\begin{aligned}
|K^{1,(c,\alpha)}_{n,(i_l,j_l)}(x)| \le\ & Cn^{1-\alpha}\prod_{l=1}^{d-2}(x_{i_l}-x_{j_l})^{-1-\beta}x_{j_{d-1}}^{\beta(d-2)-\alpha-1}1_{\{x_{j_{d-1}}\le\pi/2\}} \\
&+C\prod_{l=1}^{d-2}(x_{i_l}-x_{j_l})^{-1-\beta}x_{j_{d-1}}^{\beta(d-2)-2}1_{\{x_{j_{d-1}}\le\pi/2\}} \\
&+Cn^{1-\alpha}\prod_{l=1}^{d-2}(x_{i_l}-x_{j_l})^{-1-\beta}(\pi-x_{i_{d-1}})^{\beta(d-2)-\alpha-1}1_{\{x_{j_{d-1}}>\pi/2\}} \\
&+C\prod_{l=1}^{d-2}(x_{i_l}-x_{j_l})^{-1-\beta}(\pi-x_{i_{d-1}})^{\beta(d-2)-2}1_{\{x_{j_{d-1}}>\pi/2\}}.
\end{aligned}
$$

Lemma 10.12 *If* $0 < \alpha \le 1$ *and* $0 < \beta < \frac{\alpha+1}{d-1} \wedge \frac{d-2}{d-1}$, *then for all* $q = 1, \ldots, d$,

$$
\begin{aligned}
|\partial_q K^{1,(c,\alpha)}_{n,(i_l,j_l)}(x)| \le\ & Cn^{1-\alpha}\prod_{l=1}^{d-1}(x_{i_l}-x_{j_l})^{-1-\beta}x_{j_{d-1}}^{\beta(d-1)-\alpha-1}1_{\{x_{j_{d-1}}\le\pi/2\}} \\
&+C\prod_{l=1}^{d-1}(x_{i_l}-x_{j_l})^{-1-\beta}x_{j_{d-1}}^{\beta(d-1)-2}1_{\{x_{j_{d-1}}\le\pi/2\}} \\
&+Cn^{1-\alpha}\prod_{l=1}^{d-1}(x_{i_l}-x_{j_l})^{-1-\beta}(\pi-x_{i_{d-1}})^{\beta(d-1)-\alpha-1}1_{\{x_{j_{d-1}}>\pi/2\}} \\
&+C\prod_{l=1}^{d-1}(x_{i_l}-x_{j_l})^{-1-\beta}(\pi-x_{i_{d-1}})^{\beta(d-1)-2}1_{\{x_{j_{d-1}}>\pi/2\}}.
\end{aligned}
$$

For $q=\infty$, we obtain as in (6.14)

$$
\begin{aligned}
K_n^{\infty,(c,\alpha)}(x) &= \frac{1}{A^{\alpha}_{n-1}}\sum_{k=0}^{n-1}A^{\alpha-1}_{n-1-k}\prod_{i=1}^{d}\frac{\sin((k+1/2)x_i)}{\sin(x_i/2)} \\
&= \sum_{\epsilon'}\pm 2^{-d+1}\prod_{i=1}^{d}(\sin(x_i/2))^{-1}\frac{1}{A^{\alpha}_{n-1}}\sum_{k=0}^{n-1}A^{\alpha-1}_{n-1-k} \\
&\quad \Big(\mathrm{soc}\,((k+1/2)(\sum_{j=1}^{d-1}\epsilon_j x_j + x_d)) - \mathrm{soc}\,((k+1/2)(\sum_{j=1}^{d-1}\epsilon_j x_j - x_d))\Big) \\
&=: \sum_{\epsilon'}K^{\infty,(c,\alpha)}_{n,\epsilon'}(x).
\end{aligned}
$$

Then

$$
|K^{\infty,(c,\alpha)}_{n,\epsilon'}(x)| \le Cn^d \qquad \text{and} \qquad |K^{\infty,(c,\alpha)}_{n,\epsilon'}(x)| \le C\prod_{i=1}^{d}x_i^{-1}.
$$

Similarly to Lemmas 6.15, 6.16 and 6.17, we can show the next three lemmas (see Weisz [101]).

Lemma 10.13 *If $0 < \alpha \le 1$, then*

$$|K_{n,\epsilon'}^{\infty,(c,\alpha)}(x)| \le C \sum_{\epsilon_d} n^{-\alpha} \Big(\prod_{i=1}^d x_i^{-1} \Big) \Big| \sum_{j=1}^d \epsilon_j x_j \Big|^{-\alpha} + C \sum_{\epsilon_d} n^{-1} \Big(\prod_{i=1}^d x_i^{-1} \Big) \Big| \sum_{j=1}^d \epsilon_j x_j \Big|^{-1}.$$

Lemma 10.14 *If $0 < \alpha \le 1$, then for all $x \in \mathcal{S}_k \setminus \mathcal{S}_{\epsilon'}$, $x \in \mathcal{S}_k \setminus \mathcal{S}'$ $(k = 1, \ldots, d-1)$ and $x \in \mathcal{S}_{\epsilon',d}^c$,*

$$|K_{n,\epsilon'}^{\infty,(c,\alpha)}(x)|| \le C \sum_{\epsilon_d} n^{1-\alpha} \Big(\prod_{i=1}^{d-1} x_i^{-1} \Big) \Big| \sum_{j=1}^d \epsilon_j x_j \Big|^{-\alpha} + C \sum_{\epsilon_d} \Big(\prod_{i=1}^{d-1} x_i^{-1} \Big) \Big| \sum_{j=1}^d \epsilon_j x_j \Big|^{-1}.$$

Lemma 10.15 *If $0 < \alpha \le 1$, then for all $l = 1, \ldots, d$ and $x \in \mathcal{S}$,*

$$\begin{aligned} |\partial_l K_{n,\epsilon'}^{\infty,(c,\alpha)}(x)| \ &\le \ C \sum_{\epsilon_d} n^{1-\alpha} \Big(\prod_{i=1}^d x_i^{-1} \Big) \Big| \sum_{j=1}^d \epsilon_j x_j \Big|^{-\alpha} \\ &\quad + C \sum_{\epsilon_d} \Big(\prod_{i=1}^d x_i^{-1} \Big) \Big| \sum_{j=1}^d \epsilon_j x_j \Big|^{-1} \\ &\quad + C \sum_{\epsilon_d} \Big(\prod_{i=1}^d x_i^{-1} \Big) x_d^{-1} 1_{\cup_{k=1}^{d-1}(\mathcal{S}_k \cap \mathcal{S}_{\epsilon'}) \cup (\mathcal{S}_d \cap \mathcal{S}_{\epsilon',d})}(x). \end{aligned}$$

11 ℓ_q-summability of Fourier transforms

The above results hold also for summability means of Fourier transforms. First suppose that $f \in L_p(\mathbb{R}^d)$ for some $1 \le p \le 2$. It is known that if $\widehat{f} \in L_1(\mathbb{R}^d)$, then

$$f(x) = \int_{\mathbb{R}^d} \widehat{f}(v) \mathrm{e}^{\mathrm{i}x \cdot v} \, \mathrm{d}v \qquad (x \in \mathbb{R}^d)$$

(see e.g. Butzer and Nessel [15]). This motivates the definition of the **Dirichlet integral** $s_t^q f$,

$$s_t^q f(x) := \int_{\mathbb{R}^d} 1_{\{\|v\|_q \le t\}} \widehat{f}(v) \mathrm{e}^{\mathrm{i}x \cdot v} \, \mathrm{d}v = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(x-u) D_t^q(u) \, \mathrm{d}u \qquad (t > 0),$$

where the **Dirichlet kernel** is defined by

$$D_t^q(u) := \int_{\mathbb{R}^d} 1_{\{\|v\|_q \le t\}} \mathrm{e}^{\mathrm{i}u \cdot v} \, \mathrm{d}v.$$

Carleson's theorem also holds in this case. More exactly, replacing $\mathbb{T}$ by $\mathbb{R}$, Theorems 2.1, 2.3, 4.2, 4.3, 4.4 and 4.6 remain true (see Grafakos [43]).

Let $\theta_0^q(u) := \theta(\|u\|_q)$ and suppose that $\theta_0^q \in L_1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$. For $T > 0$, the **ℓ_q-θ-means** of a function $f \in L_p(\mathbb{R}^d)$ $(1 \leq p \leq 2)$ are defined by

$$\sigma_T^{q,\theta} f(x) := \int_{\mathbb{R}^d} \theta\Big(\frac{\|v\|_q}{T}\Big) \widehat{f}(v) e^{\imath x \cdot v} \, dv.$$

It is easy to see that

$$\sigma_T^{q,\theta} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(x-u) K_T^{q,\theta}(u) \, du$$

where the **ℓ_q-θ-kernel** is given by

$$K_T^{q,\theta}(u) = \int_{\mathbb{R}^d} \theta\Big(\frac{\|v\|_q}{T}\Big) e^{\imath u \cdot v} \, dv = (2\pi)^d T^d \widehat{\theta_0^q}(Tu). \tag{11.1}$$

On the other hand, by the first equality of (11.1),

$$K_T^{q,\theta}(u) = \frac{-1}{T} \int_{\mathbb{R}^d} \int_{\|v\|_q}^{\infty} \theta'\Big(\frac{t}{T}\Big) \, dt \, e^{\imath u \cdot v} \, dv = \frac{-1}{T} \int_0^{\infty} \theta'\Big(\frac{t}{T}\Big) D_t^q(u) \, dt.$$

Hence

$$\sigma_T^{q,\theta} f(x) = \frac{-1}{T} \int_0^{\infty} \theta'\Big(\frac{t}{T}\Big) s_t^q f(x) \, dt.$$

Note that, for the Fejér means (i.e., for $\theta(t) = \max((1-|t|), 0)$), we get the usual definition

$$\sigma_T^q f(x) = \frac{1}{T} \int_0^T s_t^q f(x) \, dt.$$

On $\mathbb{R}^d$, we will consider tempered distributions rather than distributions. In this case the **Schwartz class** $\mathcal{S}(\mathbb{R}^d)$ consists of all $f \in C^\infty(\mathbb{R}^d)$ for which

$$\|f\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta f(x)| < \infty, \tag{11.2}$$

where for the multiindices $\alpha = (\alpha_1, \ldots, \alpha_d), \beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}^d$, we use the conventional notations

$$x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \qquad \partial^\beta = \partial_1^{\beta_1} \cdots \partial_d^{\beta_d}.$$

A **tempered distribution** u (briefly $u \in \mathcal{S}'(\mathbb{R}^d)$) is a linear functional on $\mathcal{S}(\mathbb{R}^d)$ that is continuous in the topology on $\mathcal{S}(\mathbb{R}^d)$ induced by the family of seminorms (11.2). Namely, $f_n \to f$ in $\mathcal{S}(\mathbb{R}^d)$ if

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta (f_n - f)(x)| = 0 \qquad \text{for all multiindices } \alpha, \beta.$$

Moreover, if u is a tempered distribution, then it is linear and $u(f_n) \to u(f)$ if $f_n \to f$ in $\mathcal{S}(\mathbb{R}^d)$. The functions from $L_p(\mathbb{R}^d)$ $(1 \leq p \leq \infty)$ can be identified with tempered distributions

$u \in \mathcal{S}'(\mathbb{R}^d)$ as in Section 2. We say that the distributions u_j tend to the distribution u **in the sense of tempered distributions** or in $\mathcal{S}'(\mathbb{R}^d)$ if $u_j(f) \to u(f)$ for all $f \in \mathcal{S}(\mathbb{R}^d)$ as $j \to \infty$. For more about tempered distributions, see e.g. Stein and Weiss [80] or Grafakos [43].

Now the definition of the ℓ_q-θ**-means** can be extended to tempered distributions by

$$\sigma_T^{q,\theta} f := f * K_T^{q,\theta} \qquad (T > 0).$$

Indeed, the convolution $f * g$ is again well defined for all $g \in L_1(\mathbb{R}^d)$ and $f \in L_p(\mathbb{R}^d)$ $(1 \leq p \leq \infty)$ or $f \in B$, where B is a homogeneous Banach space on $\mathbb{R}^d$. For a tempered distribution $u \in \mathcal{S}'(\mathbb{T}^d)$ and $g \in \mathcal{S}(\mathbb{T}^d)$, we define the tempered distribution $u * g$ by (6.2) as in Section 6. We can show again that $u * g$ is equal to the function $x \mapsto u(T_x \tilde{g})$, which is a C^∞ function. We say that a tempered distribution u is L_r**-bounded** if $u * g \in L_r(\mathbb{R}^d)$ for all $g \in \mathcal{S}(\mathbb{R}^d)$. Then we can define $u * g$ as in (6.3) for all L_r-bounded tempered distributions u and $g \in L_{r'}(\mathbb{R}^d)$, where $1/r + 1/r' = 1$.

Since $\theta_0^q \in L_1(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$, we have $\widehat{\theta_0^q} \in L_{r'}(\mathbb{R}^d)$ $(2 \leq r' \leq \infty)$ and the same holds for $K_T^{q,\theta}$ by (11.1). Since all tempered distributions $f \in H_p^\square(\mathbb{R}^d)$ $(0 < p \leq \infty)$ are L_r-bounded for all $p \leq r \leq \infty$, $\sigma_T^{q,\theta} f$ is well defined as a tempered distribution for all $f \in H_p^\square(\mathbb{R}^d)$.

In the definition of the **homogeneous Banach space** B containing Lebesgue measurable functions on $\mathbb{R}^d$, we have to replace (iii) at the beginning of Section 6 by

(iii') for every compact set $K \subset \mathbb{R}^d$ there exists a constant C_K such that

$$\int_K |f| \,\mathrm{d}\lambda \leq C_K \|f\|_B \qquad (f \in B).$$

The **Hardy space** $H_p^\square(\mathbb{R}^d)$, which is defined with the help of the **non-periodic Poisson kernel**

$$P_t^d(x) := \frac{c_d t}{(t^2 + |x|^2)^{(d+1)/2}} \qquad (t > 0, x \in \mathbb{R}^d),$$

has the same properties as the periodic space $H_p^\square(\mathbb{T}^d)$. We get the same Hardy space with equivalent norms if we use the kernel ϕ_t instead of the Poisson kernel, where $\phi \in \mathcal{S}(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} \phi \,\mathrm{d}\lambda \neq 0$ and

$$\phi_t(x) := t^{-d} \phi(x/t) \qquad (t > 0, x \in \mathbb{R}^d).$$

Starting from this definition, we can verify that a tempered distribution from $H_p^\square(\mathbb{R}^d)$ is L_r-bounded for all $p \leq r \leq \infty$ (see Stein [78, p. 100]).

For a tempered distribution f, the **conjugate distributions** or **Riesz transforms** are defined by

$$(\tilde{f}^{(i)})^\wedge(t) := -\mathrm{i}\, \frac{t_i}{\|t\|_2} \widehat{f}(t) \qquad (t \in \mathbb{R}^d, i = 1, \ldots, d).$$

One can show that for an integrable function f,

$$\tilde{f}^{(i)}(x) := \text{p.v.} \int_{\mathbb{T}} f(x-t) \Phi_i(t) \,\mathrm{d}t := \lim_{\epsilon \to 0} \int_{\epsilon < \|t\|_2} f(x-t) \Phi_i(t) \,\mathrm{d}t \qquad \text{a.e.},$$

where

$$\widehat{\Phi}_i(t) = -\mathrm{i}\,\frac{t_i}{\|t\|_2}, \qquad \Phi_i(t) = \frac{c_d t_i}{\|t\|_2^{d+1}} \qquad (t \in \mathbb{R}^d).$$

Here, c_d is the constant appearing in the definition of the Poisson kernel. In the one-dimensional case the transform is called again the Hilbert transform and

$$\widehat{\Phi}(t) = -\mathrm{i}\ \mathrm{sign}\ t, \qquad \Phi(t) = \frac{1}{\pi t} \qquad (t \in \mathbb{R}).$$

We have proved in [104, 96, 99, 101] that the same results hold for the operator $\sigma_T^{q,\theta}$ and for the **maximal operator**

$$\sigma_*^{q,\theta} f := \sup_{T \in \mathbb{R}_+} |\sigma_T^{q,\theta} f|$$

as in Sections 6–10 with the difference that we can allow $0 < \alpha < \infty$ for $q = 1, \infty$ with the critical index $\frac{d}{d+\alpha\wedge 1}$. It is easy to see that the operators $\sigma_T^{q,\theta}$ are uniformly bounded on $L_p(\mathbb{R}^d)$ if and only if $\sigma_1^{q,\theta}$ is bounded on $L_p(\mathbb{R}^d)$. We point out that the space $C_u(\mathbb{R}^d)$ of uniformly continuous bounded functions endowed with the supremum norm is also a homogeneous Banach space:

Corollary 11.1 *Assume that (10.1)–(10.5) are satisfied. If f is a uniformly continuous and bounded function, then*

$$\lim_{T\to\infty} \sigma_T^{q,\theta} f \to f \qquad \textit{uniformly.}$$

12 Rectangular summability

We now investigate the other type of summability method, the *rectangular summability.* The **rectangular Fejér and Riesz means** of $f \in L_1(\mathbb{T}^d)$ are defined by

$$\begin{aligned}\sigma_n f(x) &= \sum_{|k_1|\le n_1} \cdots \sum_{|k_d|\le n_d} \prod_{i=1}^d \Big(1 - \frac{|k_i|}{n_i}\Big) \widehat{f}(k) e^{\mathrm{i}k\cdot x} \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-u) K_n(u)\,\mathrm{d}u\end{aligned}$$

and

$$\begin{aligned}\sigma_n^\alpha f(x) &= \sum_{|k_1|\le n_1} \cdots \sum_{|k_d|\le n_d} \prod_{i=1}^d \Big(1 - \Big(\frac{|k_i|}{n_i}\Big)^\gamma\Big)^\alpha \widehat{f}(k) e^{\mathrm{i}k\cdot x} \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-u) K_n^\alpha(u)\,\mathrm{d}u,\end{aligned}$$

respectively, where the **rectangular Fejér and Riesz kernels** are given by

$$K_n(u) := \sum_{|k_1|\le n_1} \cdots \sum_{|k_d|\le n_d} \prod_{i=1}^d \Big(1 - \frac{|k_i|}{n_i}\Big) e^{\mathrm{i}k\cdot u} = \frac{1}{\prod_{i=1}^d n_i} \sum_{k_1=1}^{n_1-1} \cdots \sum_{k_d=1}^{n_d-1} D_k(u)$$

and

$$K_n^\alpha(u) := \sum_{|k_1|\le n_1} \cdots \sum_{|k_d|\le n_d} \prod_{i=1}^d \Big(1-\Big(\frac{|k_i|}{n_i}\Big)^\gamma\Big)^\alpha e^{\imath k\cdot u},$$

respectively (see Figure 25).

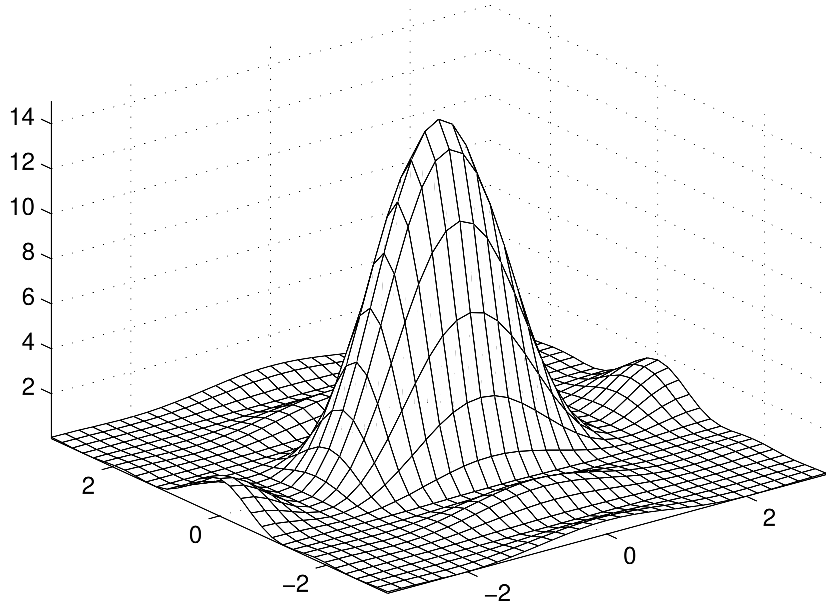

Figure 25: The rectangular Fejér kernel K_n with $d=2$, $n_1=3$, $n_2=5$, $\alpha=1$, $\gamma=1$.

We could choose also different exponents α_i and γ_i in the product. Again, the Fejér means are the arithmetic means of the partial sums:

$$\sigma_n f(x) = \frac{1}{\prod_{i=1}^d n_i} \sum_{k_1=1}^{n_1-1} \cdots \sum_{k_d=1}^{n_d-1} s_k f(x).$$

13 Norm convergence of rectangular summability means

We extend the definition of the Fejér and Riesz means to distributions by

$$\sigma_n^\alpha f(x) := f * K_n^\alpha \qquad (n\in\mathbb{N}).$$

This is well defined for all $f\in H_p(\mathbb{T}^d)$ $(0<p\le\infty)$ (see Section 15), for all $f\in L_p(\mathbb{T}^d)$ $(1\le p\le\infty)$ and for all $f\in B$, where B is a homogeneous Banach space.

The norm convergence of the rectangular means follows immediately from the one-dimensional result by iteration. Since the d-dimensional Riesz kernel is the Kronecker product of the one-dimensional kernels,

$$K_n^\alpha(u) = (K_{n_1}^\alpha \otimes \cdots \otimes K_{n_d}^\alpha)(u) := K_{n_1}^\alpha(u_1) \cdots K_{n_d}^\alpha(u_d) \qquad (n \in \mathbb{N}^d),$$

we obtain easily

Theorem 13.1 *If $\alpha > 0$, then*

$$\int_{\mathbb{T}^d} |K_n^\alpha(x)| \,\mathrm{d}x \le C \qquad (n \in \mathbb{N}^d).$$

Theorem 13.2 *If $\alpha > 0$ and B is a homogeneous Banach space on $\mathbb{T}^d$, then*

$$\|\sigma_n^\alpha f\|_B \le C \|f\|_B \qquad (n \in \mathbb{N}^d)$$

and

$$\lim_{n\to\infty} \sigma_n^\alpha f = f \qquad \text{in the } B\text{-norm for all } f \in B.$$

Here, the convergence is understood in Pringsheim's sense as in Theorem 4.1. For the almost everywhere convergence, we investigate two types of convergence, the restricted convergence (or the convergence taken in a cone) and the unrestricted convergence (or the convergence taken in Pringsheim's sense).

14 Restricted summability

14.1 Summability over a cone

For a given $\tau \ge 1$, we define a cone by

$$\mathbb{R}_\tau^d := \{x \in \mathbb{R}_+^d : \tau^{-1} \le x_i/x_j \le \tau, i, j = 1, \ldots, d\}.$$

The choice $\tau = 1$ obviously yields the diagonal. The **restricted maximal operator** is defined by

$$\sigma_\square^\alpha f := \sup_{n \in \mathbb{R}_\tau^d} |\sigma_n^\alpha f|.$$

As we can see on Figure 26, in the restricted maximal operator the supremum is taken on a cone only.

Marcinkiewicz and Zygmund [63] were the first who considered the restricted convergence. Similarly to Theorem 8.1, the restricted maximal operator is bounded from $H_p^\square(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$ (Weisz [90, 94]).

Theorem 14.1 *If $\alpha > 0$ and $\max\{d/(d+1), 1/(\alpha \wedge 1 + 1)\} < p \le \infty$, then*

$$\|\sigma_\square^\alpha f\|_p \le C_p \|f\|_{H_p^\square} \qquad (f \in H_p^\square(\mathbb{T}^d)).$$

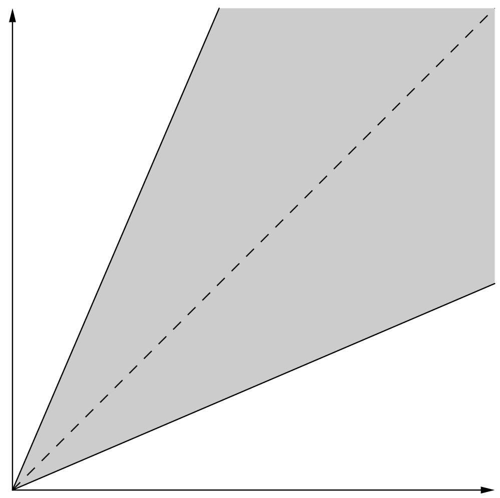

Figure 26: The cone for $d = 2$.

Proof. Let

$$\theta(s) := \begin{cases} (1-|s|^\gamma)^\alpha & \text{if } |s| \le 1 \\ 0 & \text{if } |s| > 1 \end{cases} \qquad (s \in \mathbb{R}).$$

By the one-dimensional version of Corollary 6.23,

$$|\widehat{\theta}(x)|, |(\widehat{\theta})'(x)| \le C|x|^{-\alpha-1} \qquad (x \ne 0).$$

Taking into account (6.19), we conclude that

$$|K_{n_j}^\alpha(x)| \le \frac{C}{n_j^\alpha |x|^{\alpha+1}} \qquad (x \ne 0) \tag{14.1}$$

and

$$|(K_{n_j}^\alpha)'(x)| \le \frac{C}{n_j^{\alpha-1} |x|^{\alpha+1}} \qquad (x \ne 0). \tag{14.2}$$

We will prove the result for $d = 2$, only. For $d > 2$, the verification is very similar. Instead of n_1, n_2 and I_1, I_2 we will write n, m and I, J, respectively. Let a be an arbitrary $H_p^\square$-atom with support $I \times J$ and

$$2^{-K-1} < |I|/\pi = |J|/\pi \le 2^{-K} \qquad (K \in \mathbb{N}).$$

We can suppose again that the center of $I \times J$ is zero. In this case,

$$[-\pi 2^{-K-2}, \pi 2^{-K-2}] \subset I, J \subset [-\pi 2^{-K-1}, \pi 2^{-K-1}].$$

Choose $s \in \mathbb{N}$ such that $2^{s-1} < \tau \leq 2^s$. It is easy to see that if $n \geq k$ or $m \geq k$, then we have $n, m \geq k 2^{-s}$. Indeed, since (n, m) is in a cone, $n \geq k$ implies $m \geq \tau^{-1} n \geq k 2^{-s}$. By Theorem 7.2, it is enough to prove that

$$\int_{\mathbb{T}^2 \setminus 4(I \times J)} |\sigma_{\square}^{\alpha} a(x,y)|^p \,\mathrm{d}x\,\mathrm{d}y \leq C_p. \tag{14.3}$$

First suppose that $\alpha \leq 1$ and let us integrate over $(\mathbb{T} \setminus 4I) \times 4J$. Obviously,

$$\begin{aligned}
\int_{\mathbb{T}\setminus 4I} \int_{4J} |\sigma_{\square}^{\alpha} a(x,y)|^p \,\mathrm{d}x\,\mathrm{d}y \quad &\leq \quad \sum_{|i|=1}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{4J} \sup_{n,m \geq 2^{K-s}} |\sigma_{n,m}^{\alpha} a(x,y)|^p \,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \sum_{|i|=1}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{4J} \sup_{n,m < 2^K} |\sigma_{n,m}^{\alpha} a(x,y)|^p \,\mathrm{d}x\,\mathrm{d}y \\
&=: \quad (A) + (B).
\end{aligned}$$

We can suppose that $i > 0$. Using that

$$\int_{\mathbb{T}} |K_m^{\alpha}| \,\mathrm{d}\lambda \leq C \qquad (m \in \mathbb{N}),$$

(14.1) and the definition of the atom, we conclude

$$\begin{aligned}
|\sigma_{n,m}^{\alpha} a(x,y)| \quad &= \quad \Big| \int_I \int_J a(t,u) K_n^{\alpha}(x-t) K_m^{\alpha}(y-u) \,\mathrm{d}t\,\mathrm{d}u \Big| \\
&\leq \quad C_p 2^{2K/p} \int_I \frac{1}{n^{\alpha} |x-t|^{\alpha+1}} \,\mathrm{d}t.
\end{aligned}$$

For $x \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$ $(i \geq 1)$ and $t \in I$, we have

$$\frac{1}{|x-t|^{\nu}} \leq \frac{1}{(\pi i 2^{-K} - \pi 2^{-K-1})^{\nu}} \leq \frac{C 2^{K\nu}}{i^{\nu}} \qquad (\nu > 0). \tag{14.4}$$

From this, it follows that

$$|\sigma_{n,m}^{\alpha} a(x,y)| \leq C_p 2^{2K/p + K\alpha} \frac{1}{n^{\alpha} i^{\alpha+1}}.$$

Since $n \geq 2^K 2^{-s}$, we obtain

$$(A) \leq C_p \sum_{i=1}^{2^K-1} 2^{-2K} 2^{2K + K\alpha p} \frac{1}{2^{K\alpha p} i^{(\alpha+1)p}} \leq C_p \sum_{i=1}^{2^K-1} \frac{1}{i^{(\alpha+1)p}},$$

which is a convergent series if $p > 1/(\alpha+1)$.

To consider (B), let $I = J = (-\mu, \mu)$ and

$$A_1(x,v) := \int_{-\pi}^{x} a(t,v)\,\mathrm{d}t \qquad \text{and} \qquad A_2(x,y) := \int_{-\pi}^{y} A_1(x,t)\,\mathrm{d}t. \tag{14.5}$$

Then

$$|A_k(x,y)| \le C_p 2^{K(2/p-k)}. \tag{14.6}$$

Integrating by parts, we get that

$$\int_I a(t,u) K_n^\alpha(x-t)\,\mathrm{d}t = A_1(\mu,u)K_n^\alpha(x-\mu) - \int_I A_1(t,u)(K_n^\alpha)'(x-t)\,\mathrm{d}t. \tag{14.7}$$

Recall that the one-dimensional kernel K_m^α satisfies

$$|K_m^\alpha| \le Cm \qquad (m \in \mathbb{N}).$$

For $x \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$, the inequalities (14.1), (14.4) and (14.6) imply

$$\begin{aligned} \Big| \int_J A_1(\mu,u) K_n^\alpha(x-\mu) K_m^\alpha(y-u)\,\mathrm{d}u \Big| &\le C_p 2^{2K/p-K} 2^{-K} \frac{1}{n^\alpha |x-\mu|^{\alpha+1}} m \\ &\le C_p 2^{2K/p+K\alpha-K} n^{1-\alpha} \frac{1}{i^{\alpha+1}}. \end{aligned}$$

Moreover, by (14.2), (14.4) and (14.6),

$$\begin{aligned} \Big| \int_J \int_I A_1(t,u)(K_n^\alpha)'(x-t) K_m^\alpha(y-u)\,\mathrm{d}u\,\mathrm{d}t \Big| &\le C_p 2^{2K/p-K} \int_I \frac{1}{n^{\alpha-1}|x-t|^{\alpha+1}}\,\mathrm{d}t \\ &\le C_p 2^{2K/p+K\alpha-K} n^{1-\alpha} \frac{1}{i^{\alpha+1}}. \end{aligned}$$

Consequently,

$$(B) \le C_p \sum_{i=1}^{2^K-1} 2^{-2K} 2^{2K+K\alpha p-Kp} 2^{K(1-\alpha)p} \frac{1}{i^{(\alpha+1)p}} \le C_p \sum_{i=1}^{2^K-1} \frac{1}{i^{(\alpha+1)p}} < \infty,$$

because $p > 1/(\alpha+1)$. Hence, we have proved that in this case

$$\int_{\mathbb{T}\setminus 4I} \int_{4J} |\sigma_\square^\alpha a(x,y)|^p\,\mathrm{d}x\,\mathrm{d}y \le C_p.$$

Next, we integrate over $(\mathbb{T}\setminus 4I) \times (\mathbb{T}\setminus 4J)$,

$$\begin{aligned} &\int_{\mathbb{T}\setminus 4I} \int_{\mathbb{T}\setminus 4J} |\sigma_\square^\alpha a(x,y)|^p\,\mathrm{d}x\,\mathrm{d}y \\ &\le \sum_{|i|=1}^{\infty} \sum_{|j|=1}^{\infty} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{\pi j 2^{-K}}^{\pi(j+1)2^{-K}} \sup_{n,m \ge 2^{K-s}} |\sigma_{n,m}^\alpha a(x,y)|^p\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \sum_{|i|=1}^{\infty} \sum_{|j|=1}^{\infty} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{\pi j 2^{-K}}^{\pi(j+1)2^{-K}} \sup_{n,m < 2^K} |\sigma_{n,m}^\alpha a(x,y)|^p\,\mathrm{d}x\,\mathrm{d}y \\ &=: (C) + (D). \end{aligned}$$

We may suppose again that $i,j>0$.

For $x \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$ and $y \in [\pi j 2^{-K}, \pi(j+1)2^{-K})$, we have by (14.1) and (14.4) that

$$\begin{aligned} |\sigma_{n,m}^{\alpha} a(x,y)| &\leq C_p 2^{2K/p} \int_I \frac{1}{n^{\alpha}|x-t|^{\alpha+1}} \,\mathrm{d}t \int_J \frac{1}{m^{\alpha}|y-u|^{\alpha+1}} \,\mathrm{d}u \\ &\leq C_p \frac{2^{2K/p+K\alpha+K\alpha}}{n^{\alpha} m^{\alpha} i^{\alpha+1} j^{\alpha+1}}. \end{aligned}$$

This implies that

$$\begin{aligned} (C) &\leq C_p \sum_{i=1}^{2^K-1} \sum_{j=1}^{2^K-1} 2^{-2K} \frac{2^{2K+K\alpha p+K\alpha p}}{2^{K\alpha p+K\alpha p} i^{(\alpha+1)p} j^{(\alpha+1)p}} \\ &\leq C_p \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{i^{(\alpha+1)p} j^{(\alpha+1)p}} < \infty. \end{aligned}$$

Using (14.7) and integrating by parts in both variables, we get that

$$\begin{aligned} &\int_I \int_J a(t,u) K_n^{\alpha}(x-t) K_m^{\alpha}(y-u) \,\mathrm{d}t \,\mathrm{d}u \\ &\quad = -\int_J A_2(\mu,u) K_n^{\alpha}(x-\mu) (K_m^{\alpha})'(y-u) \,\mathrm{d}u \\ &\quad\quad + \int_I A_2(t,\mu) (K_n^{\alpha})'(x-t) K_m^{\alpha}(y-\mu) \,\mathrm{d}t \\ &\quad\quad - \int_I \int_J A_2(t,u) (K_n^{\alpha})'(x-t) (K_m^{\alpha})'(y-u) \,\mathrm{d}t \,\mathrm{d}u \\ &\quad =: D_{n,m}^1(x,y) + D_{n,m}^2(x,y) + D_{n,m}^3(x,y). \end{aligned} \tag{14.8}$$

Note that $A(\mu,-\mu) = A(\mu,\mu) = 0$. Since $|K_n^{\alpha}| \leq Cn$ and (14.1) holds as well, we obtain

$$|K_n^{\alpha}(x)| \leq C \frac{n^{\eta+\alpha(\eta-1)}}{|x|^{(\alpha+1)(1-\eta)}}$$

for all $0 \leq \eta \leq 1$. It is easy to see that

$$|(K_m^{\alpha})'| \leq Cm^2 \qquad (m \in \mathbb{N}).$$

Then

$$|(K_m^{\alpha})'(y)| \leq C \frac{m^{2\zeta+(\alpha-1)(\zeta-1)}}{|y|^{(\alpha+1)(1-\zeta)}} = C \frac{m^{\zeta+1+\alpha(\zeta-1)}}{|y|^{(\alpha+1)(1-\zeta)}} \tag{14.9}$$

follows from (14.2) for all $0 \leq \zeta \leq 1$. Inequalities (14.4) and (14.6) imply

$$\begin{aligned} |D_{n,m}^1(x,y)| &\leq C_p 2^{2K/p-2K} \frac{n^{\eta+\alpha(\eta-1)}}{|x-\mu|^{(\alpha+1)(1-\eta)}} \int_J \frac{m^{\zeta+1+\alpha(\zeta-1)}}{|y-u|^{(\alpha+1)(1-\zeta)}} \,\mathrm{d}u \\ &\leq C_p 2^{2K/p-3K} n^{\eta+\alpha(\eta-1)} \left(\frac{2^K}{i}\right)^{(\alpha+1)(1-\eta)} \\ &\quad m^{\zeta+1+\alpha(\zeta-1)} \left(\frac{2^K}{j}\right)^{(\alpha+1)(1-\zeta)}, \end{aligned} \tag{14.10}$$

whenever $x \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$, $y \in [\pi j 2^{-K}, \pi(j+1)2^{-K})$ and $0 \le \eta, \zeta \le 1$. If

$$\eta + \alpha(\eta - 1) + \zeta + 1 + \alpha(\zeta - 1) \ge 0,$$

then

$$\sup_{n,m<2^K} |D^1_{n,m}(x,y)| \le C_p 2^{2K/p} \frac{1}{i^{(\alpha+1)(1-\eta)}} \frac{1}{j^{(\alpha+1)(1-\mu)}}$$

because (n,m) is in a cone. Choosing

$$\eta := \zeta := \frac{2\alpha - 1}{2(\alpha+1)} \vee 0,$$

we can see that

$$\begin{aligned}
&\int_{\mathbb{T}\setminus 4I} \int_{\mathbb{T}\setminus 4J} \sup_{n,m<2^K} |D^1_{n,m}(x,y)|^p \,\mathrm{d}x\,\mathrm{d}y \\
&\le \quad C_p \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} 2^{-2K} 2^{2K} \frac{1}{i^{3p/2 \wedge (\alpha+1)p}} \frac{1}{j^{3p/2 \wedge (\alpha+1)p}},
\end{aligned}$$

which is a convergent series. The analogous estimate for $|D^2_{n,m}(x,y)|$ can be similarly proved.

For $x \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$ and $y \in [\pi j 2^{-K}, \pi(j+1)2^{-K})$, we conclude that

$$\begin{aligned}
|D^3_{n,m}(x,y)| &\le C_p 2^{2K/p-2K} \int_I \frac{1}{n^{\alpha-1}|x-t|^{\alpha+1}} \,\mathrm{d}t \int_J \frac{1}{m^{\alpha-1}|y-u|^{\alpha+1}} \,\mathrm{d}u \\
&\le C_p \frac{2^{2K/p-2K+K\alpha+K\alpha} n^{1-\alpha} m^{1-\alpha}}{i^{\alpha+1} j^{\alpha+1}}.
\end{aligned}$$

So

$$\begin{aligned}
&\int_{\mathbb{T}\setminus 4I} \int_{\mathbb{T}\setminus 4J} \sup_{n,m<2^K} |D^3_{n,m}(x,y)|^p \,\mathrm{d}x\,\mathrm{d}y \\
&\le C_p \sum_{i=1}^{2^K-1} \sum_{j=1}^{2^K-1} 2^{-2K} \frac{2^{2K-2Kp+K\alpha p+K\alpha p} 2^{K(2-\alpha-\alpha)p}}{i^{(\alpha+1)p} j^{(\alpha+1)p}} \\
&\le C_p \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{i^{(\alpha+1)p}} \frac{1}{j^{(\alpha+1)p}} < \infty
\end{aligned}$$

by the hypothesis. The integration over $4I \times (\mathbb{T} \setminus 4J)$ can be done as above. This finishes the proof of (14.3).

Now let $\alpha > 1$. Since $|\widehat{\theta}| \le C$ and $|(\widehat{\theta})'(x)| \le C$ trivially and since $|x|^{-\alpha-1} \le |x|^{-2}$ if $|x| \ge 1$, we conclude that

$$|\widehat{\theta}(x)|, |(\widehat{\theta})'(x)| \le C|x|^{-2} \qquad (x \ne 0).$$

Hence

$$|K^\alpha_{n_j}(x)| \le \frac{C}{n_j |x|^2}, \qquad |(K^\alpha_{n_j})'(x)| \le \frac{C}{|x|^2} \qquad (x \ne 0)$$

and (14.3) can be proved as above. The theorem follows from Theorem 7.2. ∎

Remark 14.2 In the d-dimensional case, the constant $d/(d+1)$ appears if we investigate the corresponding term to $D^1_{n,m}$. More exactly, if we integrate the term

$$\int_{I_d} A(\mu, \cdots, \mu, u) K^\alpha_{n_1}(x_1 - \mu) \cdots K^\alpha_{n_{d-1}}(x_{d-1} - \mu)(K^\alpha_{n_d})'(x_d - u)\,\mathrm{d}u$$

over $(\mathbb{T} \setminus 4I_1) \times \cdots \times (\mathbb{T} \setminus 4I_d)$ similarly to (14.10), then we get that $p > d/(d+1)$.

Since $H^\square_p(\mathbb{T}^d) \sim L_p(\mathbb{T}^d)$ for $1 < p \leq \infty$, we have

$$\|\sigma^\alpha_\square f\|_p \leq C_p \|f\|_p \qquad (f \in L_p(\mathbb{T}^d), 1 < p \leq \infty).$$

As we have seen in Theorem 8.2, in the one-dimensional case, the operator $\sigma^\alpha_\square$ is not bounded from $H^\square_p(\mathbb{T})$ to $L_p(\mathbb{T})$ if $0 < p \leq 1/(\alpha+1)$ and $\alpha = 1$. Using interpolation and Theorem 14.1, we obtain the weak type $(1,1)$ inequality.

Corollary 14.3 *If $\alpha > 0$ and $f \in L_1(\mathbb{T}^d)$, then*

$$\sup_{\rho>0} \rho\lambda(\sigma^\alpha_\square f > \rho) \leq C\|f\|_1.$$

The density argument of Marcinkiewicz and Zygmund (Theorem 3.2) implies

Corollary 14.4 *If $\alpha > 0$ and $f \in L_1(\mathbb{T}^d)$, then*

$$\lim_{n\to\infty,\, n\in\mathbb{R}^d_\tau} \sigma^\alpha_n f = f \qquad \text{a.e.}$$

This result was proved by Marcinkiewicz and Zygmund [63] for the two-dimensional Fejér means. The general version of Corollary 14.4 is due to the author [90, 94].

Similarly to Theorem 9.2 and Corollary 9.3, we can prove for the **conjugate operators**

$$\tilde\sigma^{(i);\alpha}_n f(x) := \tilde f^{(i)} * K^\alpha_n \qquad (i = 0, 1, \ldots, d)$$

and

$$\tilde\sigma^{(i);\alpha}_\square f := \sup_{n\in\mathbb{R}^d_\tau} |\tilde\sigma^{(i);\alpha}_n f|$$

the following results.

Theorem 14.5 *If $\alpha > 0$ and $\max\{d/(d+1), 1/(\alpha \wedge 1 + 1)\} < p < \infty$, then for all $i = 0, 1, \ldots, d$,*

$$\|\tilde\sigma^{(i);\alpha}_\square f\|_p \leq C_p \|f\|_{H^\square_p} \qquad (f \in H^\square_p(\mathbb{T}^d))$$

and

$$\|\tilde\sigma^{(i);\alpha}_n f\|_{H^\square_p} \leq C_p \|f\|_{H^\square_p} \qquad (n \in \mathbb{N}^d, f \in H^\square_p(\mathbb{T}^d)).$$

In particular, if $f \in L_1(\mathbb{T}^d)$, then

$$\sup_{\rho>0} \rho\, \lambda(\tilde\sigma^{(i);\alpha}_\square f > \rho) \leq C\|f\|_1.$$

The proof of this result is similar to that of Theorem 9.2.

Corollary 14.6 *If $\alpha > 0$, $i = 0, 1, \ldots, d$ and $f \in L_1(\mathbb{T}^d)$, then*

$$\lim_{n\to\infty,\, n\in\mathbb{R}^d_\tau} \tilde\sigma_n^{(i);\alpha} f = \tilde f^{(i)} \qquad \text{a.e.}$$

Moreover, if $f \in H_p^\square(\mathbb{T}^d)$ with $\max\{d/(d+1), 1/(\alpha\wedge 1+1)\} < p < \infty$, then this convergence also holds in the $H_p^\square(\mathbb{T}^d)$-norm.

14.2 Summability over a cone-like set

Gát [41] generalized Corollary 14.4 to so called cone-like sets. Suppose that for all $j = 2, \ldots, d$, $\gamma_j : \mathbb{R}_+ \to \mathbb{R}_+$ are strictly increasing and continuous functions such that $\lim_{j\to\infty} \gamma_j = \infty$ and $\lim_{j\to+0} \gamma_j = 0$. Moreover, suppose that there exist $c_{j,1}, c_{j,2}, \xi > 1$ such that

$$c_{j,1}\gamma_j(x) \le \gamma_j(\xi x) \le c_{j,2}\gamma_j(x) \qquad (x > 0). \tag{14.11}$$

Note that this is satisfied if γ_j is a power function. Let us define the numbers $\omega_{j,1}$ and $\omega_{j,2}$ via the formula

$$c_{j,1} = \xi^{\omega_{j,1}} \quad \text{and} \quad c_{j,2} = \xi^{\omega_{j,2}} \qquad (j = 2, \ldots, d). \tag{14.12}$$

For convenience, we extend the notations for $j = 1$ by $\gamma_1 := \mathcal{I}$, $c_{1,1} = c_{1,2} = \xi$. Here $\mathcal{I}$ denotes the identity function $\mathcal{I}(x) = x$. Let $\gamma = (\gamma_1, \ldots, \gamma_d)$ and $\tau = (\tau_1, \ldots, \tau_d)$ with $\tau_1 = 1$ and fixed $\tau_j \ge 1$ $(j = 2, \ldots, d)$. We define the **cone-like set** (with respect to the first dimension) by

$$\mathbb{R}^d_{\tau,\gamma} := \{x \in \mathbb{R}^d_+ : \tau_j^{-1}\gamma_j(n_1) \le n_j \le \tau_j\gamma_j(n_1), j = 2, \ldots, d\}. \tag{14.13}$$

Figure 27 shows a cone-like set for $d = 2$.

The condition on γ_j seems to be natural, because Gát [41] proved in the two-dimensional case that to each cone-like set with respect to the first dimension there exists a larger cone-like set with respect to the second dimension and conversely, if and only if (14.11) holds.

For given γ, τ satisfying the above conditions, the **restricted maximal operator** is defined by

$$\sigma_\gamma^\alpha f := \sup_{n\in\mathbb{R}^d_{\tau,\gamma}} |\sigma_n^\alpha f|.$$

If $\gamma_j = \mathcal{I}$ for all $j = 2, \ldots, d$, then we get a cone investigated above. Replacing the definition of the **Hardy space** $H_p^\square(\mathbb{T}^d)$ by

$$\|f\|_{H_p^\gamma} := \|\sup_{t>0} |f * (P_t \otimes P_{\gamma_2(t)} \otimes \cdots \otimes P_{\gamma_d(t)})|\|_p,$$

we can prove all the theorems of Subsection 14.1 for $H_p^\gamma(\mathbb{T}^d)$ and σ_γ^α (see Weisz [97]). Here

$$P_t(x) := \sum_{k\in\mathbb{Z}} \mathrm{e}^{-t|k|}\mathrm{e}^{\imath kx} = \frac{1-r^2}{1+r^2-2r\cos x} \qquad (x \in \mathbb{T}, t > 0, r = \mathrm{e}^{-t})$$

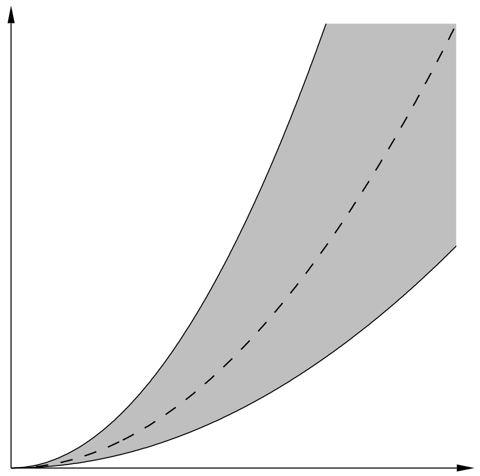

Figure 27: Cone-like set for $d = 2$.

is the **one-dimensional Poisson kernel**. If each $\gamma_j = \mathcal{I}$, we get back the Hardy spaces $H_p^{\square}(\mathbb{T}^d)$. We have to modify slightly the definition of atoms. A bounded function a is an H_p^{γ}**-atom** if there exists a rectangle $I := I_1 \times \cdots \times I_d \subset \mathbb{T}^d$ with $|I_j| = \gamma_j(|I_1|^{-1})^{-1}$ such that

(i) $\operatorname{supp} a \subset I$,

(ii) $\|a\|_\infty \leq |I|^{-1/p}$,

(iii) $\int_I a(x) x^k \,\mathrm{d}x = 0$ for all multi-indices k with $|k| \leq \lfloor d(1/p-1) \rfloor$.

Theorems 7.1 and 7.2 are valid in this case as well.

Let H be an arbitrary subset of $\{1, \ldots, d\}$, $H \neq \emptyset$, $H \neq \{1, \ldots, d\}$ and $H^c := \{1, \ldots, d\} \setminus H$. Define

$$p_1 := \sup_{H \subset \{1,\ldots,d\}} \frac{\sum_{j \in H} \omega_{j,2} + \sum_{j \in H^c} \omega_{j,1}}{\sum_{j \in H} \omega_{j,2} + 2 \sum_{j \in H^c} \omega_{j,1}},$$

where the numbers $\omega_{j,1}$ and $\omega_{j,2}$ are defined in (14.12).

Theorem 14.7 *If $\alpha > 0$ and $\max\{p_1, 1/(\alpha \wedge 1 + 1)\} < p \leq \infty$, then*

$$\|\sigma_\gamma^\alpha f\|_p \leq C_p \|f\|_{H_p^\gamma} \qquad (f \in H_p^\gamma(\mathbb{T}^d)).$$

Proof. We will again prove the result only for $d = 2$. For $d > 2$, the verification is very similar. To simplify notation, instead of n_1, n_2, $c_{2,1}, c_{2,2}$ and $\omega_{2,1}, \omega_{2,2}$, we will write n, m, c_1, c_2 and ω_1, ω_2, respectively. Let a be an arbitrary H_p^γ-atom with support $I \times J$, $|J|^{-1} = \gamma(|I|^{-1})$ and

$$2^{-K-1} < |I|/\pi \leq 2^{-K}, \qquad \gamma(2^{K+1})^{-1} < |J|/\pi \leq \gamma(2^K)^{-1}$$

for some $K \in \mathbb{N}$. We can suppose that the center of R is zero. In this case

$$[-\pi 2^{-K-2}, \pi 2^{-K-2}] \subset I \subset [-\pi 2^{-K-1}, \pi 2^{-K-1}],$$

$$[-\pi\gamma(2^{K+1})^{-1}/2, \pi\gamma(2^{K+1})^{-1}/2] \subset J \subset [-\pi\gamma(2^{K})^{-1}/2, \pi\gamma(2^{K})^{-1}/2].$$

Assume that $\alpha \le 1$. To prove (14.3), first we integrate over $\mathbb{T}^2 \setminus 4(I \times J)$,

$$\begin{aligned}
\int_{\mathbb{T}\setminus 4I}\int_{4J} |\sigma_\gamma^\alpha a(x,y)|^p \,\mathrm{d}x\,\mathrm{d}y &\le \int_{\mathbb{T}\setminus 4I}\int_{4J} \sup_{n\ge 2^K,(n,m)\in\mathbb{R}^d_{\tau,\gamma}} |\sigma^\alpha_{n,m} a(x,y)|^p \,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\mathbb{T}\setminus 4I}\int_{4J} \sup_{n< 2^K,(n,m)\in\mathbb{R}^d_{\tau,\gamma}} |\sigma^\alpha_{n,m} a(x,y)|^p \,\mathrm{d}x\,\mathrm{d}y \\
&=: (A) + (B).
\end{aligned}$$

If $n \ge 2^K$ and $x \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$ $(i \ge 1)$, then by (14.4),

$$\begin{aligned}
|\sigma^\alpha_{n,m} a(x,y)| &= \Big|\int_I\int_J a(t,u) K_n^\alpha(x-t) K_m^\alpha(y-u)\,\mathrm{d}t\,\mathrm{d}u\Big| \\
&\le C_p 2^{K/p}\gamma(2^K)^{1/p} \int_I \frac{1}{n^\alpha |x-t|^{\alpha+1}}\,\mathrm{d}t \\
&\le C_p 2^{K/p+K\alpha}\gamma(2^K)^{1/p} \frac{1}{n^\alpha i^{\alpha+1}} \\
&\le C_p 2^{K/p}\gamma(2^K)^{1/p} \frac{1}{i^{\alpha+1}}.
\end{aligned}$$

Then

$$\begin{aligned}
(A) &\le \sum_{i=1}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{4J} \sup_{n\ge 2^K} |\sigma^\alpha_{n,m} a(x,y)|^p \,\mathrm{d}x\,\mathrm{d}y \\
&\le C_p \sum_{i=1}^{2^K-1} 2^{-K}\gamma(2^K)^{-1} 2^K \gamma(2^K) \frac{1}{i^{(\alpha+1)p}} \\
&= C_p \sum_{i=1}^{2^K-1} \frac{1}{i^{(\alpha+1)p}},
\end{aligned}$$

which is a convergent series if $p > 1/(1+\alpha)$.

We estimate (B) by

$$\begin{aligned}
(B) &\le \sum_{k=0}^{\infty} \int_{\mathbb{T}\setminus 4I}\int_{4J} \sup_{\frac{2^K}{\xi^{k+1}} \le n < \frac{2^K}{\xi^k},(n,m)\in\mathbb{R}^d_{\tau,\gamma}} |\sigma^\alpha_{n,m} a(x,y)|^p \,\mathrm{d}x\,\mathrm{d}y \\
&\le \sum_{k=0}^{\infty} \Big(\int_{\mathbb{T}\setminus[-\frac{\pi\xi^k}{2^K}, \frac{\pi\xi^k}{2^K}]} \int_{4J} + \int_{[-\frac{\pi\xi^k}{2^K}, \frac{\pi\xi^k}{2^K}]} \int_{4J} \Big) \\
&\qquad \sup_{\frac{2^K}{\xi^{k+1}} \le n < \frac{2^K}{\xi^k},(n,m)\in\mathbb{R}^d_{\tau,\gamma}} |\sigma^\alpha_{n,m} a(x,y)|^p \,\mathrm{d}x\,\mathrm{d}y \\
&=: (B_1) + (B_2).
\end{aligned}$$

If $(n,m)\in\mathbb{R}^d_{\tau,\gamma}$ and $\frac{2^K}{\xi^{k+1}}\le n<\frac{2^K}{\xi^k}$, then $m<\tau\gamma(\frac{2^K}{\xi^k})$. The inequality $|K_m^\alpha|\le Cm$ and (14.1) imply

$$\begin{aligned}|\sigma^\alpha_{n,m}a(x,y)| &\le C_p2^{K/p}\gamma(2^K)^{1/p-1}m\int_I\frac{1}{n^\alpha|x-t|^{\alpha+1}}\,\mathrm{d}t\\ &\le C_p2^{K/p-K}\gamma(2^K)^{1/p-1}\gamma(\frac{2^K}{\xi^k})(\frac{2^K}{\xi^{k+1}})^{-\alpha}|x-\pi2^{-K-1}|^{-\alpha-1}.\end{aligned}$$

Hence

$$\begin{aligned}(B_1) &\le C_p\sum_{k=0}^\infty 2^{K(1-p-\alpha p)}\gamma(2^K)^{-p}\gamma(\frac{2^K}{\xi^k})^p\xi^{k\alpha p}\int_{\mathbb{T}\setminus[-\frac{\pi\xi^k}{2^K},\frac{\pi\xi^k}{2^K}]}|x-\pi2^{-K-1}|^{-(\alpha+1)p}\,\mathrm{d}x\\ &\le C_p\sum_{k=0}^\infty 2^{K(1-p-\alpha p)}\gamma(2^K)^{-p}\gamma(\frac{2^K}{\xi^k})^p\xi^{k\alpha p}(\xi^k2^{-K})^{-(\alpha+1)p+1}.\end{aligned}$$

Since $\gamma(x)\le c_1^{-1}\gamma(\xi x)$ by (14.11), we conclude

$$(B_1)\le C_p\sum_{k=0}^\infty\gamma(2^K)^{-p}\gamma(2^K)^pc_1^{-kp}\xi^{k(1-p)}=C_p\sum_{k=0}^\infty\xi^{k(1-p-\omega_1p)},$$

which is convergent if $p>1/(1+\omega_1)$. Note that $1/(1+\omega_1)<(1+\omega_1)/(1+2\omega_1)\le p_1$.

For (B_2), we obtain similarly that

$$|\sigma^\alpha_{n,m}a(x,y)|\le C_p2^{K/p-K}\gamma(2^K)^{1/p-1}nm\le C_p2^{K/p-K}\gamma(2^K)^{1/p-1}\frac{2^K}{\xi^k}\gamma(\frac{2^K}{\xi^k})\tag{14.14}$$

and

$$(B_2)\le C_p\sum_{k=0}^\infty\frac{\xi^k}{2^K}\gamma(2^K)^{-1}2^K\gamma(2^K)^{1-p}\xi^{-kp}\gamma(\frac{2^K}{\xi^k})^p\le C_p\sum_{k=0}^\infty\xi^{k(1-p)}c_1^{-kp},$$

which was just considered. Hence, we have proved that

$$\int_{\mathbb{T}\setminus4I}\int_{4J}|\sigma^\alpha_\gamma a(x,y)|^p\,\mathrm{d}x\,\mathrm{d}y\le C_p\qquad(p_1<p\le1).$$

The integral over $4I\times(\mathbb{T}\setminus4J)$ can be handled with a similar idea. Indeed, let us denote the terms corresponding to $(A),(B),(B_1),(B_2)$ by $(A'),(B'),(B_1'),(B_2')$. If we take the integrals in (A') over $4I\times[\pi j\gamma(2^K)^{-1},\pi(j+1)\gamma(2^K)^{-1}]$ $(j=1,\ldots,\gamma(2^K)/2-1)$, then we get in the same way that (A') is bounded if $p>1/(1+\alpha)$. For (B_1'), we can see that

$$\begin{aligned}(B_1') &= \sum_{k=0}^\infty\int_{4I}\int_{\mathbb{T}\setminus[-\pi\gamma(\frac{2^K}{\xi^k})^{-1},\pi\gamma(\frac{2^K}{\xi^k})^{-1}]}\sup_{\frac{2^K}{\xi^{k+1}}\le n<\frac{2^K}{\xi^k},(n,m)\in\mathbb{R}^d_{\tau,\gamma}}|\sigma^\alpha_{n,m}a(x,y)|^p\,\mathrm{d}x\,\mathrm{d}y\\ &\le C_p\sum_{k=0}^\infty 2^K\gamma(2^K)2^{-K}2^{-Kp}\int_{\mathbb{T}\setminus[-\pi\gamma(\frac{2^K}{\xi^k})^{-1},\pi\gamma(\frac{2^K}{\xi^k})^{-1}]}\end{aligned}$$

$$
\begin{aligned}
& \sup_{\frac{2^K}{\xi^{k+1}}\le n<\frac{2^K}{\xi^k},(n,m)\in\mathbb{R}^d_{\tau,\gamma}} \Big(n\int_J \frac{1}{m^\alpha|y-u|^{\alpha+1}}\,\mathrm{d}u\Big)^p\,\mathrm{d}y \\
\le\ & C_p\sum_{k=0}^\infty \xi^{-kp}\gamma(2^K)^{1-p}\gamma(\frac{2^K}{\xi^{k+1}})^{-\alpha p} \\
& \int_{\mathbb{T}\setminus[-\pi\gamma(\frac{2^K}{\xi^k})^{-1},\pi\gamma(\frac{2^K}{\xi^k})^{-1}]} |y-\pi\gamma(2^K)^{-1}/2|^{-(\alpha+1)p}\,\mathrm{d}y \\
\le\ & C_p\sum_{k=0}^\infty \xi^{-kp}\gamma(2^K)^{1-p}\gamma(\frac{2^K}{\xi^k})^{p-1} \\
\le\ & C_p\sum_{k=0}^\infty \xi^{-kp}c_2^{k(1-p)} \\
=\ & C_p\sum_{k=0}^\infty \xi^{k(\omega_2-\omega_2 p-p)}
\end{aligned}
$$

and this converges if $p>\omega_2/(1+\omega_2)$, which is less than $(1+\omega_2)/(2+\omega_2)\le p_1$. Using (14.14), we establish that

$$
\begin{aligned}
(B_2') &= \sum_{k=0}^\infty\int_{4I}\int_{[-\gamma(\frac{2^K}{\xi^k})^{-1},\gamma(\frac{2^K}{\xi^k})^{-1}]} \sup_{\frac{2^K}{\xi^{k+1}}\le n<\frac{2^K}{\xi^k},(n,m)\in\mathbb{R}^d_{\tau,\gamma}} |\sigma^\alpha_{n,m}a(x,y)|^p\,\mathrm{d}x\,\mathrm{d}y \\
&\le C_p\sum_{k=0}^\infty 2^{-K}\gamma(\frac{2^K}{\xi^k})^{-1}2^K\gamma(2^K)^{1-p}\xi^{-kp}\gamma(\frac{2^K}{\xi^k})^p \\
&\le C_p\sum_{k=0}^\infty \xi^{-kp}c_2^{k(1-p)}.
\end{aligned}
$$

Hence

$$
\int_{4I}\int_{\mathbb{T}\setminus 4J}|\sigma^\alpha_\gamma a(x,y)|^p\,\mathrm{d}x\,\mathrm{d}y\le C_p \qquad (p_1<p\le 1).
$$

Integrating over $(\mathbb{T}\setminus 4I)\times(\mathbb{T}\setminus 4J)$, we decompose the integral as

$$
\begin{aligned}
\int_{\mathbb{T}\setminus 4I}\int_{\mathbb{T}\setminus 4J}|\sigma^\alpha_\gamma a(x,y)|^p\,\mathrm{d}x\,\mathrm{d}y &\le \int_{\mathbb{T}\setminus 4I}\int_{\mathbb{T}\setminus 4J}\sup_{n\ge 2^K,(n,m)\in\mathbb{R}^d_{\tau,\gamma}}|\sigma^\alpha_{n,m}a(x,y)|^p\,\mathrm{d}x\,\mathrm{d}y \\
&\quad+\int_{\mathbb{T}\setminus 4I}\int_{\mathbb{T}\setminus 4J}\sup_{n< 2^K,(n,m)\in\mathbb{R}^d_{\tau,\gamma}}|\sigma^\alpha_{n,m}a(x,y)|^p\,\mathrm{d}x\,\mathrm{d}y \\
&=: (C)+(D)
\end{aligned}
$$

and

$$
(C)\le\sum_{i=1}^{2^K-1}\sum_{j=1}^{\gamma(2^K)/2-1}\int_{\pi i2^{-K}}^{\pi(i+1)2^{-K}}\int_{\pi j\gamma(2^K)^{-1}}^{\pi(j+1)\gamma(2^K)^{-1}}\sup_{n\ge 2^K}|\sigma^\alpha_{n,m}a(x,y)|^p\,\mathrm{d}x\,\mathrm{d}y.
$$

For $x \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$ and $y \in [\pi j\gamma(2^K)^{-1}, \pi(j+1)\gamma(2^K)^{-1})$, we have by (14.1) and (14.4) that

$$\begin{aligned} |\sigma_{n,m}^\alpha a(x,y)| &\leq C_p 2^{K/p}\gamma(2^K)^{1/p} \int_I \frac{1}{n^\alpha |x-t|^{\alpha+1}} \,\mathrm{d}t \int_J \frac{1}{m^\alpha |y-u|^{\alpha+1}} \,\mathrm{d}u \\ &\leq C_p \frac{2^{K/p+K\alpha}\gamma(2^K)^{1/p+\alpha}}{n^\alpha m^\alpha i^{\alpha+1} j^{\alpha+1}} \\ &\leq C_p \frac{2^{K/p}\gamma(2^K)^{1/p}}{i^{\alpha+1} j^{\alpha+1}}. \end{aligned} \tag{14.15}$$

Then

$$(C) \leq C_p \sum_{i=1}^{2^K-1} \sum_{j=1}^{\gamma(2^K)/2-1} \frac{1}{i^{(\alpha+1)p} j^{(\alpha+1)p}} < \infty$$

if $p > 1/(1+\alpha)$.

To consider (D) let us define $A_1(x,y)$, $A_2(x,y)$, $D_{n,m}^1(x,y)$, $D_{n,m}^2(x,y)$ and $D_{n,m}^3(x,y)$ as in (14.5) and (14.8), respectively, and let $I = [-\mu, \mu]$, $J = [-\nu, \nu]$. Then

$$|A_1(x,u)| \leq 2^{K/p-K}\gamma(2^K)^{1/p}, \qquad |A_2(x,y)| \leq 2^{K/p-K}\gamma(2^K)^{1/p-1}. \tag{14.16}$$

Obviously,

$$\begin{aligned} &\int_{\mathbb{T}\setminus 4I} \int_{\mathbb{T}\setminus 4J} \sup_{n<2^K, (n,m)\in\mathbb{R}^d_{\tau,\gamma}} |D_{n,m}^1(x,y)|^p \,\mathrm{d}x \,\mathrm{d}y \\ &\leq \sum_{k=0}^\infty \int_{\mathbb{T}\setminus 4I} \int_{\mathbb{T}\setminus 4J} \sup_{\frac{2^K}{\xi^{k+1}} \leq n < \frac{2^K}{\xi^k}, (n,m)\in\mathbb{R}^d_{\tau,\gamma}} |D_{n,m}^1(x,y)|^p \,\mathrm{d}x \,\mathrm{d}y \\ &\leq \sum_{k=0}^\infty \sum_{j=1}^{\gamma(2^K)/2-1} \Big(\int_{\mathbb{T}\setminus[-\frac{\pi\xi^k}{2^K}, \frac{\pi\xi^k}{2^K}]} \int_{\pi j\gamma(2^K)^{-1}}^{\pi(j+1)\gamma(2^K)^{-1}} + \int_{[-\frac{\pi\xi^k}{2^K}, \frac{\pi\xi^k}{2^K}]} \int_{\pi j\gamma(2^K)^{-1}}^{\pi(j+1)\gamma(2^K)^{-1}} \Big) \\ &\qquad \sup_{n<2^K, (n,m)\in\mathbb{R}^d_{\tau,\gamma}} |D_{n,m}^1(x,y)|^p \,\mathrm{d}x \,\mathrm{d}y \\ &=: (D_1) + (D_2). \end{aligned}$$

It follows from (14.4), (14.9) and (14.16) that

$$\begin{aligned} |D_{n,m}^1(x,y)| &\leq C_p 2^{K/p-K}\gamma(2^K)^{1/p-2} \frac{1}{n^\alpha |x-\mu|^{\alpha+1}} \frac{m^{\zeta+1+\alpha(\zeta-1)}}{|y-\nu|^{(\alpha+1)(1-\zeta)}} \\ &\leq C_p 2^{K/p-K}\gamma(2^K)^{1/p-2+(\alpha+1)(1-\zeta)} \frac{(\frac{2^K}{\xi^{k+1}})^{-\alpha}}{|x-\mu|^{\alpha+1}} \frac{\gamma(\frac{2^K}{\xi^k})^{\zeta+1+\alpha(\zeta-1)}}{j^{(\alpha+1)(1-\zeta)}}, \end{aligned}$$

where $0 \leq \zeta \leq 1$. This leads to

$$(D_1) \leq C_p \sum_{k=0}^\infty \sum_{j=1}^{\gamma(2^K)/2-1} \int_{\mathbb{T}\setminus[-\frac{\pi\xi^k}{2^K}, \frac{\pi\xi^k}{2^K}]}$$

$$
\begin{aligned}
& 2^{K(1-p-\alpha p)}\gamma(2^K)^{p(-2+(\alpha+1)(1-\zeta))}\xi^{k\alpha p}|x-\mu|^{-(\alpha+1)p}\frac{\gamma(\frac{2^K}{\xi^k})^{p(2+(\alpha+1)(\zeta-1))}}{j^{p(\alpha+1)(1-\zeta)}}\,\mathrm{d}x \\
\leq\ & C_p\sum_{k=0}^{\infty}\sum_{j=1}^{\gamma(2^K)/2-1} 2^{K(1-p-\alpha p)}\xi^{k\alpha p}(\xi^k 2^{-K})^{-(\alpha+1)p+1}\frac{c_1^{-kp(2+(\alpha+1)(\zeta-1))}}{j^{p(\alpha+1)(1-\zeta)}} \\
\leq\ & C_p\sum_{k=0}^{\infty}\sum_{j=1}^{\gamma(2^K)/2-1}\frac{\xi^{k(1-p-\omega_1 p(2+(\alpha+1)(\zeta-1)))}}{j^{p(\alpha+1)(1-\zeta)}},
\end{aligned}
$$

which is convergent if $p>1/(1+\omega_1(2+(\alpha+1)(\zeta-1)))$ and $p>1/(\alpha+1)(1-\zeta)$. After some computation, we can see that the optimal bound is reached if

$$
\zeta=\frac{\alpha-\omega_1+\alpha\omega_1}{1+\alpha+\omega_1+\alpha\omega_1},
$$

which means that $p>(1+\omega_1)/(1+2\omega_1)$.

Considering (D_2), we estimate as follows:

$$
\begin{aligned}
|D^1_{n,m}(x,y)| &\leq C_p 2^{K/p-K}\gamma(2^K)^{1/p-2}n\frac{m^{\zeta+1+\alpha(\zeta-1)}}{|y-\nu|^{(\alpha+1)(1-\zeta)}} \\
&\leq C_p 2^{K/p}\gamma(2^K)^{1/p-2+(\alpha+1)(1-\zeta)}\xi^{-k}\frac{\gamma(\frac{2^K}{\xi^k})^{\zeta+1+\alpha(\zeta-1)}}{j^{(\alpha+1)(1-\zeta)}}
\end{aligned}
$$

and

$$
\begin{aligned}
(D_2) \leq\ & C_p\sum_{k=0}^{\infty}\sum_{j=1}^{\gamma(2^K)/2-1} \\
& \int_{[-\frac{\pi\xi^k}{2^K},\frac{\pi\xi^k}{2^K}]} 2^K\gamma(2^K)^{p(-2+(\alpha+1)(1-\zeta))}\xi^{-kp}\frac{\gamma(\frac{2^K}{\xi^k})^{p(2+(\alpha+1)(\zeta-1))}}{j^{p(\alpha+1)(1-\zeta)}}\,\mathrm{d}x \\
\leq\ & C_p\sum_{k=0}^{\infty}\sum_{j=1}^{\gamma(2^K)/2-1}\frac{\xi^{k(1-p-\omega_1 p(2+(\alpha+1)(\zeta-1)))}}{j^{p(\alpha+1)(1-\zeta)}} \\
\leq\ & C_p
\end{aligned}
$$

as above.

The term $D^2_{n,m}$ can be handled similarly. We obtain

$$
\int_{\mathbb{T}\setminus 4I}\int_{\mathbb{T}\setminus 4J}\sup_{n<2^K,(n,m)\in\mathbb{R}^d_{\tau,\gamma}}|D^2_{n,m}(x,y)|^p\,\mathrm{d}x\,\mathrm{d}y\leq C_p
$$

if $p>(1+\omega_2)/(2+\omega_2)$.

Using (14.2), we estimate $D^3_{n,m}$ in the same way as (C) in (14.15). Now the exponents of n and m are non-negative and so they can be estimated by 2^K and $\gamma(2^K)$ as in (14.15). This proves that

$$\int_{\mathbb{T}\setminus 4I}\int_{\mathbb{T}\setminus 4J} |\sigma_\gamma^\alpha a(x,y)|^p \,\mathrm{d}x\,\mathrm{d}y \le C_p.$$

This completes the proof for $0<\alpha\le 1$. For $\alpha>1$, the proof can be finished as in Theorem 14.1. ∎

Remark 14.8 In the d-dimensional case, the constant p_1 appears if we investigate the terms corresponding to $D^1_{n,m}$ and $D^2_{n,m}$. Indeed, let $\prod_{j=1}^d I_j$ be centered at 0 and the support of the atom a, A be the integral of a, $I_j =: [-\mu_j,\mu_j]$ and

$$\overline{t}_j := \begin{cases} \mu_j, & j\in H; \\ t_j, & j\in H^c, \end{cases}$$

$H\subset\{1,\ldots,d\}$, $H\neq\emptyset$, $H\neq\{1,\ldots,d\}$. If we integrate the term

$$\int_{\prod_{j\in H^c} I_j} A(\overline{t}_1,\ldots,\overline{t}_d)\prod_{j\in H} K^\alpha_{n_j}(x_j-\mu_j)\prod_{i\in H^c}(K^\alpha_{n_i})'(x_i-t_i)\,\mathrm{d}t$$

over $\prod_{j=1}^d(\mathbb{T}\setminus 4I_j)$, then we get that

$$p > \frac{\sum_{j\in H}\omega_{j,2}+\sum_{j\in H^c}\omega_{j,1}}{\sum_{j\in H}\omega_{j,2}+2\sum_{j\in H^c}\omega_{j,1}}.$$

Moreover, considering the integral

$$\int_{\prod_{j\in H}(\mathbb{T}\setminus 4I_j)}\int_{\prod_{j\in H^c}4J_j} |\sigma_\gamma^\alpha a(x)|^p\,\mathrm{d}x,$$

we obtain

$$p > \frac{\sum_{j\in H}\omega_{j,2}}{\sum_{j\in H}\omega_{j,2}+\sum_{j\in H^c}\omega_{j,1}}.$$

However, this bound is less than p_1.

Remark 14.9 If $\omega_{j,1}=\omega_{j,2}=1$ for all $j=1,\ldots,d$, then we obtain in Theorem 14.7 the bound

$$\max\{d/(d+1), 1/(\alpha+1)\}.$$

In particular, this holds if $\gamma_j=\mathcal{I}$ for all $j=1,\ldots,d$, i.e., if we consider a cone. This bound was obtained for cones in Theorem 14.1.

Corollary 14.10 *If $\alpha>0$ and $f\in L_1(\mathbb{T}^d)$, then*

$$\sup_{\rho>0}\rho\lambda(\sigma_\gamma^\alpha f>\rho)\le C\|f\|_1.$$

Corollary 14.11 *If $\alpha > 0$ and $f \in L_1(\mathbb{T}^d)$, then*

$$\lim_{n\to\infty, n\in\mathbb{R}^d_{\tau,\gamma}} \sigma_n^\alpha f = f \qquad a.e.$$

In the two-dimensional case, Corollaries 14.10 and 14.11 were proved by Gát [41] for Fejér summability. In this case, he verified also that if the cone-like set $\mathbb{R}^d_{\tau,\gamma}$ is defined by $\tau_j(n_1)$ instead of τ_j and if $\tau_j(n_1)$ is not bounded, then Corollary 14.11 does not hold and the largest space for the elements of which we have a.e. convergence is $L \log L$. This means that under these conditions Theorem 14.7 cannot be true for any $p < 1$.

15 $H_p(\mathbb{T}^d)$ Hardy spaces

For the investigation of the unrestricted almost everywhere convergence, we need a new type of Hardy space, the so called product Hardy spaces. A function f is in the **product Hardy space** $H_p(\mathbb{T}^d)$, in the **product weak Hardy space** $H_{p,\infty}(\mathbb{T}^d)$ and in the **hybrid Hardy space** $H_p^i(\mathbb{T}^d)$ if

$$\|f\|_{H_p} := \| \sup_{t_k>0, k=1,\ldots,d} |f * (P_{t_1} \otimes \cdots \otimes P_{t_d})| \|_p < \infty,$$

$$\|f\|_{H_{p,\infty}} := \| \sup_{t_k>0, k=1,\ldots,d} |f * (P_{t_1} \otimes \cdots \otimes P_{t_d})| \|_{p,\infty} < \infty$$

and

$$\|f\|_{H_p^i} := \| \sup_{t_k>0, k=1,\ldots,d; k\neq i} |f * (P_{t_1} \otimes \cdots \otimes P_{t_{i-1}} \otimes P_{t_{i+1}} \otimes \cdots \otimes P_{t_d})| \|_p < \infty,$$

respectively, where $0 < p \leq \infty$ and P_t is the one-dimensional Poisson kernel. It is known (see Chang and Fefferman [22, 20, 21], Gundy and Stein [47] or Weisz [94]) that

$$H_p(\mathbb{T}^d) \sim H_p^i(\mathbb{T}^d) \sim L_p(\mathbb{T}^d) \qquad (1 < p \leq \infty)$$

and $H_1(\mathbb{T}^d) \subset H_1^i(\mathbb{T}^d) \subset H_{1,\infty}(\mathbb{T}^d) \cap L_1(\mathbb{T}^d)$. Moreover,

$$\|f\|_{H_{1,\infty}} \leq C \|f\|_{H_1^i} \qquad (f \in H_1^i(\mathbb{T}^d), i = 1, \ldots, d). \tag{15.1}$$

Let the set $L(\log L)^{d-1}(\mathbb{T}^d)$ contain all measurable functions for which

$$\| \, |f| (\log^+ |f|)^{d-1} \|_1 < \infty.$$

Then $H_1^i(\mathbb{T}^d) \supset L(\log L)^{d-1}(\mathbb{T}^d)$ for all $i = 1, \ldots, d$ and

$$\|f\|_{H_1^i} \leq C + C \| |f| (\log^+ |f|)^{d-1} \|_1 \qquad (f \in L(\log L)^{d-1}(\mathbb{T}^d)). \tag{15.2}$$

Considering the product Hardy spaces, we have to introduce new **conjugate distributions** defined by

$$\tilde{f}^{(j_1,\ldots,j_d)} \sim \sum_{n\in\mathbb{Z}^d} \Big(\prod_{i=1}^d (-\mathrm{i} \operatorname{sign} n_i)^{j_i} \Big) \widehat{f}(n) \mathrm{e}^{\mathrm{i} n \cdot x} \qquad (j_i = 0, 1).$$

In the case f is an integrable function,

$$\begin{aligned}\tilde{f}^{(j_1,\ldots,j_d)}(x) &= \text{p.v.} \int_{\mathbb{T}} \cdots \int_{\mathbb{T}} \frac{f(x_1 - t_1^{j_1}, \ldots, x_d - t_d^{j_d})}{\prod_{i=1}^d (2\pi \tan(t_i/2))^{j_i}} \,\mathrm{d}t^{j_1} \cdots \mathrm{d}t^{j_d} \\ &:= \lim_{\epsilon \to 0} \int_{\epsilon_1 < |t_1| < \pi} \cdots \int_{\epsilon_d < |t_d| < \pi} \frac{f(x_1 - t_1^{j_1}, \ldots, x_d - t_d^{j_d})}{\prod_{i=1}^d (2\pi \tan(t_i/2))^{j_i}} \,\mathrm{d}t^{j_1} \cdots \mathrm{d}t^{j_d} \qquad \text{a.e.},\end{aligned}$$

where $\mathrm{d}t^{j_i}$ means the ordinary integral if $j_i = 1$ while the integral with respect to the ith variable is omitted if $j_i = 0$.

If $j_i = 0$ for all $i = 1, \ldots, d$, let $\tilde{f}^{(j_1,\ldots,j_d)} = f$. The following result can be found in Gundy and Stein [47, 46], Chang and Fefferman [22] and Weisz [93, 94].

Theorem 15.1 *For $f \in H_p(\mathbb{T}^d)$,*

$$\|\tilde{f}^{(j_1,\ldots,j_d)}\|_{H_p} = C_p \|f\|_{H_p} \qquad (j_i = 0, 1)$$

and

$$\|f\|_{H_p} \sim \sum_{j_1=0}^{1} \cdots \sum_{j_d=0}^{1} \|\tilde{f}^{(j_1,\ldots,j_d)}\|_p \qquad (0 < p < \infty).$$

We note again that this theorem holds also for $L_p(\mathbb{T}^d)$ spaces when $1 < p < \infty$.

The atomic decomposition for $H_p(\mathbb{T}^d)$ is much more complicated than for $H_p^\square(\mathbb{T}^d)$. One reason for this is that the support of an atom is not a rectangle but an open set. Moreover, here we have to choose the atoms from $L_2(\mathbb{T}^d)$ instead of $L_\infty(\mathbb{T}^d)$. This atomic decomposition was proved by Chang and Fefferman [22, 33] and Weisz [92, 94]. For an open set $F \subset (\mathbb{T}^d)$, denote by $\mathcal{M}(F)$ the set of the maximal dyadic subrectangles of F.

A function $a \in L_2(\mathbb{T}^d)$ is a **H_p-atom** if

(i) supp $a \subset F$ for some open set $F \subset \mathbb{T}^d$,

(ii) $\|a\|_2 \leq |F|^{1/2-1/p}$,

(iii) a can be further decomposed into the sum of "elementary particles" $a_R \in L_2(\mathbb{T}^d)$, $a = \sum_{R \in \mathcal{M}(F)} a_R$ in $L_2(\mathbb{T}^d)$, satisfying

  (a) supp $a_R \subset 2R \subset F$,

  (b) for $i = 1, \ldots, d$, $k \leq \lfloor 2/p - 3/2 \rfloor$ and $R \in \mathcal{M}(F)$, we have

$$\int_{\mathbb{T}} a_R(x) x_i^k \,\mathrm{d}x_i = 0,$$

  (c) for every disjoint partition $\mathcal{P}_l$ $(l = 1, 2, \ldots)$ of $\mathcal{M}(F)$,

$$\Big(\sum_l \| \sum_{R \in \mathcal{P}_l} a_R \|_2^2 \Big)^{1/2} \leq |F|^{1/2-1/p}.$$

The basic result about the *atomic decomposition* was proved by Chang and Fefferman [20, 21, 22] (see also Weisz [94]).

Theorem 15.2 *A function f is in $H_p(\mathbb{T}^d)$ $(0 < p \le 1)$ if and only if there exist a sequence $(a^k, k \in \mathbb{N})$ of H_p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that*

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty \quad \text{and} \quad \sum_{k=0}^{\infty} \mu_k a^k = f \quad \text{in the sense of distributions.}$$

Moreover,

$$\|f\|_{H_p} \sim \inf \Big(\sum_{k=0}^{\infty} |\mu_k|^p \Big)^{1/p}.$$

The result corresponding to Theorem 7.2 for the $H_p(\mathbb{T}^d)$ space is much more complicated. Since the definition of the H_p-atom is very complex, to obtain a usable condition about the boundedness of the operators, we have to introduce simpler atoms.

If $d = 2$, a function a is a **simple H_p-atom** if

(i) $\operatorname{supp} a \subset R$ for some rectangle $R \subset \mathbb{T}^2$,

(ii) $\|a\|_2 \le |R|^{1/2-1/p}$,

(iii) $\int_{\mathbb{T}} a(x) x_i^k \,\mathrm{d}x_i = 0$ for all $i = 1, 2$ and $k \le \lfloor 2/p - 3/2 \rfloor$.

Note that not every $f \in H_p(\mathbb{T}^2)$ can be decomposed into simple H_p-atoms. A counterexample can be found in Weisz [89]. However, the following result says that for an operator V to be bounded from $H_p(\mathbb{T}^2)$ to $L_p(\mathbb{T}^2)$ $(0 < p \le 1)$, it is enough to check V on simple H_p-atoms and the boundedness of V on $L_2(\mathbb{T}^2)$. It can be proved with the help of an idea due to Fefferman [33] (see the proof of Theorem 7.2 and [94]).

Theorem 15.3 *For each $n \in \mathbb{N}^2$, let $V_n : L_1(\mathbb{T}^2) \to L_1(\mathbb{T}^2)$ be a bounded linear operator and*

$$V_* f := \sup_{n \in \mathbb{N}^2} |V_n f|.$$

Let $d = 2$ and $0 < p_0 \le 1$. Suppose that there exists $\eta > 0$ such that for every simple H_{p_0}-atom a and for every $r \ge 1$

$$\int_{\mathbb{T}^2 \setminus R^r} |V_* a|^{p_0} \,\mathrm{d}\lambda \le C_p 2^{-\eta r},$$

where R is the support of a. If V_* is bounded from $L_2(\mathbb{T}^2)$ to $L_2(\mathbb{T}^2)$, then

$$\|V_* f\|_p \le C_{p_0} \|f\|_{H_p} \qquad (f \in H_p(\mathbb{T}^2) \cap H_1^i(\mathbb{T}^2)) \tag{15.3}$$

for all $p_0 \le p \le 2$ and $i = 1, \ldots, d$. If $\lim_{k\to\infty} f_k = f$ in the H_p-norm implies that $\lim_{k\to\infty} V_n f_k = V_n f$ in the sense of distributions $(n \in \mathbb{N}^d)$, then (15.3) holds for all $f \in H_p(\mathbb{T}^2)$.

Unfortunately, the preceding theorem is not true for higher dimensions (Journé [52]). So, there are fundamental differences between the theory in the two-parameter and three- or multi-parameter cases. Fefferman asked in [34] whether one can find sufficient conditions for an operator to be bounded from $H_p(T^d)$ to $L_p(\mathbb{T}^d)$ in higher dimensions. The following theorem answers this problem.

If $d \geq 3$, a is called a **simple** H_p**-atom** if there exist intervals $I_i \subset \mathbb{T}$, $i = 1, \ldots, j$ for some $1 \leq j \leq d-1$, such that

(i) $\operatorname{supp} a \subset I_1 \times \cdots \times I_j \times A$ for some measurable set $A \subset \mathbb{T}^{d-j}$,

(ii) $\|a\|_2 \leq (|I_1| \cdots |I_j||A|)^{1/2-1/p}$,

(iii) $\int_{\mathbb{T}} a(x) x_i^k \, dx_i = \int_A a \, d\lambda = 0$ for all $i = 1, \ldots, j$ and $k \leq \lfloor 2/p - 3/2 \rfloor$.

If $j = d-1$, we may suppose that $A = I_d$ is also an interval. Of course if $a \in L_2(\mathbb{T}^d)$ satisfies these conditions for another subset of $\{1, \ldots, d\}$ than $\{1, \ldots, j\}$, then it is also called a simple H_p-atom. As for the $H_p^\square(\mathbb{T}^d)$ spaces, we could suppose that the integrals in (iii) of all definitions of atoms or simple atoms are zero for all k for which $k \leq N$, where $N \geq \lfloor 2/p - 3/2 \rfloor$.

As in the two-parameter case, not every $f \in H_p(\mathbb{T}^d)$ can be decomposed into simple H_p-atoms. The next theorem is due to the author [92, 94]. Let H^c denote the complement of the set H.

Theorem 15.4 *For each $n \in \mathbb{N}^d$, let $V_n : L_1(\mathbb{T}^d) \to L_1(\mathbb{T}^d)$ be a bounded linear operator and*

$$V_* f := \sup_{n \in \mathbb{N}^d} |V_n f|.$$

Let $d \geq 3$ and $0 < p_0 \leq 1$. Suppose that there exist $\eta_1, \ldots, \eta_d > 0$ such that for every simple H_{p_0}-atom a and for every $r_1 \ldots, r_d \geq 1$

$$\int_{(I_1^{r_1})^c \times \cdots \times (I_j^{r_j})^c} \int_A |V_* a|^{p_0} \, d\lambda \leq C_{p_0} 2^{-\eta_1 r_1} \cdots 2^{-\eta_j r_j},$$

where $I_1 \times \cdots \times I_j \times A$ is the support of a. If $j = d-1$ and $A = I_d$ is an interval, then we also assume that

$$\int_{(I_1^{r_1})^c \times \cdots \times (I_{d-1}^{r_{d-1}})^c} \int_{(I_d)^c} |V_* a|^{p_0} \, d\lambda \leq C_{p_0} 2^{-\eta_1 r_1} \cdots 2^{-\eta_{d-1} r_{d-1}}.$$

If V_* is bounded from $L_2(\mathbb{T}^d)$ to $L_2(\mathbb{T}^d)$, then

$$\|V_* f\|_p \leq C_p \|f\|_{H_p} \qquad (f \in H_p(\mathbb{T}^d) \cap H_1^i(\mathbb{T}^d)) \tag{15.4}$$

for all $p_0 \leq p \leq 2$ and $i = 1, \ldots, d$. If $\lim_{k\to\infty} f_k = f$ in the H_p-norm implies that $\lim_{k\to\infty} V_n f_k = V_n f$ in the sense of distributions ($n \in \mathbb{N}^d$), then (15.4) holds for all $f \in H_p(\mathbb{T}^d)$.

Inequalities (15.3) or (15.4) imply by interpolation that the operator

$$V_* \quad \text{is bounded from} \quad H_{p,\infty}(\mathbb{T}^d) \quad \text{to} \quad L_{p,\infty}(\mathbb{T}^d) \tag{15.5}$$

when $p_0 < p < 2$. If $p_0 < 1$ in Theorems 15.3 or 15.4, then (15.5) holds also for $p = 1$. Thus V_* is of weak type (H_1^i, L_1) by (15.1):

$$\sup_{\rho>0} \rho\, \lambda(|V_* f| > \rho) = \|V_* f\|_{1,\infty} \le C\|f\|_{H_{1,\infty}} \le C\|f\|_{H_1^i}$$

for all $f \in H_1^i(\mathbb{T}^d)$, $i = 1, \ldots, d$.

Corollary 15.5 *If $p_0 < 1$ in Theorems 15.3 or 15.4, then for all $f \in H_1^i(\mathbb{T}^d)$ and $i = 1, \ldots, d$*

$$\sup_{\rho>0} \rho\, \lambda(|V_* f| > \rho) \le C\|f\|_{H_1^i}.$$

16 Unrestricted summability

Let us define the **unrestricted maximal operator** by

$$\sigma_*^\alpha f := \sup_{n \in \mathbb{N}^d} |\sigma_n^\alpha f|.$$

We will first prove that the operator σ_*^α is bounded from $L_p(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$ $(1 < p \le \infty)$ and then that it is bounded from $H_p(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$ $(1/(\alpha+1) < p \le 1)$. To this end, we introduce the one-dimensional operators

$$\tau_n^\alpha f(x) := \frac{1}{2\pi} \int_{\mathbb{T}} f(x-u)|K_n^\alpha(u)|\, \mathrm{d}u = f * |K_n^\alpha|(x)$$

and

$$\tau_*^\alpha f := \sup_{n \in \mathbb{N}} |\tau_n^\alpha f|.$$

Obviously,

$$|\sigma_n^\alpha f| \le \tau_n^\alpha |f| \quad (n \in \mathbb{N}) \qquad \text{and} \qquad \sigma_*^\alpha f \le \tau_*^\alpha |f|. \tag{16.1}$$

The next result can be proved as was Theorem 14.1.

Theorem 16.1 *If $\alpha > 0$ and $1/(\alpha \wedge 1 + 1) < p \le \infty$, then*

$$\|\tau_*^\alpha f\|_p \le C_p \|f\|_{H_p} \qquad (f \in H_p(\mathbb{T})).$$

Proof. It is easy to see that

$$\|\tau_*^\alpha f\|_\infty \le C\|f\|_\infty \qquad (f \in L_\infty(\mathbb{T})).$$

Let $\alpha \le 1$ and a be an arbitrary $H_p^\square$-atom with support $I \subset \mathbb{T}$ and

$$[-\pi 2^{-K-2}, \pi 2^{-K-2}] \subset I \subset [-\pi 2^{-K-1}, \pi 2^{-K-1}].$$

Then

$$\begin{aligned} \int_{\mathbb{T}\setminus 4I} |\tau_*^\alpha a(x)|^p \,\mathrm{d}x &\le \sum_{|i|=1}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \sup_{n\ge 2^K} |\tau_n^\alpha a(x)|^p \,\mathrm{d}x \\ &\quad + \sum_{|i|=1}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \sup_{n< 2^K} |\tau_n^\alpha a(x)|^p \,\mathrm{d}x \\ &=: (A)+(B). \end{aligned}$$

Using (14.1), (14.2) and (14.4), we can see that

$$|\tau_n^\alpha a(x)| = \Big| \int_I a(t)|K_n^\alpha(x-t)|\,\mathrm{d}t \Big| \le C_p 2^{K/p} \int_I \frac{1}{n^\alpha |x-t|^{\alpha+1}}\,\mathrm{d}t \le C_p 2^{K/p} \frac{1}{i^{\alpha+1}}$$

and

$$(A) \le C_p \sum_{i=1}^{2^K-1} 2^{-K} 2^K \frac{1}{i^{(\alpha+1)p}} \le C_p$$

as in Theorem 14.1.

To estimate (B), observe that by (iii) of the definition of the atom,

$$\tau_n^\alpha a(x) = \int_I a(t)|K_n^\alpha(x-t)|\,\mathrm{d}t = \int_I a(t)(|K_n^\alpha(x-t)| - |K_n^\alpha(x)|)\,\mathrm{d}t.$$

Thus,

$$|\tau_n^\alpha a(x)| \le \int_I |a(t)||K_n^\alpha(x-t) - K_n^\alpha(x)|\,\mathrm{d}t.$$

Using Lagrange's mean value theorem and (14.2), we conclude

$$|K_n^\alpha(x-t) - K_n^\alpha(x)| = |(K_n^\alpha)'(x-\xi)||t| \le \frac{C_p 2^{-K}}{n^{\alpha-1}|x-\xi|^{\alpha+1}} \le \frac{C_p 2^K}{i^{\alpha+1}},$$

where $\xi \in I$ and $x \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$. Consequently,

$$|\tau_n^\alpha a(x)| \le C_p 2^{K/p-K} \frac{2^K}{i^{\alpha+1}}$$

and

$$(B) \le C_p \sum_{i=1}^{2^K-1} 2^{-K} 2^K \frac{1}{i^{(\alpha+1)p}} \le C_p.$$

If $\alpha > 1$, then the theorem can be proved in the same way. ■

We get by interpolation that

$$\sup_{\rho>0} \rho\, \lambda(\tau_*^\alpha f > \rho) \leq C\|f\|_1 \qquad (f \in L_1(\mathbb{T})).$$

This, (16.1) and Theorem 16.1 yield

$$\|\sigma_*^\alpha f\|_p \leq C_p \|f\|_p \qquad (f \in L_p(\mathbb{T}), 1 < p < \infty) \tag{16.2}$$

and

$$\sup_{\rho>0} \rho\, \lambda(\sigma_*^\alpha f > \rho) \leq C\|f\|_1 \qquad (f \in L_1(\mathbb{T})).$$

Note that the last inequality is exactly Theorem 3.1.

Now, we return to the higher dimensional case and verify the $L_p(\mathbb{T}^d)$ boundedness of σ_*^α.

Theorem 16.2 *If $\alpha > 0$ and $1 < p \leq \infty$, then*

$$\|\sigma_*^\alpha f\|_p \leq C_p \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

Proof. We may suppose that $d = 2$. Applying Theorem 16.1 and (16.2), we have

$$\begin{aligned}
&\int_{\mathbb{T}}\int_{\mathbb{T}} \sup_{n,m\in\mathbb{N}} \Big| \int_{\mathbb{T}}\int_{\mathbb{T}} f(t,u) K_n^\alpha(x-t) K_m^\alpha(y-u)\,\mathrm{d}t\,\mathrm{d}u \Big|^p \,\mathrm{d}x\,\mathrm{d}y \\
\leq\ & \int_{\mathbb{T}}\int_{\mathbb{T}} \sup_{m\in\mathbb{N}} \Big(\int_{\mathbb{T}} \Big(\sup_{n\in\mathbb{N}} \Big| \int_{\mathbb{T}} f(t,u) K_n^\alpha(x-t)\,\mathrm{d}t \Big| \Big) |K_m^\alpha(y-u)|\,\mathrm{d}u \Big)^p \,\mathrm{d}y\,\mathrm{d}x \\
\leq\ & C_p \int_{\mathbb{T}}\int_{\mathbb{T}} \sup_{n\in\mathbb{N}} \Big| \int_{\mathbb{T}} f(t,y) K_n^\alpha(x-t)\,\mathrm{d}t \Big|^p \,\mathrm{d}x\,\mathrm{d}y \\
\leq\ & C_p \int_{\mathbb{T}}\int_{\mathbb{T}} |f(x,y)|^p \,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

which proves the theorem. ∎

The next result is due to the author ([92, 94]).

Theorem 16.3 *If $\alpha > 0$ and $1/(\alpha+1) < p \leq \infty$, then*

$$\|\sigma_*^\alpha f\|_p \leq C_p \|f\|_{H_p} \qquad (f \in H_p(\mathbb{T}^d)).$$

Proof. We sketch the proof by giving only the main ideas. Similarly to (14.1) and (14.2), we have for the Riesz kernels that

$$|(K_{n_j}^\alpha)^{(k)}(x)| \leq \frac{C}{n_j^{\alpha-k} |x|^{\alpha+1}} \qquad (x \neq 0, k \in \mathbb{N}). \tag{16.3}$$

We will prove the theorem only for $d = 3$, because the proof is similar for larger d or for $d = 2$. Choose a simple H_p-atom a with support $R = I_1 \times I_2 \times A$ where I_1 and I_2 are intervals with

$$2^{-K_i-1} < |I_i|/\pi \leq 2^{-K_i} \qquad (K_i \in \mathbb{N}, i = 1, 2)$$

and

$$[-\pi 2^{-K_i-2}, \pi 2^{-K_i-2}] \subset I_i \subset [-\pi 2^{-K_i-1}, \pi 2^{-K_i-1}].$$

We assume that $r_i \geq 2$ are arbitrary integers. Theorem 16.2 implies that the operator σ_*^α is bounded from $L_2(\mathbb{T}^d)$ to $L_2(\mathbb{T}^d)$. By Theorem 15.4, it is enough to show that

$$\int_{(I_1^{r_1})^c} \int_{(I_2^{r_2})^c} \int_A |\sigma_*^\alpha(x)|^p \,\mathrm{d}x \leq C_p 2^{-\eta_1 r_1} 2^{-\eta_2 r_2}, \tag{16.4}$$

and, if $A = I_3$ is also an interval,

$$\int_{(I_1^{r_1})^c} \int_{(I_2^{r_2})^c} \int_{(I_2)^c} |\sigma_*^\alpha(x)|^p \,\mathrm{d}x \leq C_p 2^{-\eta_1 r_1} 2^{-\eta_2 r_2} \tag{16.5}$$

for all $1/(\alpha+1) < p \leq 1$.

First, we decompose the supremum as

$$\begin{aligned} \sigma_*^\alpha a \quad \leq \quad & \sup_{\substack{n_1 < 2^{K_1}, n_2 < 2^{K_2} \\ n_3 \in \mathbb{N}}} |\sigma_n^\alpha a| + \sup_{\substack{n_1 \geq 2^{K_1}, n_2 < 2^{K_2} \\ n_3 \in \mathbb{N}}} |\sigma_n^\alpha a| \\ & + \sup_{\substack{n_1 < 2^{K_1}, n_2 \geq 2^{K_2} \\ n_3 \in \mathbb{N}}} |\sigma_n^\alpha a| + \sup_{\substack{n_1 \geq 2^{K_1}, n_2 \geq 2^{K_2} \\ n_3 \in \mathbb{N}}} |\sigma_n^\alpha a|. \end{aligned} \tag{16.6}$$

We will investigate only the second term. Obviously,

$$\begin{aligned} & \int_{(I_1^{r_1})^c} \int_{(I_2^{r_2})^c} \int_A \sup_{\substack{n_1 \geq 2^{K_1}, n_2 < 2^{K_2} \\ n_3 \in \mathbb{N}}} |\sigma_n^\alpha a(x)|^p \,\mathrm{d}x \\ & \leq \sum_{|i_1| = 2^{r_1-2}}^{2^{K_1}-1} \sum_{|i_2| = 2^{r_2-2}}^{2^{K_2}-1} \int_{\pi i_1 2^{-K_1}}^{\pi(i_1+1)2^{-K_1}} \int_{\pi i_2 2^{-K_2}}^{\pi(i_2+1)2^{-K_2}} \int_A \sup_{\substack{n_1 \geq 2^{K_1}, n_2 < 2^{K_2} \\ n_3 \in \mathbf{N}}} |\sigma_n^\alpha a(x)|^p \,\mathrm{d}x, \end{aligned}$$

where we may suppose that $i_l > 0$. Let $A_{0,0,0}(x) := a(x)$ and

$$A_{k_1+1,k_2,k_3}(x) := \int_{-\pi}^{x_1} A_{k_1,k_2,k_3}(t, x_2, x_3) \,\mathrm{d}t \qquad (k_i \in \mathbb{N}).$$

In the other indices, we use the same definition. By (iii) of the definition of the simple H_p-atom, we can show that $\operatorname{supp} A_{k_1,k_2,0} \subset R$ and $A_{k_1,k_2,0}(x)$ is zero if x_1 is at the boundary of I_1 or x_2 is at the boundary of I_2 for $k_i = 0, \ldots, N(p)+1$ $(i = 1,2)$, where $N(p) \geq \lfloor 2/p - 3/2 \rfloor$. Moreover, using (ii), we can compute that

$$\|A_{k_1,k_2,0}\|_2 \leq |I_1|^{k_1} |I_2|^{k_2} (|I_1||I_2||A|)^{1/2-1/p} \qquad (k_i = 0, \ldots, N(p)+1). \tag{16.7}$$

We may suppose that $N(p) \geq \alpha + 1$ and choose $N \in \mathbb{N}$ such that $N < \alpha \leq N+1$. For $x_l \in [\pi i_l 2^{-K_l}, \pi(i_l+1)2^{-K_l})$, $t_l \in [-\pi 2^{-K_l-1}, \pi 2^{-K_l-1})$ $(l = 1,2)$ inequality (16.3) implies

$$|(K_{n_1}^\alpha)^{(N)}(x_1 - t_1)| \leq \frac{C n_1^{N-\alpha} 2^{K_1(\alpha+1)}}{i_1^{\alpha+1}} \leq \frac{C 2^{K_1(N+1)}}{i_1^{\alpha+1}}$$

and

$$|(K_{n_2}^\alpha)^{(N+1)}(x_2-t_2)| \le \frac{Cn_2^{N+1-\alpha}2^{K_2(\alpha+1)}}{i_2^{\alpha+1}} \le \frac{C2^{K_2(N+2)}}{i_2^{\alpha+1}}.$$

Recall that in the first case $n_1 \ge 2^{K_1}$ and in the second one $n_2 < 2^{K_2}$.

Integrating by parts, we can see that

$$\begin{aligned}
&|\sigma_n^\alpha a(x)| \\
&= |\int_{I_1}\int_{I_2}\int_A A_{N,N+1,0}(t)(K_{n_1}^\alpha)^{(N)}(x_1-t_1)(K_{n_2}^\alpha)^{(N+1)}(x_2-t_2)K_{n_3}^\alpha(x_3-t_3)\,\mathrm{d}t| \\
&\le \frac{C2^{K_1(N+1)}2^{K_2(N+2)}}{i_1^{\alpha+1}i_2^{\alpha+1}}\int_{I_1}\int_{I_2}|\int_A A_{N,N+1,0}(t)K_{n_3}^\alpha(x_3-t_3)\,\mathrm{d}t_3|\,\mathrm{d}t_1\,\mathrm{d}t_2
\end{aligned}$$

whenever $x_l \in [\pi i_l 2^{-K_l}, \pi(i_l+1)2^{-K_l})$. Hence, by Hölder's inequality,

$$\begin{aligned}
&\int_{(I_1^{r_1})^c}\int_{(I_2^{r_2})^c}\int_A \sup_{\substack{n_1\ge 2^{K_1},n_2<2^{K_2}\\ n_3\in\mathbb{N}}}|\sigma_n^\alpha a(x)|^p\,\mathrm{d}x \\
&\le C_p\sum_{i_1=2^{r_1-2}}^{2^{K_1}-1}\sum_{i_2=2^{r_2-2}}^{2^{K_2}-1}2^{-K_1}2^{-K_2}\frac{2^{K_1(N+1)p}2^{K_2(N+2)p}}{i_1^{(\alpha+1)p}i_2^{(\alpha+1)p}} \\
&\qquad \int_A\Big(\int_{I_1}\int_{I_2}\sup_{n_3\in\mathbb{N}}|\int_A A_{N,N+1,0}(t)K_{n_3}^\alpha(x_3-t_3)\,\mathrm{d}t_3|\,\mathrm{d}t_1\,\mathrm{d}t_2\Big)^p\,\mathrm{d}x_3 \\
&\le C_p|A|^{1-p}\sum_{i_1=2^{r_1-2}}^{2^{K_1}-1}\sum_{i_2=2^{r_2-2}}^{2^{K_2}-1}\frac{2^{K_1((N+1)p-1)}2^{K_2((N+2)p-1)}}{i_1^{(\alpha+1)p}i_2^{(\alpha+1)p}} \\
&\qquad \Big(\int_A\int_{I_1}\int_{I_2}\sup_{n_3\in\mathbb{N}}|\int_A A_{N,N+1,0}(t)K_{n_3}^\alpha(x_3-t_3)\,\mathrm{d}t_3|\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}x_3\Big)^p.
\end{aligned}$$

Using again Hölder's inequality and the fact that σ_*^α is bounded on $L_2(\mathbb{T}^d)$ for all $d \ge 1$, we conclude

$$\begin{aligned}
&\int_{(I_1^{r_1})^c}\int_{(I_2^{r_2})^c}\int_A \sup_{\substack{n_1\ge 2^{K_1},n_2<2^{K_2}\\ n_3\in\mathbb{N}}}|\sigma_n^\alpha a(x)|^p\,\mathrm{d}x \\
&\le C_p|A|^{1-p/2}\sum_{i_1=2^{r_1-2}}^{2^{K_1}-1}\sum_{i_2=2^{r_2-2}}^{2^{K_2}-1}\frac{2^{K_1((N+1)p-1)}2^{K_2((N+2)p-1)}}{i_1^{(\alpha+1)p}i_2^{(\alpha+1)p}} \\
&\qquad \Big(\int_{I_1}\int_{I_2}\Big(\int_{\mathbb{T}}\sup_{n_3\in\mathbb{N}}|\int_A A_{N,N+1,0}(t)K_{n_3}^\alpha(x_3-t_3)\,\mathrm{d}t_3|^2\,\mathrm{d}x_3\Big)^{1/2}\,\mathrm{d}t_1\,\mathrm{d}t_2\Big)^p \\
&\le C_p|A|^{1-p/2}\sum_{i_1=2^{r_1-2}}^{2^{K_1}-1}\sum_{i_2=2^{r_2-2}}^{2^{K_2}-1}\frac{2^{K_1((N+1)p-1)}2^{K_2((N+2)p-1)}}{i_1^{(\alpha+1)p}i_2^{(\alpha+1)p}} \\
&\qquad \Big(\int_{I_1}\int_{I_2}\Big(\int_{\mathbb{T}}|A_{N,N+1,0}(t_1,t_2,x_3)|^2\,\mathrm{d}x_3\Big)^{1/2}\,\mathrm{d}t_1\,\mathrm{d}t_2\Big)^p.
\end{aligned}$$

Then (16.7) implies

$$\begin{aligned}
&\int_{(I_1^{r_1})^c}\int_{(I_2^{r_2})^c}\int_A \sup_{\substack{n_1\geq 2^{K_1},n_2<2^{K_2}\\ n_3\in\mathbb{N}}} |\sigma_n^\alpha a(x)|^p \,\mathrm{d}x \\
&\leq \; C_p|A|^{1-p/2}\sum_{i_1=2^{r_1-2}}^{2^{K_1}-1}\sum_{i_2=2^{r_2-2}}^{2^{K_2}-1} 2^{-K_1p/2}2^{-K_2p/2}\frac{2^{K_1((N+1)p-1)}2^{K_2((N+2)p-1)}}{i_1^{(\alpha+1)p}i_2^{(\alpha+1)p}} \\
&\qquad \Big(\int_{I_1}\int_{I_2}\int_{\mathbb{T}} |A_{N,N+1,0}(t_1,t_2,x_3)|^2 \,\mathrm{d}x_3\,\mathrm{d}t_1\,\mathrm{d}t_2\Big)^{p/2} \\
&\leq \; C_p\sum_{i_1=2^{r_1-2}}^{2^{K_1}-1}\sum_{i_2=2^{r_2-2}}^{2^{K_2}-1}\frac{1}{i_1^{(\alpha+1)p}i_2^{(\alpha+1)p}} \leq C_p 2^{-r_1((\alpha+1)p-1)}2^{-r_2((\alpha+1)p-1)}.
\end{aligned}$$

The other terms of (16.6) can be handled in the same way, which shows (16.4). Obviously, the same ideas show (16.5). ■

Corollary 15.5 implies

Corollary 16.4 *If $\alpha>0$ and $f\in H_1^i(\mathbb{T}^d)$ for some $i=1,\ldots,d$, then*

$$\sup_{\rho>0}\rho\lambda(\sigma_*^\alpha f>\rho)\leq C\|f\|_{H_1^i}.$$

By the density argument, we get here almost everywhere convergence for functions from the spaces $H_1^i(\mathbb{T}^d)$ instead of $L_1(\mathbb{T}^d)$. In some sense, the Hardy space $H_1^i(\mathbb{T}^d)$ plays the role of $L_1(\mathbb{T}^d)$ in higher dimensions.

Corollary 16.5 *If $\alpha>0$ and $f\in H_1^i(\mathbb{T}^d)$ for some $i=1,\ldots,d$, then*

$$\lim_{n\to\infty}\sigma_n^\alpha f=f \qquad \text{a.e.}$$

The almost everywhere convergence is not true for all $f\in L_1(\mathbb{T}^d)$.

A counterexample, which shows that the almost everywhere convergence is not true for all integrable functions, is due to Gát [41]. Recall that

$$L_1(\mathbb{T}^d)\supset H_1^i(\mathbb{T}^d)\supset L(\log L)^{d-1}(\mathbb{T}^d)\supset L_p(\mathbb{T}^d) \qquad (1<p\leq\infty).$$

Let the **conjugate Riesz means** and **conjugate maximal operators** of a distribution f be defined by

$$\tilde{\sigma}_n^{(j_1,\ldots,j_d);\alpha}f(x):=\tilde{f}^{(j_1,\ldots,j_d)}*K_n^\alpha \qquad (j_i=0,1)$$

and

$$\tilde{\sigma}_*^{(j_1,\ldots,j_d);\alpha}f:=\sup_{n\in\mathbb{N}^d}|\tilde{\sigma}_n^{(j_1,\ldots,j_d);\alpha}f|.$$

Then the following results hold.

Theorem 16.6 *If $\alpha > 0$ and $1/(\alpha+1) < p < \infty$, then for all $j_i = 0, 1$,*

$$\|\tilde\sigma_*^{(j_1,\ldots,j_d);\alpha} f\|_p \le C_p \|f\|_{H_p} \qquad (f \in H_p(\mathbb{T}^d))$$

and

$$\|\tilde\sigma_n^{(j_1,\ldots,j_d);\alpha} f\|_{H_p} \le C_p \|f\|_{H_p} \qquad (n \in \mathbb{N}^d, f \in H_p(\mathbb{T}^d)).$$

In particular, if $f \in H_1^i(\mathbb{T}^d)$ for some $i = 1, \ldots, d$, then

$$\sup_{\rho>0} \lambda(\tilde\sigma_*^{(j_1,\ldots,j_d);\alpha} f > \rho) \le C \|f\|_{H_1^i}.$$

Corollary 16.7 *If $\alpha > 0$, $j_i = 0, 1$ and $f \in H_1^i(\mathbb{T}^d)$, then*

$$\lim_{n\to\infty} \tilde\sigma_n^{(j_1,\ldots,j_d);\alpha} f = \tilde f^{(j_1,\ldots,j_d)} \qquad \text{a.e.}$$

Moreover, if $f \in H_p(\mathbb{T}^d)$ with $1/(\alpha+1) < p < \infty$, then this convergence also holds in the $H_p(\mathbb{T}^d)$-norm.

The proofs of the last two results are similar to those of Theorem 9.2 and Corollary 9.3.

17 Rectangular θ-summability

Given the d-dimensional function θ, the **rectangular θ-means** of $f \in L_1(\mathbb{T}^d)$ are defined by

$$\begin{aligned} \sigma_n^\theta f(x) &:= \sum_{k_1\in\mathbb{Z}} \cdots \sum_{k_d\in\mathbb{Z}} \theta\Big(\frac{-k_1}{n_1}, \ldots, \frac{-k_d}{n_d}\Big) \widehat{f}(k) e^{ik\cdot x} \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-u) K_n^\theta(u)\, du \end{aligned} \qquad (17.1)$$

($x \in \mathbb{T}^d, n \in \mathbb{N}^d$), where the **$\theta$-kernels** K_n^θ are given by

$$K_n^\theta(u) := \sum_{k_1\in\mathbb{Z}} \cdots \sum_{k_d\in\mathbb{Z}} \theta\Big(\frac{-k_1}{n_1}, \ldots, \frac{-k_d}{n_d}\Big) e^{ik\cdot u} \qquad (u \in \mathbb{T}^d).$$

Define the **restricted** and **unrestricted maximal operators** by

$$\sigma_\square^\theta f := \sup_{n\in\mathbb{R}_\tau^d} |\sigma_n^\theta f|, \qquad \sigma_*^\theta f := \sup_{n\in\mathbb{N}^d} |\sigma_n^\theta f|.$$

If $d = 1$, then, instead of the third condition of (10.1), we may suppose that $\theta \in W(C, \ell_1)(\mathbb{R})$ (see the definition below). A measurable function f belongs to the **Wiener amalgam space** $W(L_\infty, \ell_1)(\mathbb{R}^d)$ if

$$\|f\|_{W(L_\infty,\ell_1)} := \sum_{k\in\mathbb{Z}^d} \sup_{x\in[0,1)^d} |f(x+k)| < \infty.$$

It is easy to see that

$$W(L_\infty,\ell_1)(\mathbb{R}^d) \subset L_p(\mathbb{R}^d) \quad \text{for all} \quad 1 \le p \le \infty.$$

The smallest closed subspace of $W(L_\infty,\ell_1)(\mathbb{R}^d)$ containing continuous functions is denoted by $W(C,\ell_1)(\mathbb{R}^d)$ and is called the **Wiener algebra**. It is used quite often in Gabor analysis, because it provides a convenient and general class of windows (see e.g. Walnut [87] and Gröchenig [45]).

If θ is continuous and $|\theta|$ can be estimated by an integrable function η which is non-decreasing on $(-\infty,c)$ and non-increasing on (c,∞), then $\theta \in W(C,\ell_1)(\mathbb{R})$. Since

$$\begin{aligned} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} \Big|\theta\Big(\frac{k_1}{n_1},\ldots,\frac{k_d}{n_d}\Big)\Big| &\le \sum_{l\in\mathbb{Z}^d} \Big(\prod_{j=1}^d n_j\Big) \sup_{x\in[0,1)^d} |\theta(x+l)| \\ &= \Big(\prod_{j=1}^d n_j\Big) \|\theta\|_{W(C,\ell_1)} < \infty, \end{aligned} \tag{17.2}$$

the θ-kernels K_n^θ and the θ-means $\sigma_n^\theta f$ are well defined.

We introduce **Feichtinger's algebra** $\mathbf{S}_0(\mathbb{R}^d)$, which is a subspace of the Wiener algebra. The **short-time Fourier transform** of $f \in L_2(\mathbb{R}^d)$ with respect to a window function $g \in L_2(\mathbb{R}^d)$ is defined by

$$S_g f(x,\omega) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(t)\overline{g(t-x)} \mathrm{e}^{-\mathrm{i}\omega\cdot t}\,\mathrm{d}t \qquad (x,\omega\in\mathbb{R}^d).$$

Using the short-time Fourier transform with respect to the Gauss function $g_0(x) := \mathrm{e}^{-\pi\|x\|_2^2}$, we define $\mathbf{S}_0(\mathbb{R}^d)$ by

$$\mathbf{S}_0(\mathbb{R}^d) := \left\{ f \in L^2(\mathbb{R}^d) : \|f\|_{\mathbf{S}_0} := \|S_{g_0} f\|_{L_1(\mathbb{R}^{2d})} < \infty \right\}.$$

Any other non-zero Schwartz function defines the same space and an equivalent norm. It is known that $\mathbf{S}_0(\mathbb{R}^d)$ is isometrically invariant under translation, modulation and Fourier transform (see Feichtinger [35]). Actually, $\mathbf{S}_0$ is the minimal Banach space having this property (see Feichtinger [35]). Furthermore, the embedding $\mathbf{S}_0(\mathbb{R}^d) \hookrightarrow W(C,\ell_1)(\mathbb{R}^d)$ is dense and continuous and

$$\mathbf{S}_0(\mathbb{R}^d) \subsetneq W(C,\ell_1)(\mathbb{R}^d) \cap \mathcal{F}(W(C,\ell_1)(\mathbb{R}^d)),$$

where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}(W(C,\ell_1)(\mathbb{R}^d))$ the set of Fourier transforms of the functions from $W(C,\ell_1)(\mathbb{R}^d)$ (see Feichtinger and Zimmermann [38], Losert [60] and Gröchenig [44]).

17.1 Norm convergence

First, we investigate the L_2-norm convergence of $\sigma_n^\theta f$ as $n \to \infty$.

Theorem 17.1 *If $\theta \in W(C,\ell_1)(\mathbb{R}^d)$ and $\theta(0)=1$, then*

$$\lim_{n\to\infty} \sigma_n^\theta f = f \quad \text{in the } L_2(\mathbb{T}^d)\text{-norm for all } f \in L_2(\mathbb{T}^d).$$

Proof. It is easy to see that the norm of the operator $\sigma_n^\theta : L_2(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$ is

$$\begin{aligned}
\sup_{f\in L_2(\mathbb{T}^d),\, \|f\|_2\le 1} \|f * K_n^\theta\|_2 &= \sup_{f\in L_2(\mathbb{T}^d),\, \|f\|_2\le 1} \|\widehat{f}\widehat{K}_n^\theta\|_2 \\
&= \sup_{\widehat{f}\in \ell_2(\mathbb{Z}^d),\, \|\widehat{f}\|_2\le 1} \|\widehat{f}\widehat{K}_n^\theta\|_2 \\
&= \|\widehat{K}_n^\theta\|_\infty \\
&= \sup_{k\in\mathbb{Z}^d} \Big|\theta\Big(\frac{-k_1}{n_1+1},\ldots,\frac{-k_d}{n_d+1}\Big)\Big| \\
&\le C.
\end{aligned}$$

Thus the norms of σ_n^θ $(n \in \mathbb{N}^d)$ are uniformly bounded. Since θ is continuous, the convergence holds for all trigonometric polynomials. The set of the trigonometric polynomials are dense in $L_2(\mathbb{T}^d)$, so the usual density theorem proves Theorem 17.1. ∎

The θ-means can be written as a singular integral of f and of the Fourier transform of θ in the following way (Feichtinger and Weisz [36]).

Theorem 17.2 *If $\theta \in W(C,\ell_1)(\mathbb{R}^d)$ and $\widehat{\theta} \in L_1(\mathbb{R}^d)$, then*

$$\sigma_n^\theta f(x) = \Big(\prod_{j=1}^d n_j\Big) \int_{\mathbb{R}^d} f(x-t)\widehat{\theta}(n_1t_1,\ldots,n_dt_d)\,\mathrm{d}t$$

for a.e. $x \in \mathbb{T}^d$ and for all $n \in \mathbb{N}^d$ and $f \in L_1(\mathbb{T}^d)$.

Proof. If $f(t) = \mathrm{e}^{\mathrm{i}k\cdot t}$ $(k \in \mathbb{Z}^d, t \in \mathbb{T}^d)$, then

$$\begin{aligned}
\Big(\prod_{j=1}^d n_j\Big) \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}k\cdot(x-t)}\widehat{\theta}(n_1t_1,\ldots,n_dt_d)\,\mathrm{d}t &= \mathrm{e}^{\mathrm{i}k\cdot x}\int_{\mathbb{R}^d} \Big(\prod_{j=1}^d \mathrm{e}^{-\mathrm{i}k_jt_j/n_j}\Big)\widehat{\theta}(t)\,\mathrm{d}t \\
&= \theta\Big(\frac{-k_1}{n_1},\ldots,\frac{-k_d}{n_d}\Big)\mathrm{e}^{\mathrm{i}k\cdot x} \\
&= \sigma_n^\theta f(x),
\end{aligned}$$

thus the theorem holds also for trigonometric polynomials. The proof can be finished as in Theorem 6.24. ∎

Now, we give a sufficient and necessary condition for the uniform and L_1 convergence $\sigma_n^\theta f \to f$ (see Feichtinger and Weisz [36]). Note that the statement $(i) \Leftrightarrow (ii)$ in the next theorem was shown in the one-dimensional case by Natanson and Zuk [66] for θ having compact support. The situation in our general case is much more complicated.

Theorem 17.3 *If $\theta \in W(C,\ell_1)(\mathbb{R}^d)$ and $\theta(0)=1$, then the following conditions are equivalent:*

(i) $\widehat{\theta} \in L_1(\mathbb{R}^d)$,

(ii) $\sigma_n^\theta f \to f$ *uniformly for all* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$,

(iii) $\sigma_n^\theta f(x) \to f(x)$ *for all* $x \in \mathbb{T}^d$ *and* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$,

(iv) $\sigma_n^\theta f \to f$ *in the* $L_1(\mathbb{T}^d)$*-norm for all* $f \in L_1(\mathbb{T}^d)$ *as* $n \to \infty$,

(v) $\sigma_n^\theta f \to f$ *uniformly for all* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$ *and* $n \in \mathbb{R}_\tau^d$,

(vi) $\sigma_n^\theta f(x) \to f(x)$ *for all* $x \in \mathbb{T}^d$ *and* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$ *and* $n \in \mathbb{R}_\tau^d$,

(vii) $\sigma_n^\theta f \to f$ *in the* $L_1(\mathbb{T}^d)$*-norm for all* $f \in L_1(\mathbb{T}^d)$ *as* $n \to \infty$ *and* $n \in \mathbb{R}_\tau^d$.

Recall the definition of R_τ^d from (14.13).

Proof. First, we verify the equivalence between *(i)*, *(ii)*, *(iii)* and *(iv)*. We may suppose that $d=1$, since the multi-dimensional case is similar. If *(i)* holds, then by Theorem 17.2,

$$\|\sigma_n^\theta f\|_\infty \le \|f\|_\infty \|\widehat{\theta}\|_1 \qquad (f \in C(\mathbb{T}), n \in \mathbb{N})$$

and so $\sigma_n : C(\mathbb{T}) \to C(\mathbb{T})$ are uniformly bounded. Since *(ii)* holds for all trigonometric polynomials and the set of the trigonometric polynomials are dense in $C(\mathbb{T})$, *(ii)* follows easily. *(ii)* implies *(iii)* trivially.

Suppose that *(iii)* is satisfied. We are going to prove *(i)*. For a fixed $x \in \mathbb{T}$, the operators

$$U_n : C(\mathbb{T}) \to \mathbb{R}, \qquad U_n f := \sigma_n^\theta f(x) \qquad (n \in \mathbb{N})$$

are uniformly bounded by the Banach-Steinhaus theorem. We get by (17.1) that

$$\|U_n\| = \int_{\mathbb{T}} |K_n^\theta(x-t)| \,\mathrm{d}t = \|K_n^\theta\|_1 \qquad (n \in \mathbb{N}).$$

Hence

$$\sup_{n\in\mathbb{N}} \|K_n^\theta\|_1 \le C.$$

Since K_n^θ is 2π-periodic, we have for $\alpha \le (n+1)/2$ that

$$\begin{aligned}
&\int_{-2\alpha\pi}^{2\alpha\pi} \frac{1}{n+1} \Big| \sum_{k=-\infty}^{\infty} \theta\Big(\frac{-k}{n+1}\Big) \mathrm{e}^{\mathrm{i}t\frac{k}{n+1}} \Big| \,\mathrm{d}t \\
&\le \int_{-(n+1)\pi}^{(n+1)\pi} \frac{1}{n+1} \Big| \sum_{k=-\infty}^{\infty} \theta\Big(\frac{-k}{n+1}\Big) \mathrm{e}^{\mathrm{i}t\frac{k}{n+1}} \Big| \,\mathrm{d}t \\
&= \int_{-\pi}^{\pi} \Big| \sum_{k=-\infty}^{\infty} \theta\Big(\frac{-k}{n+1}\Big) \mathrm{e}^{\mathrm{i}kx} \Big| \,\mathrm{d}x \\
&= \int_{\mathbb{T}} |K_n^\theta(x)| \,\mathrm{d}x \le C. \qquad (17.3)
\end{aligned}$$

For a fixed $t \in \mathbb{R}$, let

$$h_n(t) := \frac{1}{n+1} \sum_{k=-\infty}^{\infty} \theta\Big(\frac{-k}{n+1}\Big) \mathrm{e}^{\mathrm{i}t\frac{k}{n+1}}$$

and

$$\varphi_n(t,u) := \sum_{k=-\infty}^{\infty} \theta\Big(\frac{-k}{n+1}\Big) \mathrm{e}^{\mathrm{i}t\frac{k}{n+1}} 1_{[\frac{k}{n+1},\frac{k+1}{n+1})}(u).$$

It is easy to see that

$$\lim_{n\to\infty} \varphi_n(t,u) = \theta(-u)\mathrm{e}^{\mathrm{i}tu}.$$

Moreover,

$$|\varphi_n(t,u)| \leq \sum_{l=-\infty}^{\infty} \sup_{x\in[0,1)} |\theta(x-l-1)| 1_{[l,l+1)}(u)$$

and

$$\int_{-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \sup_{x\in[0,1)} |\theta(x-l-1)| 1_{[l,l+1)}(u)\,\mathrm{d}u = \sum_{l=-\infty}^{\infty} \sup_{x\in[0,1)} |\theta(x-l-1)| = \|\theta\|_{W(C,\ell_1)}.$$

Lebesgue's dominated convergence theorem implies that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \varphi_n(t,u)\,\mathrm{d}u = \int_{-\infty}^{\infty} \theta(-u)\mathrm{e}^{\mathrm{i}tu}\,\mathrm{d}u = \widehat{\theta}(t).$$

Obviously,

$$\int_{-\infty}^{\infty} \varphi_n(t,u)\,\mathrm{d}u = h_n(t)$$

and so

$$\lim_{n\to\infty} h_n(t) = \widehat{\theta}(t).$$

Of course, this holds for all $t \in \mathbb{R}$. We have by (17.2) that $|h_n(t)| \leq \|\theta\|_{W(C,\ell_1)}$. Thus

$$\lim_{n\to\infty} \int_{-2\alpha\pi}^{2\alpha\pi} |h_n(t)|\,\mathrm{d}t = \int_{-2\alpha\pi}^{2\alpha\pi} |\widehat{\theta}(t)|\,\mathrm{d}t.$$

Inequality (17.3) yields that

$$\int_{-2\alpha\pi}^{2\alpha\pi} |\widehat{\theta}(t)|\,\mathrm{d}t \leq C \qquad \text{for all} \qquad \alpha > 0$$

and so

$$\int_{-\infty}^{\infty} |\widehat{\theta}(t)|\,\mathrm{d}t \leq C,$$

which shows *(i)*.

If $\widehat{\theta} \in L_1(\mathbb{R})$, then Theorem 17.2 implies

$$\|\sigma_n^\theta f\|_1 \le \|f\|_1 \|\widehat{\theta}\|_1 \qquad (f \in L_1(\mathbb{T}), n \in \mathbb{N}).$$

Hence *(iv)* follows from *(i)* because the set of the trigonometric polynomials are dense in $L_1(\mathbb{T})$. The fact that *(iv)* implies *(i)* can be proved similarly as $(iii) \Rightarrow (i)$, since, by duality, the norm of the operator $\sigma_n^\theta : L_1(\mathbb{T}) \to L_1(\mathbb{T})$ is again $\|\sigma_n^\theta\| = \|K_n^\theta\|_1$.

It is easy to see that the equivalence between *(i)*, *(v)*, *(vi)* and *(vii)* can be proved in the same way. ■

One part of the preceding result is generalized for homogeneous Banach spaces.

Theorem 17.4 *Assume that B is a homogeneous Banach space on $\mathbb{T}^d$. If $\theta(0) = 1$, $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ and $\widehat{\theta} \in L_1(\mathbb{R}^d)$, then*

$$\|\sigma_n^\theta f\|_B \le C \|f\|_B \qquad (n \in \mathbb{N}^d)$$

and

$$\lim_{n\to\infty} \sigma_n^\theta f = f \qquad \text{in the } B\text{-norm for all } f \in B.$$

Proof. For simplicity, we show the theorem for $d = 1$. Using Theorem 17.2, we conclude

$$\begin{aligned}\sigma_n^\theta f(x) - f(x) &= n \int_{\mathbb{R}} (f(x-t) - f(x)) \widehat{\theta}(nt) \,\mathrm{d}t \\ &= \int_{\mathbb{R}} \Big(f(x - \frac{t}{n}) - f(x)\Big) \widehat{\theta}(t) \,\mathrm{d}t\end{aligned}$$

and

$$\|\sigma_n^\theta f - f\|_B = \int_{\mathbb{R}} \Big\| T_{\frac{t}{n}} f - f \Big\|_B |\widehat{\theta}(t)| \,\mathrm{d}t.$$

The theorem follows from the definition of the homogeneous Banach spaces and from the Lebesgue dominated convergence theorem. ■

Since $\theta \in \mathbf{S}_0(\mathbb{R}^d)$ implies $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ and $\widehat{\theta} \in \mathbf{S}_0(\mathbb{R}^d) \subset L_1(\mathbb{R}^d)$, the next corollary follows from Theorems 17.3 and 17.4.

Corollary 17.5 *If $\theta \in \mathbf{S}_0(\mathbb{R}^d)$ and $\theta(0) = 1$, then*

- *(i)* $\sigma_n^\theta f \to f$ *uniformly for all* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$,
- *(ii)* $\sigma_n^\theta f \to f$ *in the* $L_1(\mathbb{T}^d)$*-norm for all* $f \in L_1(\mathbb{T}^d)$ *as* $n \to \infty$,
- *(iii)* $\sigma_n^\theta f \to f$ *in the* B*-norm for all* $f \in B$ *as* $n \to \infty$ *if* B *is a homogeneous Banach space.*

The next corollary follows from the fact that $\theta \in \mathbf{S}_0(\mathbb{R}^d)$ is equivalent to $\widehat{\theta} \in L_1(\mathbb{R}^d)$, provided that θ has compact support (see e.g. Feichtinger and Zimmermann [38]).

Corollary 17.6 *If $\theta \in C(\mathbb{R}^d)$ has compact support and $\theta(0) = 1$, then the following conditions are equivalent:*

(i) $\theta \in \mathbf{S}_0(\mathbb{R}^d)$,

(ii) $\sigma_n^\theta f \to f$ *uniformly for all* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$,

(iii) $\sigma_n^\theta f(x) \to f(x)$ *for all* $x \in \mathbb{T}^d$ *and* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$,

(iv) $\sigma_n^\theta f \to f$ *in the* $L_1(\mathbb{T}^d)$*-norm for all* $f \in L_1(\mathbb{T}^d)$ *as* $n \to \infty$,

(v) $\sigma_n^\theta f \to f$ *uniformly for all* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$ *and* $n \in \mathbb{R}_\tau^d$,

(vi) $\sigma_n^\theta f(x) \to f(x)$ *for all* $x \in \mathbb{T}^d$ *and* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$ *and* $n \in \mathbb{R}_\tau^d$,

(vii) $\sigma_n^\theta f \to f$ *in the* $L_1(\mathbb{T}^d)$*-norm for all* $f \in L_1(\mathbb{T}^d)$ *as* $n \to \infty$ *and* $n \in \mathbb{R}_\tau^d$.

In the rest of this subsection, we give some sufficient conditions for a function θ to satisfy $\widehat{\theta} \in L_1(\mathbb{R}^d)$, resp. $\theta \in \mathbf{S}_0(\mathbb{R}^d)$. Several such conditions are already known. For example, if $\theta \in L_\infty(\mathbb{R}^d)$ and $\widehat{\theta} \geq 0$, then $\widehat{\theta} \in L_1(\mathbb{R}^d)$ (see Bachman, Narici and Beckenstein [4]). As mentioned before, $\theta \in \mathbf{S}_0(\mathbb{R}^d)$ implies also that $\widehat{\theta} \in L_1(\mathbb{R}^d)$. Recall that $\mathbf{S}_0(\mathbb{R}^d)$ contains all Schwartz functions. If $\theta \in L_1(\mathbb{R}^d)$ and $\widehat{\theta}$ has compact support or if $\theta \in L_1(\mathbb{R}^d)$ has compact support and $\widehat{\theta} \in L_1(\mathbb{R}^d)$, then $\theta \in \mathbf{S}_0(\mathbb{R}^d)$. If $\theta \in L_1(\mathbb{R})$ has compact support and $\theta \in \mathrm{Lip}(\alpha)$ for some $\alpha > 1/2$, then $\widehat{\theta} \in L_1(\mathbb{R})$ (see Natanson and Zuk [66, p. 176]) and so $\theta \in \mathbf{S}_0(\mathbb{R})$. If $\theta v_s, \widehat{\theta} v_s \in L_2(\mathbb{R}^d)$ for some $s > d$ or if $\theta v_s, \widehat{\theta} v_s \in L_\infty(\mathbb{R}^d)$ for some $s > 3d/2$, then $\theta \in \mathbf{S}_0(\mathbb{R}^d)$. The weight function v_s is given by $v_s(\omega) := (1 + |\omega|)^s$ ($\omega \in \mathbb{R}^d, s \in \mathbb{R}$).

Sufficient conditions can also be given with the help of Sobolev, fractional Sobolev and Besov spaces. For a detailed description of these spaces, see Triebel [85], Runst and Sickel [72], Stein [77] and Grafakos [43]. The **Sobolev space** $W_p^k(\mathbb{R}^d)$ ($1 \leq p \leq \infty, k \in \mathbb{N}$) is defined by

$$W_p^k(\mathbb{R}^d) := \{\theta \in L_p(\mathbb{R}^d) : D^\alpha \theta \in L_p(\mathbb{R}^d), |\alpha| \leq k\}$$

and endowed with the norm

$$\|\theta\|_{W_p^k} := \sum_{|\alpha| \leq k} \|D^\alpha \theta\|_p,$$

where D denotes the distributional derivative.

This definition can be extended to every real s in the following way. The **fractional Sobolev space** $\mathcal{L}_p^s(\mathbb{R}^d)$ ($1 \leq p \leq \infty, s \in \mathbb{R}$) consists of all tempered distributions θ for which

$$\|\theta\|_{\mathcal{L}_p^s} := \|\mathcal{F}^{-1}((1 + |\cdot|^2)^{s/2} \widehat{\theta})\|_p < \infty,$$

where $\mathcal{F}$ denotes the Fourier transform. It is known that

$$\mathcal{L}_p^s(\mathbb{R}^d) = W_p^k(\mathbb{R}^d) \quad \text{if} \quad s = k \in \mathbb{N} \quad \text{and} \quad 1 < p < \infty$$

with equivalent norms.

In order to define the Besov spaces, take a non-negative Schwartz function $\psi \in \mathcal{S}(\mathbb{R})$ with support $[1/2, 2]$ that satisfies

$$\sum_{k=-\infty}^{\infty} \psi(2^{-k}s) = 1 \quad \text{for all} \quad s \in \mathbb{R} \setminus \{0\}.$$

For $x \in \mathbb{R}^d$, let

$$\phi_k(x) := \psi(2^{-k}|x|) \quad \text{for} \quad k \geq 1 \quad \text{and} \quad \phi_0(x) = 1 - \sum_{k=1}^{\infty} \phi_k(x).$$

The **Besov space** $B^s_{p,r}(\mathbb{R}^d)$ $(0 < p, r \leq \infty, s \in \mathbb{R})$ is the space of all tempered distributions f for which

$$\|f\|_{B^s_{p,r}} := \Big(\sum_{k=0}^{\infty} 2^{ksr} \|(\mathcal{F}^{-1}\phi_k) * f\|_p^r \Big)^{1/r} < \infty.$$

The Sobolev, fractional Sobolev and Besov spaces are all quasi-Banach spaces, and if $1 \leq p, r \leq \infty$, then they are Banach spaces. All these spaces contain the Schwartz functions. The following facts are known: in the case $1 \leq p, r \leq \infty$ one has

$$W^m_p(\mathbb{R}^d), B^s_{p,r}(\mathbb{R}^d) \hookrightarrow L_p(\mathbb{R}^d) \qquad \text{if} \quad s > 0, m \in \mathbb{N},$$

$$W^{m+1}_p(\mathbb{R}^d) \hookrightarrow B^s_{p,r}(\mathbb{R}^d) \hookrightarrow W^m_p(\mathbb{R}^d) \qquad \text{if} \quad m < s < m+1, \tag{17.4}$$

$$B^s_{p,r}(\mathbb{R}^d) \hookrightarrow B^s_{p,r+\epsilon}(\mathbb{R}^d), B^{s+\epsilon}_{p,\infty}(\mathbb{R}^d) \hookrightarrow B^s_{p,r}(\mathbb{R}^d) \qquad \text{if} \quad \epsilon > 0, \tag{17.5}$$

$$B^{d/p_1}_{p_1,1}(\mathbb{R}^d) \hookrightarrow B^{d/p_2}_{p_2,1}(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d) \qquad \text{if} \quad 1 \leq p_1 \leq p_2 < \infty. \tag{17.6}$$

For two quasi-Banach spaces $\mathbf{X}$ and $\mathbf{Y}$, the embedding $\mathbf{X} \hookrightarrow \mathbf{Y}$ means that $\mathbf{X} \subset \mathbf{Y}$ and $\|f\|_{\mathbf{Y}} \leq C\|f\|_{\mathbf{X}}$.

The connection between Besov spaces and Feichtinger's algebra is summarized in the next theorem.

Theorem 17.7 *We have*

(i) If $1 \leq p \leq 2$ *and* $\theta \in B^{d/p}_{p,1}(\mathbb{R}^d)$, *then* $\widehat{\theta} \in L_1(\mathbb{R}^d)$ *and*

$$\|\widehat{\theta}\|_1 \leq C\|\theta\|_{B^{d/p}_{p,1}}.$$

(ii) If $s > d$, *then* $\mathcal{L}^s_1(\mathbb{R}^d) \hookrightarrow \mathbf{S}_0(\mathbb{R}^d)$.

(iii) If d' *denotes the smallest even integer which is larger than* d *and* $s > d'$, *then*

$$B^s_{1,\infty}(\mathbb{R}^d) \hookrightarrow W^{d'}_1(\mathbb{R}^d) \hookrightarrow \mathbf{S}_0(\mathbb{R}^d).$$

Proof. (i) was proved in Girardi and Weis [42] and (ii) in Okoudjou [67]. The first embedding of (iii) follows from (17.4) and (17.5). If k is even, then $W_1^k(\mathbb{R}^d) \hookrightarrow \mathcal{L}_1^k(\mathbb{R}^d)$ (see Stein [77, p. 160]). Then (ii) proves (iii). ∎

It follows from (i) and (17.4) that $\theta \in W_p^j(\mathbb{R}^d)$ $(j > d/p, j \in \mathbb{N})$ implies $\widehat{\theta} \in L_1(\mathbb{R}^d)$. If $j \geq d'$, then even $W_1^j(\mathbb{R}^d) \hookrightarrow \mathbf{S}_0(\mathbb{R}^d)$ (see (iii)). Moreover, if $s > d'$ as in (iii), then

$$B_{1,\infty}^s(\mathbb{R}^d) \hookrightarrow B_{1,1}^d(\mathbb{R}^d) \hookrightarrow B_{p,1}^{d/p}(\mathbb{R}^d) \qquad (1 < p < \infty)$$

by (17.5) and (17.6). Theorem 17.7 says that $B_{1,\infty}^s(\mathbb{R}^d) \subset \mathbf{S}_0(\mathbb{R}^d)$ $(s > d')$ and if we choose θ from the larger space $B_{p,1}^{d/p}(\mathbb{R}^d)$ $(1 \leq p \leq 2)$, then $\widehat{\theta}$ is still integrable.

The embedding $W_1^2(\mathbb{R}) \hookrightarrow \mathbf{S}_0(\mathbb{R})$ follows from (iii). With the help of the usual derivative, we give another useful sufficient condition for a function to be in $\mathbf{S}_0(\mathbb{R}^d)$.

Definition 17.8 *A function θ is in $V_1^k(\mathbb{R})$ if there are numbers $-\infty = a_0 < a_1 < \cdots < a_n < a_{n+1} = \infty$, where $n = n(\theta)$ depends on θ and*

$$\theta \in C^{k-2}(\mathbb{R}), \qquad \theta \in C^k(a_i, a_{i+1}), \qquad \theta^{(j)} \in L_1(\mathbb{R})$$

for all $i = 0, \ldots, n$ and $j = 0, \ldots, k$. Here, C^k denotes the set of k times continuously differentiable functions. The norm of this space is defined by

$$\|\theta\|_{V_1^k} := \sum_{j=0}^{k} \|\theta^{(j)}\|_1 + \sum_{i=1}^{n} |\theta^{(k-1)}(a_i + 0) - \theta^{(k-1)}(a_i - 0)|,$$

where $\theta^{(k-1)}(a_i \pm 0)$ denotes the right and left limits of $\theta^{(k-1)}$.

These limits do exist and are finite because $\theta^{(k)} \in C(a_i, a_{i+1}) \cap L_1(\mathbb{R})$ implies

$$\theta^{(k-1)}(x) = \theta^{(k-1)}(a) + \int_a^x \theta^{(k)}(t)\,\mathrm{d}t$$

for some $a \in (a_i, a_{i+1})$. Since $\theta^{(k-1)} \in L_1(\mathbb{R})$ we establish that

$$\lim_{x \to -\infty} \theta^{(k-1)}(x) = \lim_{x \to \infty} \theta^{(k-1)}(x) = 0.$$

Similarly, $\theta^{(j)} \in C_0(\mathbb{R})$ for $j = 0, \ldots, k-2$.

Of course, $W_1^2(\mathbb{R})$ and $V_1^2(\mathbb{R})$ are not identical. For $\theta \in V_1^2(\mathbb{R})$, we have $\theta' = D\theta$, however, $\theta'' = D^2\theta$ only if $\lim_{x \to a_i+0} \theta'(x) = \lim_{x \to a_i-0} \theta'(x)$ $(i = 1, \ldots, n)$.

Theorem 17.9 *We have $V_1^2(\mathbb{R}) \hookrightarrow \mathbf{S}_0(\mathbb{R})$.*

Proof. Integrating by parts, we have

$$
\begin{aligned}
S_{g_0}\theta(x,\omega) &= \frac{1}{2\pi}\int_{\mathbb{R}} \theta(t)\overline{g_0(t-x)}\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \\
&= \frac{1}{2\pi}\sum_{i=0}^{n}\int_{a_i}^{a_{i+1}} \theta(t)\mathrm{e}^{-\pi(t-x)^2}\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \\
&= \frac{1}{2\pi}\sum_{i=0}^{n}\Big[\theta(t)\mathrm{e}^{-\pi(t-x)^2}\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{-\mathrm{i}\omega}\Big]_{a_i}^{a_{i+1}} \\
&\quad - \frac{1}{2\pi}\sum_{i=0}^{n}\int_{a_i}^{a_{i+1}}\Big(\theta'(t)\mathrm{e}^{-\pi(t-x)^2} - 2\pi\theta(t)\mathrm{e}^{-\pi(t-x)^2}(t-x)\Big)\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{-\mathrm{i}\omega}\,\mathrm{d}t.
\end{aligned}
$$

Observe that the first sum is 0. In the second sum, we integrate by parts again to obtain

$$
\begin{aligned}
S_{g_0}\theta(x,\omega) &= \frac{1}{2\pi}\sum_{i=0}^{n}\Big[\Big(\theta'(t)\mathrm{e}^{-\pi(t-x)^2} - 2\pi\theta(t)\mathrm{e}^{-\pi(t-x)^2}(t-x)\Big)\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\omega^2}\Big]_{a_i}^{a_{i+1}} \\
&\quad - \frac{1}{2\pi}\sum_{i=0}^{n}\int_{a_i}^{a_{i+1}}\Big(\theta''(t)\mathrm{e}^{-\pi(t-x)^2} - 4\pi\theta'(t)\mathrm{e}^{-\pi(t-x)^2}(t-x) \\
&\quad - 2\pi\theta(t)\Big(-2\pi\mathrm{e}^{-\pi(t-x)^2}(t-x)^2 + \mathrm{e}^{-\pi(t-x)^2}\Big)\Big)\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\omega^2}\,\mathrm{d}t.
\end{aligned}
$$

The first sum is equal to

$$
\frac{1}{2\pi}\sum_{i=1}^{n}\Big(\theta'(a_i+0)-\theta'(a_i-0)\Big)\mathrm{e}^{-\pi(a_i-x)^2}\frac{\mathrm{e}^{-\mathrm{i}\omega a_i}}{\omega^2}\ .
$$

Hence

$$
\int_{\mathbb{R}}\int_{\{|\omega|\ge 1\}} |S_{g_0}\theta(x,\omega)|\,\mathrm{d}x\,\mathrm{d}\omega \le C_s\|\theta\|_{V_1^2}.
$$

On the other hand,

$$
\int_{\mathbb{R}}\int_{\{|\omega|<1\}} |S_{g_0}\theta(x,\omega)|\,\mathrm{d}x\,\mathrm{d}\omega \le C_s\int_{\mathbb{R}}\int_{\{|\omega|<1\}}\int_{\mathbb{R}}|\theta(t)|g_0(t-x)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\omega \le C_s\|\theta\|_{V_1^2},
$$

which finishes the proof of Theorem 17.9. ■

The next Corollary follows from the definition of $\mathbf{S}_0(\mathbb{R}^d)$ and from Theorem 17.9.

Corollary 17.10 *If each $\theta_j \in V_1^2(\mathbb{R})$ $(j=1,\ldots,d)$, then*

$$
\theta := \prod_{j=1}^{d}\theta_j \in \mathbf{S}_0(\mathbb{R}^d).
$$

It is easy to see that $\theta \in V_1^2(\mathbb{R}) \subset \mathbf{S}_0(\mathbb{R})$ in all examples of Subsection 10.1. Moreover, in Example 10.2 (the Riesz summation), $\theta \in \mathbf{S}_0(\mathbb{R})$ for all $0 < \alpha < \infty$. In the next examples, θ has d variables and $\theta \in \mathbf{S}_0(\mathbb{R}^d)$.

Example 17.11 (Riesz summation) Let

$$\theta(t) = \begin{cases} (1 - \|t\|_2^\gamma)^\alpha & \text{if } \|t\|_2 \leq 1 \\ 0 & \text{if } \|t\|_2 > 1 \end{cases} \qquad (t \in \mathbb{R}^d)$$

for some $(d-1)/2 < \alpha < \infty, \gamma \in \mathbb{N}_+$ (see Figure 28).

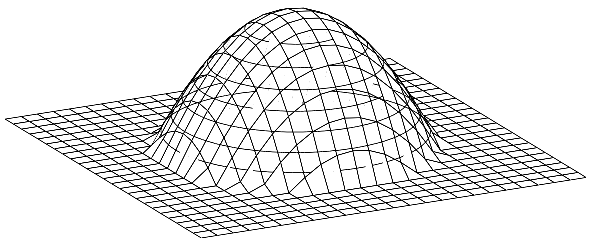

Figure 28: Riesz summability function with $d = 2$, $\alpha = 1$, $\gamma = 2$.

Example 17.12 (Weierstrass summation) Let

$$\theta(t) = \mathrm{e}^{-\|t\|_2} \quad \text{or} \quad \theta(t) = \mathrm{e}^{-\|t\|_2^2} \quad \text{or even} \quad \theta(t) = \mathrm{e}^{-\|t\|_{2l}^{2l}}$$

for some $l \in \mathbb{N}_+$ $(t \in \mathbb{R}^d)$ (see Figure 29).

Example 17.13 Let

$$\theta(t) = \mathrm{e}^{-(1+\|t\|_{2l}^{2l})^\gamma} \qquad (l \in \mathbb{N}_+, 0 < \gamma < \infty)$$

(see Figure 30).

Example 17.14 (Picard and Bessel summations) Let

$$\theta(t) = (1 + \|t\|_\gamma^\gamma)^{-\alpha} \qquad (t \in \mathbb{R}^d, 0 < \alpha < \infty, 1 \leq \gamma < \infty, \alpha\gamma > d)$$

(see Figure 31).

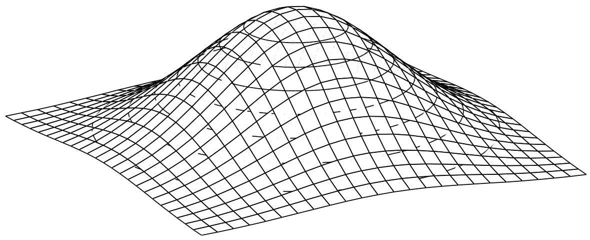

Figure 29: Weierstrass summability function $\theta(t) = \mathrm{e}^{-\|t\|_2^2}$.

17.2 Almost everywhere convergence

In this subsection, we suppose that

$$\theta(0) = 1, \qquad \theta = \theta_1 \otimes \cdots \otimes \theta_d, \qquad \theta_j \in W(C, \ell_1)(\mathbb{R}), \qquad j = 1, \ldots, d.$$

For the restricted convergence, we suppose in addition that

$$\mathcal{I}\,\theta_j \in W(C, \ell_1)(\mathbb{R}), \qquad j = 1, \ldots, d.$$

Note that $\mathcal{I}$ denotes the identity function, so $\mathcal{I}(x) = x$ and $(\mathcal{I}\,\theta_j)(x) = x\theta_j(x)$. Then (17.2) implies that

$$|K_n^{\theta_j}| \leq Cn \qquad (n \in \mathbb{N}).$$

Similarly,

$$\sum_{k=-\infty}^{\infty} \Big| \frac{k}{n} \theta_j(\frac{k}{n}) \Big| \leq n \|\mathcal{I}\,\theta_j\|_{W(C,\ell_1)} < \infty,$$

from which we get immediately that

$$|(K_n^{\theta_j})'| \leq Cn^2 \qquad (n \in \mathbb{N}).$$

These two inequalities were used several times in the proofs of Theorems 14.1 and 14.7. By Theorem 17.2,

$$K_{n_j}^{\theta_j}(x) = 2\pi n_j \sum_{k=-\infty}^{\infty} \widehat{\theta}_j(n_j(x + 2k\pi)) \qquad (x \in \mathbb{T})$$

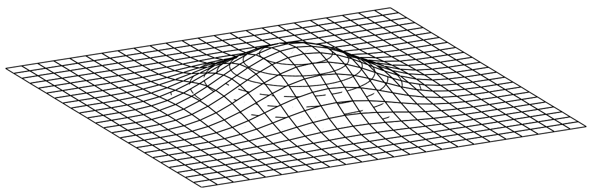

Figure 30: The summability function in Example 17.13 with $d = 2$, $l = 1$, $\gamma = 2$.

as in (6.19). If each θ_j satisfies (10.4) and (10.5) with $d = 1$, $N = 0$ and $0 < \beta_j \le 1$, then

$$|K_{n_j}^{\theta_j}(x)| \le \frac{C}{n_j^{\beta_j}|x|^{\beta_j+1}} \qquad (x \ne 0)$$

and

$$|(K_{n_j}^{\theta_j})'(x)| \le \frac{C}{n_j^{\beta_j-1}|x|^{\beta_j+1}} \qquad (x \ne 0).$$

Under these conditions, one can verify that the generalizations of the results in the restricted sense of Section 14 hold with

$$\max\{d/(d+1), 1/(\beta_j+1)\} < p < \infty$$

(see Weisz [97, 98]). As we have seen in Section 14, the Riesz summation in Example 10.2 satisfies all conditions just mentioned with $\beta_j = \alpha \wedge 1$.

Lemma 17.15 *Let $\theta \in W(C,\ell_1)(\mathbb{R})$, $\mathcal{I}\theta \in W(C,\ell_1)(\mathbb{R})$ and θ be even and twice differentiable on the interval $(0,c)$, where $[-c,c]$ is the support of θ $(0 < c \le \infty)$. Suppose that*

$$\lim_{x\to c-0} x\theta(x) = 0, \quad \lim_{x\to+0} \theta' \in \mathbb{R}, \quad \lim_{x\to c-0} \theta' \in \mathbb{R} \quad \text{and} \quad \lim_{x\to\infty} x\theta'(x) = 0.$$

If θ' and $(\mathcal{I} \vee 1)\theta''$ are integrable, then

$$|\widehat{\theta}(x)| \le \frac{C}{x^2}, \qquad |\widehat{\theta'}(x)| \le \frac{C}{x^2} \qquad (x \ne 0).$$

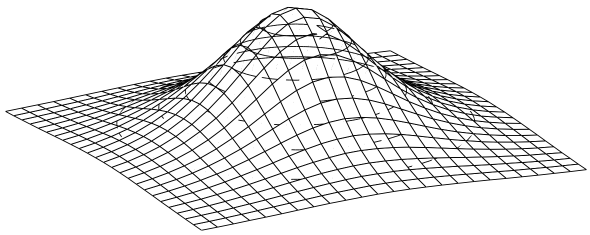

Figure 31: Picard-Bessel summability function with $d = 2$, $\alpha = 2$, $\gamma = 2$.

Proof. By integrating by parts, we have

$$\begin{aligned}\widehat{\theta}(x) &= \frac{2}{2\pi}\int_0^c \theta(t)\cos tx\,\mathrm{d}t \\ &= \frac{1}{\pi x}\int_0^c \theta'(t)\sin tx\,\mathrm{d}t \\ &= \frac{-1}{\pi x^2}[\theta'(t)\cos tx]_0^c + \frac{1}{\pi x^2}\int_0^c \theta''(t)\cos tx\,\mathrm{d}t.\end{aligned}$$

Similarly,

$$\begin{aligned}(\widehat{\theta})'(x) &= \frac{2}{2\pi}\int_0^c t\theta(t)\cos tx\,\mathrm{d}t \\ &= \frac{1}{\pi x}\int_0^c (t\theta(t))'\sin tx\,\mathrm{d}t \\ &= \frac{-1}{\pi x^2}[(t\theta(t))'\cos tx]_0^c + \frac{1}{\pi x^2}\int_0^c (t\theta(t))''\cos tx\,\mathrm{d}t,\end{aligned}$$

which proves the lemma. ∎

Examples 10.6–10.9 all satisfy Lemma 17.15, thus $\beta_j = 1$. In Example 10.9, let $\alpha\gamma > 2$. One can easily see that the same holds for Examples 10.3–10.5.

For the unrestricted convergence, we can allow more general conditions for θ_i. If each θ_i satisfies (10.1) and (10.2) or (10.4) and (10.5) with $d = 1$, then the generalizations of Theorems 16.3, 16.6 and Corollaries 16.4, 16.5 and 16.7 hold with

$$\max\{1/(\alpha_i + 1)\} < p < \infty$$

(see Weisz [95, 94]). All examples in Section 10 satisfy these conditions.

17.3 Hardy-Littlewood maximal functions

Let $\mathbb{X}$ denote either $\mathbb{T}$ or $\mathbb{R}$. The **Hardy-Littlewood maximal function** is defined by

$$Mf(x) := \sup_{x \in B} \frac{1}{|B|} \int_B |f| \, \mathrm{d}\lambda \qquad (x \in \mathbb{X}^d),$$

where the supremum is taken over all Euclidean balls $B = B_2(c,h)$ containing x, and $f \in L_p(\mathbb{X}^d)$ $(1 \le p \le \infty)$. We denote by $B_r(c,h)$ $(c \in \mathbb{R}^d, h > 0)$ the ball

$$B_r(c,h) := \{x \in \mathbb{R}^d : \|x-c\|_r < h\} \qquad (1 \le r \le \infty).$$

For $r = 2$, we omit the index and write simply $B = B_2$. We can also define the centered version of the maximal function:

$$\tilde{M}f(x) := \sup_{h>0} \frac{1}{|B(x,h)|} \int_{B(x,h)} |f| \, \mathrm{d}\lambda \qquad (x \in \mathbb{X}^d).$$

Of course, $\tilde{M}f \le Mf$. On the other hand, if $x \in B(y,h)$, then $B(y,h) \subset B(x,2h)$ and so $Mf \le 2^d \tilde{M}f$. For a ball $B(x,h)$, let $2B(x,h) := B(x,2h)$. We need the following covering lemma.

Lemma 17.16 (Vitali covering lemma) *Let E be a measurable subset of $\mathbb{X}^d$ that is the union of a finite collection of Euclidean balls $\{B_j\}$. Then we can choose a disjoint subcollection $B_1, \ldots, B_m$ such that*

$$\sum_{k=1}^m |B_k| \ge 2^{-d}|E|.$$

Proof. Let B_1 be a ball of the collection $\{B_j\}$ with maximal radius. Next choose B_2 to have maximal radius among the subcollection of balls disjoint from B_1. We continue this process until we can go no further. Then the balls $B_1, \ldots, B_m$ are disjoint. Observe that $2B_k$ contains all balls of the original collection that intersect B_k $(k = 1, \ldots, m)$. From this, it follows that $\cup_{k=1}^m 2B_k$ contains all balls from the original collection. Thus

$$|E| \le \left| \bigcup_{k=1}^m 2B_k \right| \le \sum_{k=1}^m |2B_k| \le 2^d \sum_{k=1}^m |B_k|,$$

which shows the lemma. ∎

Theorem 17.17 *The maximal operator M is of weak type (1,1), i.e.,*

$$\sup_{\rho>0} \rho\lambda(Mf > \rho) \le C\|f\|_1 \qquad (f \in L_1(\mathbb{X}^d)). \tag{17.7}$$

Moreover, if $1 < p \le \infty$, then

$$\|Mf\|_p \le C_p\|f\|_p \qquad (f \in L_p(\mathbb{X}^d)). \tag{17.8}$$

Proof. Let $E_\rho := \{Mf > \rho\}$ and $E \subset E_\rho$ be a compact subset. For each $x \in E$, there exists a ball B_x such that $x \in B_x$ and

$$|B_x| \leq \frac{1}{\rho} \int_{B_x} |f| \, \mathrm{d}\lambda. \tag{17.9}$$

Since $x \in B_x$ and E is compact, we can select a finite collection of these balls covering E. By Lemma 17.16, we can choose a finite disjoint subcollection $B_1, \ldots, B_m$ of this covering with

$$|E| \leq 2^d \sum_{k=1}^{m} |B_k|.$$

Since each ball B_k satisfies (17.9), adding these inequalities, we obtain

$$|E| \leq \frac{C}{\rho} \int_{\mathbb{X}^d} |f| \, \mathrm{d}\lambda.$$

Taking the supremum over all compact $E \subset E_\rho$, we get (17.7). Since M is evidently bounded on $L_\infty(\mathbb{X}^d)$, we get from (17.7) and from interpolation that (17.8) holds. ∎

Note that the inequality $\|f\|_p \leq \|Mf\|_p$ $(1 < p \leq \infty)$ is trivial. If we use in the definition of the Hardy-Littlewood maximal function the r-norm and the balls $B_r(c,h)$, then we get an equivalent maximal function. In the special case when $r = \infty$, we have

$$M_c f(x) := \sup_{x \in I} \frac{1}{|I|} \int_I |f| \, \mathrm{d}\lambda \qquad (x \in \mathbb{X}^d),$$

where the supremum is taken over all cubes with sides parallel to the axes. Of course,

$$C_1 M_c f \leq Mf \leq C_2 M_c f.$$

Let

$$M_\square f(x) := \sup_{\substack{x \in I, \tau^{-1} \leq |I_i|/|I_j| \leq \tau \\ i,j=1,\ldots,d}} \frac{1}{|I|} \int_I |f| \, \mathrm{d}\lambda \qquad (x \in \mathbb{X}^d)$$

for some $\tau \geq 1$, where the supremum is taken over all appropriate rectangles

$$I = I_1 \times \cdots \times I_d$$

with sides parallel to the axes. Again, it is easy to see that

$$C_1 M_\square f \leq Mf \leq C_2 M_\square f.$$

From this follows

Corollary 17.18 *We have*

$$\sup_{\rho > 0} \rho \lambda(M_\square f > \rho) \leq C \|f\|_1 \qquad (f \in L_1(\mathbb{X}^d))$$

and, for $1 < p \leq \infty$,

$$\|M_\square f\|_p \leq C_p \|f\|_p \qquad (f \in L_p(\mathbb{X}^d)).$$

Corollary 17.19 *If $f \in L_1(\mathbb{X}^d)$ and $\tau \geq 1$, then*

$$\lim_{\substack{x \in I, |I_j| \to 0 \\ \tau^{-1} \leq |I_i|/|I_j| \leq \tau, i,j=1,\ldots,d}} \frac{1}{|I|} \int_I f \, \mathrm{d}\lambda = f(x)$$

for a.e. $x \in \mathbb{X}^d$.

Proof. The result is clear for continuous functions. Since the continuous functions are dense in $L_1(\mathbb{X}^d)$, the corollary follows from the density Theorem 3.2. ∎

Let us consider the **strong maximal function**

$$M_s f(x) := \sup_{x \in I} \frac{1}{|I|} \int_I |f| \, \mathrm{d}\lambda \qquad (x \in \mathbb{X}^d),$$

where $f \in L_p(\mathbb{X}^d)$ $(1 \leq p \leq \infty)$ and the supremum is taken over all rectangles with sides parallel to the axes.

Theorem 17.20 *If $f \in L(\log L)^{d-1}(\mathbb{X}^d)$ and $C_0 > 0$, then*

$$\sup_{\rho > 0} \rho \lambda(x : M_s f(x) > \rho, \|x\|_\infty \leq C_0) \leq C + C \| \, |f| (\log^+ |f|)^{d-1} \|_1.$$

Moreover, for $1 < p \leq \infty$, we have

$$\|M_s f\|_p \leq C_p \|f\|_p \qquad (f \in L_p(\mathbb{X}^d)).$$

To the proof, we need the following lemma (see e.g. Weisz [94, p. 12]).

Lemma 17.21 *If a sublinear operator T is bounded on $L_\infty(\mathbb{T})$ and of weak type $(1,1)$, then for every $k = 1, 2, \ldots,$*

$$\| \, |Tf| (\log^+ |Tf|)^{k-1} \|_1 \leq C + C \| \, |f| (\log^+ |f|)^k \|_1 \qquad (f \in L(\log L)^k).$$

Proof of Theorem 17.20. Let us denote the one-dimensional Hardy-Littlewood maximal function with respect to the jth coordinate by $M^{(j)}$. By Theorem 17.17,

$$\begin{aligned}
&\sup_{\rho > 0} \rho \lambda(x : M_s f(x) > \rho, \|x\|_\infty \leq C_0) \\
&= \sup_{\rho > 0} \rho \lambda(x : M^{(1)} \circ M^{(2)} \circ \cdots \circ M^{(d)} f(x) > \rho, \|x\|_\infty \leq C_0) \\
&\leq \|M^{(2)} \circ \cdots \circ M^{(d)} f \, 1_{B_\infty(0,C_0)}\|_1 \\
&\leq C + C \| \, |M^{(3)} \circ \cdots \circ M^{(d)} f| (\log^+ |M^{(3)} \circ \cdots \circ M^{(d)} f|) \, 1_{B_\infty(0,C_0)} \|_1 \\
&\leq \cdots \leq C + C \| \, |f| (\log^+ |f|)^{d-1} \, 1_{B_\infty(0,C_0)} \|_1.
\end{aligned}$$

The second inequality of Theorem 17.20 follows similarly. ∎

Note that the condition $\|x\|_\infty \le C_0$ in Theorem 17.20 is important, because the measure space in Lemma 17.21 has finite measure. The operators $M_\square$ and M_s are not bounded from $L_1(\mathbb{X}^d)$ to $L_1(\mathbb{X}^d)$.

Similarly to Corollary 17.19, we obtain

Corollary 17.22 *If $f \in L(\log L)^{d-1}(\mathbb{X}^d)$, then*

$$\lim_{\substack{x\in I, |I_j|\to 0\\ j=1,\ldots,d}} \frac{1}{|I|}\int_I f \,\mathrm{d}\lambda = f(x)$$

for a.e. $x \in \mathbb{X}^d$.

17.4 Restricted convergence at Lebesgue points

Under some conditions on θ, we can characterize the set of almost everywhere convergence. The well known theorem of Lebesgue [58] says that, for the one-dimensional Fejér means and for all $f \in L_1(\mathbb{T})$,

$$\lim_{n\to\infty} \sigma_n f(x) = f(x) \tag{17.10}$$

at each Lebesgue point of f. In this subsection, we generalize this result to higher dimensions. Here, we investigate the almost everywhere convergence for other summability functions than in the preceding subsection. We do not suppose that $\theta = \theta_1 \otimes \cdots \otimes \theta_d$.

First of all, we introduce the Herz spaces. We say that a function belongs to the **homogeneous Herz space** $E_q(\mathbb{R}^d)$ $(1 \le q \le \infty)$ if

$$\|f\|_{E_q} := \sum_{k=-\infty}^{\infty} 2^{kd(1-1/q)} \|f 1_{P_k}\|_q < \infty, \tag{17.11}$$

where

$$P_k := \{x \in \mathbb{R}^d : 2^{k-1}\pi \le \|x\|_\infty < 2^k\pi\} = B_\infty(0, 2^k\pi) \setminus B_\infty(0, 2^{k-1}\pi) \qquad (k \in \mathbb{Z}).$$

It is easy to see that, using other norms of $\mathbb{R}^d$ in the definition of P_k, like

$$P_k^r := \{x \in \mathbb{R}^d : 2^{k-1}\pi \le \|x\|_r < 2^k\pi\} = B_r(0, 2^k\pi) \setminus B_r(0, 2^{k-1}\pi) \qquad (k \in \mathbb{Z}),$$

we obtain the same spaces $E_q(\mathbb{R}^d)$ with equivalent norms for all $1 \le r \le \infty$. If we modify the definition of P_k^r,

$$P_k^r = \{x \in \mathbb{R}^d : 2^{k-1}\pi \le \|x\|_r < 2^k\pi\} \cap \mathbb{T}^d \qquad (k \in \mathbb{Z}),$$

then we get the definition of the space $E_q(\mathbb{T}^d)$. This means that for $r = \infty$, we have to take the sum in (17.11) only for $k \le 0$. These spaces are special cases of the Herz spaces [49] (see also Garcia-Cuerva and Herrero [40]). It is easy to see that

$$L_1(\mathbb{X}^d) = E_1(\mathbb{X}^d) \hookleftarrow E_q(\mathbb{X}^d) \hookleftarrow E_{q'}(\mathbb{X}^d) \hookleftarrow E_\infty(\mathbb{X}^d) \qquad (1 < q < q' < \infty),$$

where $\mathbb{X}$ denotes either $\mathbb{R}$ or $\mathbb{T}$. Moreover,

$$E_\infty(\mathbb{T}^d) \hookleftarrow L_\infty(\mathbb{T}^d). \tag{17.12}$$

It is known in the one-dimensional case (see e.g. Torchinsky [84]) that if there exists an even function η such that η is non-increasing on $\mathbb{R}_+$, $|\widehat{\theta}| \leq \eta$, $\eta \in L_1(\mathbb{R})$, then σ_*^θ is of weak type $(1,1)$. Under similar conditions, we will generalize this result to the multi-dimensional setting. First, we introduce an equivalent condition (see Feichtinger and Weisz [37]).

Theorem 17.23 *For a measurable function f, let the non-increasing majorant be defined by*

$$\eta(x) := \sup_{\|t\|_r \geq \|x\|_r} |f(t)|$$

for some $1 \leq r \leq \infty$. Then $f \in E_\infty(\mathbb{R}^d)$ if and only if $\eta \in L_1(\mathbb{R}^d)$ and

$$C^{-1}\|\eta\|_1 \leq \|f\|_{E_\infty} \leq C\|\eta\|_1.$$

Proof. If $\eta \in L_1(\mathbb{R}^d)$, then

$$\|f\|_{E_\infty} \leq \|\eta\|_{E_\infty} = \sum_{k=-\infty}^{\infty} 2^{kd}\|\eta 1_{P_k}\|_\infty = \sum_{k=-\infty}^{\infty} 2^{kd}\eta(2^{k-1}\pi) \leq C\|\eta\|_1.$$

For the converse, denote by

$$a_k := \sup_{B_r(0,2^k\pi)\setminus B_r(0,2^{k-1}\pi)} |f| \quad \text{and} \quad \nu' := \sum_{k=-\infty}^{\infty} a_k 1_{B_r(0,2^k\pi)\setminus B_r(0,2^{k-1}\pi)}.$$

Let

$$\nu(x) := \sup_{\|t\|_r \geq \|x\|_r} \nu'(t) \qquad (x \in \mathbb{R}^d).$$

Since $f \in E_\infty(\mathbb{R}^d)$ implies $\lim_{k\to\infty} a_k = 0$, we conclude that there exists an increasing sequence $(n_k)_{k\in\mathbb{Z}}$ of integers such that $(a_{n_k})_{k\in\mathbb{Z}}$ is decreasing and ν can be written in the form

$$\nu = \sum_{k=-\infty}^{\infty} a_{n_k} 1_{B_r(0,2^{n_k}\pi)\setminus B_r(0,2^{n_{k-1}}\pi)}.$$

Thus

$$\|\eta\|_1 \leq \|\nu\|_1 = \sum_{k=-\infty}^{\infty} a_{n_k} \int_{B_r(0,2^{n_k}\pi)\setminus B_r(0,2^{n_{k-1}}\pi)} \mathrm{d}\lambda = C \sum_{k=-\infty}^{\infty} \left(2^{dn_k} - 2^{dn_{k-1}}\right) a_{n_k}.$$

By Abel rearrangement,

$$\|\eta\|_1 \leq C \sum_{k=-\infty}^{\infty} 2^{dn_{k-1}}(a_{n_{k-1}} - a_{n_k}) \leq C\|f\|_{E_\infty},$$

which proves the theorem. ∎

Obviously, the result holds for the space $E_\infty(\mathbb{T}^d)$ as well.

Theorem 17.24 *If $\theta \in W(C,\ell_1)(\mathbb{R}^d)$ and*

$$\sup_{n\in\mathbb{R}^d_\tau} \|K_n^\theta\|_{E_\infty(\mathbb{T}^d)} \le C, \tag{17.13}$$

then

$$\sigma_\square^\theta f \le C\Big(\sup_{n\in\mathbb{R}^d_\tau} \|K_n^\theta\|_{E_\infty(\mathbb{T}^d)}\Big) M_\square f \qquad \text{a.e.} \tag{17.14}$$

for all $f \in L_1(\mathbb{T}^d)$.

Proof. By (17.1),

$$|\sigma_n^\theta f(x)| = \frac{1}{(2\pi)^d}\Big|\int_{\mathbb{T}^d} f(x-t)K_n^\theta(t)\,\mathrm{d}t\Big| \le \frac{1}{(2\pi)^d}\sum_{k=-\infty}^{0}\int_{P_k} |f(x-t)||K_n^\theta(t)|\,\mathrm{d}t.$$

Then

$$|\sigma_n^\theta f(x)| \le \frac{1}{(2\pi)^d}\sum_{k=-\infty}^{0}\sup_{P_k}|K_n^\theta|\int_{P_k}|f(x-t)|\,\mathrm{d}t.$$

It is easy to see that if

$$G(u) := \int_{|t_j|<u_j, j=1,\ldots,d} |f(x-t)|\,\mathrm{d}t \qquad (u\in\mathbb{R}^d_+),$$

then

$$\frac{G(u)}{\prod_{j=1}^d u_j} \le CM_\square f(x) \qquad (u\in\mathbb{R}^d_\tau).$$

Therefore

$$\begin{aligned} |\sigma_n^\theta f(x)| &\le C\sum_{k=-\infty}^{0}\sup_{P_k}|K_n^\theta|\,G(2^k\pi,\ldots,2^k\pi) \\ &\le C\sum_{k=-\infty}^{0}2^{kd}\sup_{P_k}|K_n^\theta|\,M_\square f(x) \\ &= C\|K_n^\theta\|_{E_\infty(\mathbb{T}^d)}M_\square f(x), \end{aligned}$$

which shows the theorem. ■

Note that $\theta \in W(C,\ell_1)(\mathbb{R}^d)$ implies $K_n^\theta \in L_\infty(\mathbb{T}^d) \subset E_\infty(\mathbb{T}^d)$ for all $n\in\mathbb{N}^d$, because of (17.2). Corollary 17.18 implies immediately

Theorem 17.25 *If $\theta \in W(C,\ell_1)(\mathbb{R}^d)$ and*

$$\sup_{n\in\mathbb{R}^d_\tau} \|K_n^\theta\|_{E_\infty(\mathbb{T}^d)} \leq C$$

holds, then for every $1 < p \leq \infty$

$$\|\sigma_\square^\theta f\|_p \leq C_p\Big(\sup_{n\in\mathbb{R}^d_\tau} \|K_n^\theta\|_{E_\infty(\mathbb{T}^d)}\Big)\|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

Moreover,

$$\sup_{\rho>0} \rho\lambda(\sigma_\square^\theta f > \rho) \leq C\Big(\sup_{n\in\mathbb{R}^d_\tau} \|K_n^\theta\|_{E_\infty(\mathbb{T}^d)}\Big)\|f\|_1 \qquad (f \in L_1(\mathbb{T}^d)).$$

Corollary 17.26 *If $\theta \in W(C,\ell_1)(\mathbb{R}^d)$, $\theta(0) = 1$ and*

$$\sup_{n\in\mathbb{R}^d_\tau} \|K_n^\theta\|_{E_\infty(\mathbb{T}^d)} \leq C$$

holds, then

$$\lim_{n\to\infty,\, n\in\mathbb{R}^d_\tau} \sigma_n^\theta f = f \qquad \text{a.e.}$$

for all $f \in L_1(\mathbb{T}^d)$.

In the next theorem, we suppose a little bit more than in Theorem 17.3, namely instead of $\widehat{\theta} \in L_1(\mathbb{R}^d)$, we suppose that $\widehat{\theta} \in E_\infty(\mathbb{R}^d)$.

Theorem 17.27 *If $\theta \in W(C,\ell_1)(\mathbb{R}^d)$ and $\widehat{\theta} \in E_\infty(\mathbb{R}^d)$, then*

$$\sigma_\square^\theta f \leq C\|\widehat{\theta}\|_{E_\infty(\mathbb{R}^d)} M_\square f \qquad \text{a.e.}$$

for all $f \in L_1(\mathbb{T}^d)$.

Proof. Since by Theorem 17.2

$$\sigma_n^\theta f(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f(x-t)K_n^\theta(t)\,\mathrm{d}t = \Big(\prod_{j=1}^d n_j\Big)\int_{\mathbb{R}^d} f(x-t)\widehat{\theta}(n_1t_1,\ldots,n_dt_d)\,\mathrm{d}t,$$

we get similarly to (6.19) that

$$K_n^\theta(t) = (2\pi)^d\Big(\prod_{j=1}^d n_j\Big)\sum_{j\in\mathbb{Z}^d}\widehat{\theta}(n_1(t_1+2j_1\pi),\ldots,n_d(t_d+2j_d\pi)).$$

We will prove that $\widehat{\theta} \in E_\infty(\mathbb{R}^d)$ implies

$$\|K_n^\theta\|_{E_\infty(\mathbb{T}^d)} \leq C\|\widehat{\theta}\|_{E_\infty(\mathbb{R}^d)} \quad \text{for all} \quad n \in \mathbb{R}^d_\tau.$$

Suppose that $2^{s-1} < \tau \leq 2^s$ and $2^{l-1} < n_1 \leq 2^l$. Since $n \in \mathbb{R}^d_\tau$, we have

$$2^{l-s-1} < \frac{1}{\tau} n_1 \leq n_j \leq \tau n_1 \leq 2^{l+s} \quad \text{for all} \quad j = 1, \ldots, d.$$

First, we investigate the term $j = 0$ from the sum:

$$\begin{aligned}
&\Big\| \Big(\prod_{j=1}^d n_j \Big) \widehat{\theta}(n_1 t_1, \ldots, n_d t_d) \Big\|_{E_\infty(\mathbb{T}^d)} \\
&= \sum_{k=-\infty}^{0} 2^{kd} \Big(\prod_{j=1}^d n_j \Big) \sup_{2^{k-1}\pi \leq \|t\|_\infty < 2^k \pi} |\widehat{\theta}(n_1 t_1, \ldots, n_d t_d)| \\
&\leq C \sum_{k=-\infty}^{0} 2^{(k+l)d} \sup_{2^{k-2+l-s}\pi \leq \|t\|_\infty < 2^{k+l+s}\pi} |\widehat{\theta}(t_1, \ldots, t_d)| \\
&\leq C \sum_{i=-\infty}^{l+s} 2^{id} \sup_{2^{i-1}\pi \leq \|t\|_\infty < 2^i \pi} |\widehat{\theta}(t_1, \ldots, t_d)| \\
&\leq C \|\widehat{\theta}\|_{E_\infty(\mathbb{R}^d)}. \qquad (17.15)
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\Big\| \Big(\prod_{j=1}^d n_j \Big) \sum_{j \in \mathbb{Z}^d, j \neq 0} \widehat{\theta}(n_1(t_1 + 2j_1\pi), \ldots, n_d(t_d + 2j_d\pi)) \Big\|_{E_\infty(\mathbb{T}^d)} \\
&= \sum_{k=-\infty}^{0} 2^{kd} \Big(\prod_{j=1}^d n_j \Big) \sup_{2^{k-1}\pi \leq \|t\|_\infty < 2^k\pi} \sum_{j \in \mathbb{Z}^d, j \neq 0} |\widehat{\theta}(n_1(t_1 + 2j_1\pi), \ldots, n_d(t_d + 2j_d\pi))| \\
&\leq C \Big(\prod_{j=1}^d n_j \Big) \sup_{\|t\|_\infty < \pi} \sum_{j \in \mathbb{Z}^d, j \neq 0} |\widehat{\theta}(n_1(t_1 + 2j_1\pi), \ldots, n_d(t_d + 2j_d\pi))| \\
&\leq C \Big(\prod_{j=1}^d n_j \Big) \sum_{j \in \mathbb{Z}^d, j \neq 0} \sup_{\|t\|_\infty < \pi} |\widehat{\theta}(n_1(t_1 + 2j_1\pi), \ldots, n_d(t_d + 2j_d\pi))|.
\end{aligned}$$

Since

$$|n_j(t_j + 2j_j\pi)| \geq \frac{1}{\tau} n_1 \pi > 2^{l-s-1}\pi \qquad (j = 1, \ldots, d),$$

we conclude

$$\begin{aligned}
&\Big\| \Big(\prod_{j=1}^d n_j \Big) \sum_{j \in \mathbb{Z}^d, j \neq 0} \widehat{\theta}(n_1(t_1 + 2j_1\pi), \ldots, n_d(t_d + 2j_d\pi)) \Big\|_{E_\infty(\mathbb{T}^d)} \\
&\leq C \Big(\prod_{j=1}^d n_j \Big) \sum_{i=(l-s)\vee 0}^{\infty} \sum_{j \in \mathbb{Z}^d, j \neq 0, n_1(\mathbb{T}+2j_1\pi) \times \cdots \times n_d(\mathbb{T}+2j_d\pi) \cap \{t \in \mathbb{R}^d : 2^i\pi \leq \|t\|_\infty < 2^{i+1}\pi\} \neq 0}
\end{aligned}$$

$$
\begin{aligned}
& \sup_{2^i\pi \le \|t\|_\infty < 2^{i+1}\pi} |\widehat{\theta}(t)| \\
\le\ & C \sum_{i=(l-s)\vee 0}^{\infty} 2^{id} \sup_{2^i\pi \le \|t\|_\infty < 2^{i+1}\pi} |\widehat{\theta}(t)| \\
\le\ & C \|\widehat{\theta}\|_{E_\infty(\mathbb{R}^d)},
\end{aligned}
$$

which yields indeed that $\|K_n^\theta\|_{E_\infty(\mathbb{T}^d)} \le C\|\widehat{\theta}\|_{E_\infty(\mathbb{R}^d)}$ for all $n \in \mathbb{R}^d_\tau$. The theorem follows from Theorem 17.24. ∎

Theorem 17.28 *If $\theta \in W(C,\ell_1)(\mathbb{R}^d)$ and $\widehat{\theta} \in E_\infty(\mathbb{R}^d)$, then for every $1 < p \le \infty$*

$$\|\sigma_\square^\theta f\|_p \le C_p \|\widehat{\theta}\|_{E_\infty(\mathbb{R}^d)} \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

Moreover,

$$\sup_{\rho>0} \rho\lambda(\sigma_\square^\theta f > \rho) \le C\|\widehat{\theta}\|_{E_\infty(\mathbb{R}^d)} \|f\|_1 \qquad (f \in L_1(\mathbb{T}^d)).$$

Corollary 17.29 *If $\theta \in W(C,\ell_1)(\mathbb{R}^d)$, $\theta(0) = 1$ and $\widehat{\theta} \in E_\infty(\mathbb{R}^d)$, then*

$$\lim_{n\to\infty,\, n\in\mathbb{R}^d_\tau} \sigma_n^\theta f = f \qquad \text{a.e.}$$

for all $f \in L_1(\mathbb{T}^d)$.

If $f \in L_1(\mathbb{T}^d)$ implies the almost everywhere convergence of Corollary 17.26, then $\sigma_\square^\theta$ is bounded from $L_1(\mathbb{T}^d)$ to $L_{1,\infty}(\mathbb{T}^d)$, as in Theorem 17.25 (see Stein [76]). The partial converse of Theorem 17.24 is given in the next result. More exactly, if $\sigma_\square^\theta f$ can be estimated pointwise by $M_\square f$, then (17.13) holds.

Theorem 17.30 *If $\theta \in W(C,\ell_1)(\mathbb{R}^d)$ and*

$$\sigma_\square^\theta f(x) \le C M_\square f(x) \tag{17.16}$$

for all $f \in L_1(\mathbb{T}^d)$ and $x \in \mathbb{T}^d$, then

$$\sup_{n\in\mathbb{R}^d_\tau} \|K_n^\theta\|_{E_\infty(\mathbb{T}^d)} \le C.$$

Proof. We define the space $D_p(\mathbb{T}^d)$ $(1 \le p < \infty)$ by the norm

$$\|f\|_{D_p(\mathbb{T}^d)} := \sup_{0<r\le\pi} \Big(\frac{1}{r^d}\int_{[-r,r]^d} |f|^p \,\mathrm{d}\lambda\Big)^{1/p}. \tag{17.17}$$

One can show that the norm

$$\|f\|_* = \sup_{k\le 0} 2^{-kd/p} \|f 1_{P_k}\|_p \tag{17.18}$$

is an equivalent norm on $D_p(\mathbb{T}^d)$. Indeed, choosing $r = 2^k\pi$ $(k \le 0)$, we conclude $\|f\|_* \le C\|f\|_{D_p}$. On the other hand, for $n \le 0$

$$2^{-nd} \int_{[-2^n\pi, 2^n\pi]^d} |f|^p \,\mathrm{d}\lambda = 2^{-nd} \sum_{k=-\infty}^{n} \int_{P_k} |f|^p \,\mathrm{d}\lambda \le 2^{-nd} \sum_{k=-\infty}^{n} 2^{kd} \|f\|_*^p \le C\|f\|_*^p,$$

or, in other words $\|f\|_{D_p(\mathbb{T}^d)} \le C\|f\|_*$. Choosing $n = 0$, we can see that $D_p(\mathbb{T}^d) \subset L_p(\mathbb{T}^d)$ and $\|f\|_p \le C\|f\|_{D_p(\mathbb{T}^d)}$. Taking the suprema in (17.17) and (17.18) for all $0 < r < \infty$ and $k \in \mathbb{Z}$, we obtain the space $D_p(\mathbb{R}^d)$.

It is easy to see by (17.18) that

$$\sup_{\|f\|_{D_1(\mathbb{T}^d)} \le 1} \Big| \int_{\mathbb{T}^d} f(-t) K_n^\theta(t) \,\mathrm{d}t \Big| = \|K_n^\theta\|_{E_\infty(\mathbb{T}^d)}. \tag{17.19}$$

There exists a function $f \in D_1(\mathbb{T}^d)$ with $\|f\|_{D_1} \le 1$ such that

$$\frac{\|K_n^\theta\|_{E_\infty(\mathbb{T}^d)}}{2} \le \Big| \int_{\mathbb{T}^d} f(-t) K_n^\theta(t) \,\mathrm{d}t \Big|.$$

Since $f \in L_1(\mathbb{R}^d)$, by (17.16),

$$|\sigma_n^\theta f(0)| = \Big| \int_{\mathbb{T}^d} f(-t) K_n^\theta(t) \,\mathrm{d}t \Big| \le C M_\square f(0) \qquad (n \in \mathbb{R}_\tau^d),$$

which implies

$$\|K_n^\theta\|_{E_\infty(\mathbb{T}^d)} \le C M_\square f(0) \le C M_c f(0) \le C\|f\|_{D_1} \le C.$$

This proves the result. ■

Note that the norm of $D_p(\mathbb{T}^d)$ in (17.17) is equivalent to

$$\|f\| = \sup_{\substack{r \in [0,\pi]^d,\, \tau^{-1} \le r_i/r_j \le \tau \\ i,j=1,\ldots,d}} \Big(\frac{1}{\prod_{j=1}^d r_j} \int_{-r_1}^{r_1} \cdots \int_{-r_d}^{r_d} |f|^p \,\mathrm{d}\lambda \Big)^{1/p}.$$

Now, we introduce the concept of Lebesgue points in higher dimensions. Corollary 17.19 says that

$$\lim_{\substack{h \to 0,\, \tau^{-1} \le h_i/h_j \le \tau \\ i,j=1,\ldots,d}} \frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} f(x+u) \,\mathrm{d}u = f(x) \tag{17.20}$$

for a.e. $x \in \mathbb{T}^d$, where $f \in L_1(\mathbb{T}^d)$. A point $x \in \mathbb{T}^d$ is called a **Lebesgue point** of $f \in L_1(\mathbb{T}^d)$ if

$$\lim_{\substack{h \to 0,\, \tau^{-1} \le h_i/h_j \le \tau \\ i,j=1,\ldots,d}} \frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} |f(x+u) - f(x)| \,\mathrm{d}u = 0.$$

One can see that this definition is equivalent to

$$\lim_{h \to 0} \frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} |f(x+u) - f(x)| \,\mathrm{d}u = 0.$$

Theorem 17.31 *Almost every point is a Lebesgue point of $f \in L_1(\mathbb{T}^d)$.*

Proof. Since f is integrable, we may suppose that $f(x)$ is finite. Let q be a rational number for which $|f(x) - q| < \epsilon$. Then

$$\begin{aligned}\frac{1}{(2h)^d}\int_{-h}^{h}\cdots\int_{-h}^{h}|f(x+u)-f(x)|\,\mathrm{d}u \quad &\leq \quad \frac{1}{(2h)^d}\int_{-h}^{h}\cdots\int_{-h}^{h}|f(x+u)-q|\,\mathrm{d}u \\ &\quad + \frac{1}{(2h)^d}\int_{-h}^{h}\cdots\int_{-h}^{h}|q-f(x)|\,\mathrm{d}u.\end{aligned}$$

The second integral is equal to $|q - f(x)| < \epsilon$. Applying (17.20) to the function $|f(\cdot) - q|$, we can see that for a.e. x, the first integral is less than ϵ if h is small enough. ■

The next theorem generalizes Lebesgue's theorem (17.10) (see Feichtinger and Weisz [37]).

Theorem 17.32 *Suppose that $\theta \in W(C, \ell_1)(\mathbb{R}^d)$, $\theta(0) = 1$ and*

$$\sup_{n\in\mathbb{R}^d_\tau} \|K_n^\theta\|_{E_\infty(\mathbb{T}^d)} \leq C.$$

If there exists $1 \leq r \leq \infty$ such that for all $\delta > 0$

$$\lim_{n\to\infty, n\in\mathbb{R}^d_\tau} \sup_{\mathbb{T}^d\setminus B_r(0,\delta)} |K_n^\theta| = 0, \tag{17.21}$$

then

$$\lim_{n\to\infty,\, n\in\mathbb{R}^d_\tau} \sigma_n^\theta f(x) = f(x)$$

for all Lebesgue points of $f \in L_1(\mathbb{T}^d)$.

Proof. Now, set

$$G(u) := \int_{|t_j|<u_j, j=1,\ldots,d} |f(x-t) - f(x)|\,\mathrm{d}t \qquad (u \in \mathbb{R}_+).$$

Since x is a Lebesgue point of f, for all $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$\frac{G(u)}{\prod_{j=1}^d u_j} \leq \epsilon \qquad \text{if} \qquad 0 < u_j \leq 2^{-m}, j = 1, \ldots, d, u \in \mathbb{R}^d_\tau. \tag{17.22}$$

Note that

$$\sigma_n^\theta f(x) - f(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} (f(x-t) - f(x)) K_n^\theta(t)\,\mathrm{d}t.$$

Thus

$$\begin{aligned}
|\sigma_n^\theta f(x)-f(x)| &\leq C\int_{\mathbb{T}^d}|f(x-t)-f(x)||K_n^\theta(t)|\,\mathrm{d}t \\
&= C\int_{(-2^{-m}\pi,2^{-m}\pi)^d}|f(x-t)-f(x)||K_n^\theta(t)|\,\mathrm{d}t \\
&\quad + C\int_{\mathbb{T}^d\setminus(-2^{-m}\pi,2^{-m}\pi)^d}|f(x-t)-f(x)||K_n^\theta(t)|\,\mathrm{d}t \\
&=: A_0(x)+A_1(x).
\end{aligned}$$

We estimate $A_0(x)$ by

$$\begin{aligned}
A_0(x) &= C\sum_{k=-\infty}^{-m}\int_{P_k}|f(x-t)-f(x)||K_n^\theta(t)|\,\mathrm{d}t \\
&\leq C\sum_{k=-\infty}^{-m}\sup_{P_k}|K_n^\theta|\int_{P_k}|f(x-t)-f(x)|\,\mathrm{d}t \\
&\leq C\sum_{k=-\infty}^{-m}\sup_{P_k}|K_n^\theta|\,G(2^k\pi,\ldots,2^k\pi).
\end{aligned}$$

Then, by (17.22),

$$A_0(x)\leq C\epsilon\sum_{k=-\infty}^{-m}2^{kd}\sup_{P_k}|K_n^\theta|\leq C\epsilon\|K_n^\theta\|_{E_\infty(\mathbb{T}^d)}.$$

Let $B_r(0,\delta)$ be the largest ball for which $B_r(0,\delta)\subset(-2^{-m}\pi,2^{-m}\pi)^d$. Then

$$\begin{aligned}
A_1(x) &\leq C\int_{\mathbb{T}^d\setminus B_r(0,\delta)}|f(x-t)-f(x)||K_n^\theta(t)|\,\mathrm{d}t \\
&\leq C\sup_{\mathbb{T}^d\setminus B_r(0,\delta)}|K_n^\theta|(\|f\|_1+|f(x)|),
\end{aligned}$$

which tends to 0 as $n\to\infty, n\in\mathbb{R}_\tau^d$. ■

Observe that (17.12) and $B_r(0,\delta')\subset(-2^k\pi,2^k\pi)^d\subset B_r(0,\delta)$ imply

$$\|K_n^\theta\|_{E_\infty(\mathbb{T}^d\setminus B_r(0,\delta))}\leq\|K_n^\theta\|_{L_\infty(\mathbb{T}^d\setminus(-2^k\pi,2^k\pi)^d)}\leq C_\delta\|K_n^\theta\|_{E_\infty(\mathbb{T}^d\setminus B_r(0,\delta'))}. \tag{17.23}$$

In other words, condition (17.21) is equivalent to

$$\lim_{n\to\infty,n\in\mathbb{R}_\tau^d}\|K_n^\theta\|_{E_\infty(\mathbb{T}^d\setminus B_r(0,\delta))}=0.$$

In the case $\widehat{\theta}\in E_\infty(\mathbb{R}^d)$, we can formulate a somewhat simpler version of the preceding theorem.

Theorem 17.33 *Suppose that $\theta \in W(C,\ell_1)(\mathbb{R}^d)$, $\theta(0)=1$ and $\widehat{\theta} \in E_\infty(\mathbb{R}^d)$. Then*

$$\lim_{n\to\infty,\, n\in\mathbb{R}^d_\tau} \sigma_n^\theta f(x) = f(x)$$

for all Lebesgue points of $f \in L_1(\mathbb{T}^d)$.

Proof. We have seen in Theorem 17.27 that $\widehat{\theta} \in E_\infty(\mathbb{R}^d)$ implies

$$\|K_n^\theta\|_{E_\infty(\mathbb{T}^d)} \leq C\|\widehat{\theta}\|_{E_\infty(\mathbb{R}^d)} \quad \text{for all} \quad n \in \mathbb{R}^d_\tau,$$

so the first condition of Theorem 17.32 is satisfied. On the other hand, let $B_\infty(0,2^{k_0})$ be the largest ball for which $B_\infty(0,2^{k_0}) \subset B_r(0,\delta)$, let $2^{s-1} < \tau \leq 2^s$ and $2^{l-1} < n_1 \leq 2^l$. Obviously, if $n \to \infty$, $n \in \mathbb{R}^d_\tau$, then $l \to \infty$. We get similarly to (17.15) that

$$\|K_n^\theta\|_{E_\infty(\mathbb{T}^d \setminus B_r(0,\delta))} \leq C \sum_{i=k_0+l-s-1}^{\infty} 2^{id} \sup_{2^{i-1}\pi \leq \|t\|_\infty < 2^i\pi} |\widehat{\theta}(t_1,\ldots,t_d)|,$$

which tends to 0 as $n \to \infty$, $n \in \mathbb{R}^d_\tau$, since $\widehat{\theta} \in E_\infty(\mathbb{R}^d)$. Then (17.21) follows by (17.23). ∎

Since each point of continuity is a Lebesgue point, we have

Corollary 17.34 *If the conditions of Theorem 17.32 or Theorem 17.33 are satisfied and if $f \in L_1(\mathbb{T}^d)$ is continuous at a point x, then*

$$\lim_{n\to\infty,\, n\in\mathbb{R}^d_\tau} \sigma_n^\theta f(x) = f(x).$$

The converse of Theorem 17.32 holds also.

Theorem 17.35 *Suppose that $\theta \in W(C,\ell_1)(\mathbb{R}^d)$, $\theta(0)=1$ and (17.21) holds. If*

$$\lim_{n\to\infty,\, n\in\mathbb{R}^d_\tau} \sigma_n^\theta f(x) = f(x)$$

for all Lebesgue points of $f \in L_1(\mathbb{T}^d)$, then

$$\sup_{n\in\mathbb{R}^d_\tau} \|K_n^\theta\|_{E_\infty(\mathbb{T}^d)} \leq C.$$

Proof. The space $D_1^0(\mathbb{T}^d)$ consists of all functions $f \in D_1(\mathbb{T}^d)$ for which $f(0)=0$ and 0 is a Lebesgue point of f, in other words

$$\lim_{h\to 0} \frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} |f(u)| \,\mathrm{d}u = 0.$$

We will show that $D_1^0(\mathbb{T}^d)$ is a Banach space. Let (f_n) be a Cauchy sequence in $D_1^0(\mathbb{T}^d)$, i.e., $\|f_n - f_m\|_{D_1^0(\mathbb{T}^d)} \to 0$ as $n, m \to \infty$. Then there exists a subsequence (f_{ν_n}) such that

$$\|f_{\nu_{n+1}} - f_{\nu_n}\|_{D_1^0(\mathbb{T}^d)} \leq 2^{-n}.$$

Then (f_{ν_n}) is a.e. convergent, let $f = \lim_{n\to\infty} f_{\nu_n}$ and $f(0) = 0$. For all $\epsilon > 0$, there exists N such that

$$\|f - f_{\nu_N}\|_{D_1^0(\mathbb{T}^d)} \leq \sum_{n=N}^{\infty} \|f_{\nu_{n+1}} - f_{\nu_n}\|_{D_1^0(\mathbb{T}^d)} \leq \sum_{n=N}^{\infty} 2^{-n} < \epsilon.$$

If $h > 0$ is small enough, then

$$\frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} |f_{\nu_N}(u)| \,\mathrm{d}u < \epsilon.$$

Hence

$$\frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} |f(u)| \,\mathrm{d}u \leq \|f - f_{\nu_N}\|_{D_1^0(\mathbb{T}^d)} + \frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} |f_{\nu_N}(u)| \,\mathrm{d}u < 2\epsilon,$$

whenever h is small enough. From this it follows that $f \in D_1^0(\mathbb{T}^d)$ and 0 is a Lebesgue point of f. Thus $D_1^0(\mathbb{T}^d)$ is a Banach space.

We get from the conditions of the theorem that

$$\lim_{n\to\infty,\, n\in\mathbb{R}_\tau^d} \sigma_n^\theta f(0) = 0 \qquad \text{for all} \qquad f \in D_1^0(\mathbb{T}^d).$$

Thus the operators

$$U_n : D_1^0(\mathbb{T}^d) \to \mathbb{R}, \qquad U_n f := \sigma_n^\theta f(0) \qquad (n \in \mathbb{R}_\tau^d)$$

are uniformly bounded by the Banach-Steinhaus theorem. Observe that in (17.19), we may suppose that f is 0 in a neighborhood of 0. Then

$$\begin{aligned} C &\geq \|U_n\| \\ &= \sup_{\|f\|_{D_1^0(\mathbb{T}^d)} \leq 1} \Big| \int_{\mathbb{T}^d} f(-t) K_n^\theta(t) \,\mathrm{d}t \Big| \\ &= \sup_{\|f\|_{D_1(\mathbb{T}^d)} \leq 1} \Big| \int_{\mathbb{T}^d} f(-t) K_n^\theta(t) \,\mathrm{d}t \Big| \\ &= \|K_n^\theta\|_{E_\infty(\mathbb{T}^d)} \end{aligned}$$

for all $n \in \mathbb{R}_\tau^d$. ■

Corollary 17.36 *Suppose that $\theta \in W(C, \ell_1)(\mathbb{R}^d)$, $\theta(0) = 1$ and (17.21) holds. Then*

$$\lim_{n\to\infty,\, n\in\mathbb{R}_\tau^d} \sigma_n^\theta f(x) = f(x)$$

for all Lebesgue points of $f \in L_1(\mathbb{T}^d)$ if and only if

$$\sup_{n\in\mathbb{R}_\tau^d} \|K_n^\theta\|_{E_\infty(\mathbb{T}^d)} \leq C.$$

A one-dimensional version of this theorem can be found in the book of Alexits [1]. Now, we present some sufficient condition on θ such that $\widehat{\theta} \in E_\infty(\mathbb{R}^d)$. The next theorem was proved in Herz [49], Peetre [69] and Girardi and Weis [42].

Lemma 17.37 *If $f \in B_{1,1}^d(\mathbb{R}^d)$, then $\widehat{f} \in E_\infty(\mathbb{R}^d)$ and*

$$\|\widehat{f}\|_{E_\infty} \leq C_p \|f\|_{B_{1,1}^d}.$$

A function f belongs to the **weighted Wiener amalgam space** $W(L_\infty, \ell_1^{v_s})(\mathbb{R}^d)$ if

$$\|f\|_{W(L_\infty,\ell_1^{v_s})} := \sum_{k\in\mathbb{Z}^d} \sup_{x\in[0,1)^d} |f(x+k)| v_s(k) < \infty,$$

where $v_s(x) := (1+\|x\|_\infty)^s$ $(x \in \mathbb{R}^d)$.

Lemma 17.38 *If $f \in W(L_\infty, \ell_1^{v_d})(\mathbb{R}^d)$, then $f \in E_\infty(\mathbb{R}^d)$ and*

$$\|f\|_{E_\infty} \leq C \|f\|_{W(L_\infty,\ell_1^{v_d})}.$$

Proof. The inequalities

$$\begin{aligned}
\|f\|_{E_\infty} &= \sum_{k=-\infty}^{\infty} 2^{kd} \sup_{P_k} |f| \\
&\leq C \sum_{k=0}^{\infty} 2^{kd} \sum_{j:[-\pi,\pi]^d+2j\pi \cap P_k \neq \emptyset} \sup_{[-\pi,\pi]^d+2j\pi} |f| \\
&\leq C \sum_{j\in\mathbb{Z}^d} (1+\|j\|_\infty)^d \sup_{[-\pi,\pi]^d+2j\pi} |f| \\
&= C \|f\|_{W(L_\infty,\ell_1^{v_d})}
\end{aligned}$$

prove the result. ∎

Note that if $\theta \in W(C,\ell_1)(\mathbb{R}^d)$ and $\widehat{\theta}$ has compact support, then all the results above hold. Actually, $\theta \in \mathbf{S}_0(\mathbb{R}^d)$ in this case (see e.g. Feichtinger and Zimmerman [38]). We can generalize these results if θ is in a suitable modulation space. We define the **weighted Feichtinger's algebra** or **modulation space** $M_1^{v_s}(\mathbb{R}^d)$ (see e.g. Feichtinger [35] and Gröchenig [44]) by

$$M_1^{v_s}(\mathbb{R}^d) := \left\{ f \in L^2(\mathbb{R}^d) : \|f\|_{M_1^{v_s}} := \|S_{g_0} f \cdot v_s\|_{L_1(\mathbb{R}^{2d})} < \infty \right\} \qquad (s \geq 0),$$

where $v_s(x,\omega) := v_s(\omega) = (1+\|\omega\|_\infty)^s$ $(x,\omega \in \mathbb{R}^d)$.

Lemma 17.39 *If $f \in M_1^{v_d}(\mathbb{R}^d)$, then $\widehat{f} \in E_\infty(\mathbb{R}^d)$ and*

$$\|\widehat{f}\|_{E_\infty} \leq C \|f\|_{M_1^{v_d}}.$$

Proof. By Lemma 17.38,

$$\|\widehat{f}\|_{E_\infty} \leq C\|\widehat{f}\|_{W(L_\infty,\ell_1^{v_d})} \leq C\|f\|_{M_1^{v_d}},$$

where the second inequality can be found in Gröchenig [44, p. 249]. ∎

Corollary 17.40 *Suppose that $\theta \in W(C,\ell_1)(\mathbb{R}^d)$ and $\theta(0) = 1$. If $\theta \in M_1^{v_d}(\mathbb{R}^d)$, then*

$$\lim_{n\to\infty,\, n\in\mathbb{R}_\tau^d} \sigma_n^\theta f(x) = f(x)$$

for all Lebesgue points of $f \in L_1(\mathbb{R}^d)$. Moreover,

$$\sup_{\rho>0} \rho\, \lambda(\sigma_\square^\theta > \rho) \leq C\|\theta\|_{M_1^{v_d}}\|f\|_1 \qquad (f \in L_1(\mathbb{R}^d))$$

and, for every $1 < p \leq \infty$,

$$\|\sigma_\square^\theta f\|_p \leq C_p\|\theta\|_{M_1^{v_d}}\|f\|_p \qquad (f \in L_p(\mathbb{R}^d)).$$

Theorem 17.41 *If $\theta \in V_1^k(\mathbb{R})$ for some $k \geq 2$, then $\theta \in M_1^{v_s}(\mathbb{R})$ for all $0 \leq s < k-1$ and*

$$\|\theta\|_{M_1^{v_s}} \leq C_s\|\theta\|_{V_1^k}.$$

This theorem can be proved as was Theorem 17.9. Note that $V_1^k(\mathbb{R}^d)$ was defined in Definition 17.8.

Corollary 17.42 *If each $\theta_j \in V_1^k(\mathbb{R})$ $(j = 1,\ldots,d)$, then*

$$\theta := \prod_{j=1}^d \theta_j \in M_1^{v_s}(\mathbb{R}^d) \qquad (0 \leq s < k-1).$$

The space $V_1^2(\mathbb{R})$ is not contained in $M_1^{v_1}(\mathbb{R})$. However, the same results hold as in Corollary 17.40.

Corollary 17.43 *If $\theta \in V_1^2(\mathbb{R})$, then $\widehat{\theta} \in E_\infty(\mathbb{R})$.*

Proof. The inequality

$$|\widehat{\theta}(x)| \leq C/x^2 \qquad (x \neq 0)$$

can be shown similarly to Theorem 17.9. $\widehat{\theta} \in E_\infty(\mathbb{R})$ follows from Theorem 17.23. ∎

All examples of Subsection 10.1, respectively Subsection 17.1, satisfy the condition $\widehat{\theta} \in E_\infty(\mathbb{R})$, respectively $\widehat{\theta} \in E_\infty(\mathbb{R}^d)$.

17.5 Unrestricted convergence at Lebesgue points

To formulate the unrestricted version of the preceding theorems, we have to modify slightly the definition of the space $E_q(\mathbb{R}^d)$. The **homogeneous Herz space** $E'_q(\mathbb{R}^d)$ contains all functions f for which

$$\|f\|_{E'_q} := \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} \Big(\prod_{j=1}^{d} 2^{k_j(1-1/q)}\Big) \|f 1_{P'_k}\|_q,$$

where

$$P'_k := \{x \in \mathbb{R}^d : 2^{k_j-1} \le |x_j| < 2^{k_j}, j = 1, \ldots, d\} \qquad (k \in \mathbb{Z}^d).$$

The spaces $E'_q(\mathbb{T}^d)$ can be defined analogously. Again,

$$L_1(\mathbb{X}^d) = E'_1(\mathbb{X}^d) \hookleftarrow E'_q(\mathbb{X}^d) \hookleftarrow E'_{q'}(\mathbb{X}^d) \hookleftarrow E'_\infty(\mathbb{X}^d), \qquad 1 < q < q' < \infty,$$

where $\mathbb{X} = \mathbb{R}$ or $\mathbb{T}$ and

$$E'_\infty(\mathbb{T}^d) \hookleftarrow L_\infty(\mathbb{T}^d).$$

It is easy to see that $E'_q(\mathbb{X}^d) \supset E_q(\mathbb{X}^d)$ and

$$\|f\|_{E'_q} \le C\|f\|_{E_q} \qquad (1 \le q \le \infty).$$

Except for converse type results, all theorems of the preceding subsection can also be proved for the unrestricted convergence (see Feichtinger and Weisz [37]). We point out some of these theorems.

Theorem 17.44 *If $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ and*

$$\sup_{n\in\mathbb{N}^d} \|K_n^\theta\|_{E'_\infty(\mathbb{T}^d)} \le C,$$

then

$$\sigma_*^\theta f \le C\Big(\sup_{n\in\mathbb{N}^d} \|K_n^\theta\|_{E'_\infty(\mathbb{T}^d)}\Big) M_s f \qquad \text{a.e.}$$

for all $f \in L_1(\mathbb{T}^d)$.

Theorem 17.45 *If $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ and $\widehat{\theta} \in E'_\infty(\mathbb{R}^d)$, then*

$$\sigma_*^\theta f \le C\|\widehat{\theta}\|_{E'_\infty(\mathbb{R}^d)} M_s f \qquad \text{a.e.}$$

for all $f \in L_1(\mathbb{T}^d)$.

Theorem 17.46 *If $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ and $\widehat{\theta} \in E'_\infty(\mathbb{R}^d)$, then for every $1 < p \le \infty$,*

$$\|\sigma_*^\theta f\|_p \le C_p \|\widehat{\theta}\|_{E'_\infty(\mathbb{R}^d)} \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

Moreover,

$$\sup_{\rho>0} \rho\lambda(\sigma_*^\theta f > \rho) \le C\|\widehat{\theta}\|_{E'_\infty(\mathbb{R}^d)} (1 + \|f\|_{L(\log L)^{d-1}}) \qquad (f \in L(\log L)^{d-1}(\mathbb{T}^d)).$$

Corollary 17.47 *If $\theta \in W(C,\ell_1)(\mathbb{R}^d)$, $\theta(0)=1$ and $\widehat{\theta} \in E'_\infty(\mathbb{R}^d)$, then*

$$\lim_{n\to\infty} \sigma_n^\theta f = f \qquad \text{a.e.}$$

for all $f \in L(\log L)^{d-1}(\mathbb{T}^d)$.

Now, we also modify the definition of Lebesgue points. By Corollary 17.22,

$$\lim_{h\to 0} \frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} f(x+u)\,\mathrm{d}u = f(x)$$

for a.e. $x \in \mathbb{T}^d$, where $f \in L(\log L)^{d-1}(\mathbb{T}^d)$. A point $x \in \mathbb{T}^d$ is called a **strong Lebesgue point** of f if $M_s f(x)$ is *finite* and

$$\lim_{h\to 0} \frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} |f(x+u)-f(x)|\,\mathrm{d}u = 0.$$

Similarly to Theorem 17.31, one can show that almost every point $x \in \mathbb{T}^d$ is a strong Lebesgue point of $f \in L(\log L)^{d-1}(\mathbb{T}^d)$.

Theorem 17.48 *Suppose that $\theta \in W(C,\ell_1)(\mathbb{R}^d)$, $\theta(0)=1$ and*

$$\sup_{n\in\mathbb{N}^d} \|K_n^\theta\|_{E'_\infty(\mathbb{T}^d)} \le C.$$

If for all $\delta > 0$

$$\lim_{n\to\infty} \|K_n^\theta\|_{E'_\infty(\mathbb{T}^d \setminus B_\infty(0,\delta))} = 0,$$

then

$$\lim_{n\to\infty} \sigma_n^\theta f(x) = f(x)$$

for all strong Lebesgue points of $f \in L(\log L)^{d-1}(\mathbb{T}^d)$.

Theorem 17.49 *Suppose that $\theta \in W(C,\ell_1)(\mathbb{R}^d)$, $\theta(0)=1$ and $\widehat{\theta} \in E'_\infty(\mathbb{R}^d)$. Then*

$$\lim_{n\to\infty} \sigma_n^\theta f(x) = f(x)$$

for all Lebesgue points of $f \in L(\log L)^{d-1}(\mathbb{T}^d)$.

Corollary 17.50 *If the conditions of Theorem 17.48 or Theorem 17.49 are satisfied and if $f \in L(\log L)^{d-1}(\mathbb{T}^d)$ is continuous at a point x, then*

$$\lim_{n\to\infty} \sigma_n^\theta f(x) = f(x).$$

If $\theta_j \in M_1^{v_1}(\mathbb{R})$ for each $j=1,\ldots,d$ (for example $\theta_j \in V_1^k(\mathbb{R})$ $(k>2)$) and $\theta = \prod_{j=1}^d \theta_j$, then $\widehat{\theta_j} \in E'_\infty(\mathbb{R})$ and so $\widehat{\theta} \in E'_\infty(\mathbb{R}^d)$. Corollary 17.43 shows that also $\theta_j \in V_1^2(\mathbb{R})$ implies $\widehat{\theta_j} \in E'_\infty(\mathbb{R})$. This verifies the next corollary.

Corollary 17.51 *If $\theta = \prod_{j=1}^d \theta_j$ and each $\theta_j \in V_1^2(\mathbb{R})$ or $\theta_j \in M_1^{v_1}(\mathbb{R})$ $(j=1,\ldots,d)$, then the theorems of this subsection hold.*

The converse to Theorems 17.44 and 17.48 do not hold in this case. However, converse type results can be found in Feichtinger and Weisz [37].

17.6 Cesàro summability

We define the **rectangular Cesàro (or (C,α))-means** of a function $f \in L_1(\mathbb{T}^d)$ by

$$\begin{aligned} \sigma_n^{(c,\alpha)} f(x) &:= \frac{1}{\prod_{i=1}^d A_{n_i-1}^\alpha} \sum_{|k_1|\le n_1} \cdots \sum_{|k_d|\le n_d} \prod_{i=1}^d A_{n_i-1-|k_i|}^\alpha \widehat{f}(k) \mathrm{e}^{\mathrm{i}k\cdot x} \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^2} f(x-u) K_n^{(c,\alpha)}(u)\,\mathrm{d}u, \end{aligned}$$

where

$$\begin{aligned} K_n^{(c,\alpha)}(u) &:= \frac{1}{\prod_{i=1}^d A_{n_i-1}^\alpha} \sum_{|k_1|\le n_1} \cdots \sum_{|k_d|\le n_d} \prod_{i=1}^d A_{n_i-1-|k_i|}^\alpha \mathrm{e}^{\mathrm{i}k\cdot x} \\ &= \frac{1}{\prod_{i=1}^d A_{n_i-1}^\alpha} \sum_{j_1=0}^{n_1-1} \cdots \sum_{j_d=0}^{n_d-1} \prod_{i=1}^d A_{n_i-1-j_i}^{\alpha-1} D_j(u). \end{aligned}$$

Thus

$$\sigma_n^{(c,\alpha)} f(x) = \frac{1}{\prod_{i=1}^d A_{n_i-1}^\alpha} \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_d=0}^{n_d-1} \prod_{i=1}^d A_{n_i-1-k_i}^{\alpha-1} s_k f(x).$$

The results of Subsection 17.2 for *Cesàro summation* can be found in Weisz [91, 92, 94].

18 θ-summability of Fourier transforms

Analogously to Sections 11 and 17, we introduce now the **Dirichlet integral** by

$$s_t f(x) := \int_{-t_1}^{t_1} \cdots \int_{-t_d}^{t_d} \widehat{f}(v) \mathrm{e}^{\mathrm{i}x\cdot v}\,\mathrm{d}v \qquad (t=(t_1,\ldots,t_d)\in\mathbb{R}_+^d).$$

For $T>0$, the **rectangular θ-means** of a function $f \in L_p(\mathbb{R}^d)$ $(1 \le p \le 2)$ are defined by

$$\sigma_T^\theta f(x) := \int_{\mathbb{R}^d} \theta\Big(\frac{-v_1}{T_1},\ldots,\frac{-v_d}{T_d}\Big) \widehat{f}(v) \mathrm{e}^{\mathrm{i}x\cdot v}\,\mathrm{d}v.$$

It is easy to see that

$$\sigma_T^\theta f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(x-u) K_T^\theta(u)\,\mathrm{d}u$$

where

$$K_T^\theta(u) = \int_{\mathbb{R}^d} \theta\Big(\frac{-v_1}{T_1},\ldots,\frac{-v_d}{T_d}\Big) \mathrm{e}^{\mathrm{i}u\cdot v}\,\mathrm{d}v = (2\pi)^d \Big(\prod_{i=1}^d T_i\Big) \widehat{\theta}(T_1u_1,\ldots,T_du_d).$$

Note that for the Fejér means (i.e., if each $\theta_i(t) = \max((1-|t|), 0)$), we obtain

$$\sigma_T^\theta f(x) = \frac{1}{\prod_{i=1}^d T_i} \int_0^{T_1} \cdots \int_0^{T_d} s_t f(x)\,\mathrm{d}t.$$

We extend the definition of the θ**-means** to tempered distributions by

$$\sigma_T^\theta f := f * K_T^\theta \qquad (T > 0).$$

Again, $\sigma_T^\theta f$ is well defined for all tempered distributions $f \in H_p(\mathbb{R}^d)$ $(0 < p \le \infty)$, for all functions $f \in L_p(\mathbb{R}^d)$ $(1 \le p \le \infty)$ and for all $f \in B$, where B is a homogeneous Banach space on $\mathbb{R}^d$.

Now, the **product Hardy space** $H_p(\mathbb{R}^d)$ is defined with the help of the one-dimensional **non-periodic Poisson kernel**

$$P_t(x) := \frac{ct}{t^2 + |x|^2} \qquad (t > 0, x \in \mathbb{R}).$$

The Hardy space $H_p(\mathbb{R}^d)$ satisfies the same properties as the periodic space $H_p(\mathbb{T}^d)$, except (15.2) (see Weisz [94]). The **conjugate distributions** are defined by

$$(\tilde{f}^{(j_1,\ldots,j_d)})^\wedge(t) := \Big(\prod_{i=1}^d (-\mathrm{i}\ \mathrm{sign}\ t_i)^{j_i}\Big)\widehat{f}(t) \qquad (j_i = 0, 1, t \in \mathbb{R}^d).$$

If f is integrable, then

$$\begin{aligned} \tilde{f}^{(j_1,\ldots,j_d)}(x) &= \text{p.v.} \int_{\mathbb{T}} \cdots \int_{\mathbb{T}} \frac{f(x_1 - t_1^{j_1}, \ldots, x_d - t_d^{j_d})}{\prod_{i=1}^d (\pi t_i)^{j_i}}\,\mathrm{d}t^{j_1} \cdots \mathrm{d}t^{j_d} \\ &= \lim_{\epsilon \to 0} \int_{\epsilon_1 < |t_1| < \pi} \cdots \int_{\epsilon_d < |t_d| < \pi} \frac{f(x_1 - t_1^{j_1}, \ldots, x_d - t_d^{j_d})}{\prod_{i=1}^d (\pi t_i)^{j_i}}\,\mathrm{d}t^{j_1} \cdots \mathrm{d}t^{j_d} \qquad \text{a.e.} \end{aligned}$$

The same results are true for the **maximal operators**

$$\sigma_\square^\theta f := \sup_{T \in \mathbb{R}_\tau^d} |\sigma_T^\theta f|, \qquad \sigma_\gamma^\theta f := \sup_{T \in \mathbb{R}_{\tau,\gamma}^d} |\sigma_T^\theta f|, \qquad \sigma_*^\theta f := \sup_{T \in \mathbb{R}^d} |\sigma_T^\theta f|$$

as in Sections 13–17. We point out some of them.

Theorem 18.1 *If $\theta \in L_1(\mathbb{R}^d)$ and $\widehat{\theta} \in E_\infty(\mathbb{R}^d)$, then*

$$\sigma_\square^\theta f \le C \|\widehat{\theta}\|_{E_\infty} M_\square f \qquad \text{a.e.}$$

for all $f \in L_1(\mathbb{R}^d)$.

Theorem 18.2 *If* $\theta \in L_1(\mathbb{R}^d)$, $\widehat{\theta} \in L_1(\mathbb{R}^d)$ *and*

$$\sigma_{\Box}^{\theta} f(x) \leq C M_{\Box} f(x)$$

for all $f \in L_1(\mathbb{R}^d)$ *and* $x \in \mathbb{R}^d$, *then* $\widehat{\theta} \in E_\infty(\mathbb{R}^d)$.

Theorem 18.3 *Suppose that* $\theta \in L_1(\mathbb{R}^d)$, $\theta(0) = 1$ *and* $\widehat{\theta} \in E_\infty(\mathbb{R}^d)$. *Then*

$$\lim_{T \to \infty,\, T \in \mathbb{R}^d_\tau} \sigma_T^\theta f(x) = f(x)$$

for all Lebesgue points of $f \in L_1(\mathbb{R}^d)$.

Theorem 18.4 *Suppose that* $\theta \in L_1(\mathbb{R}^d)$, $\theta(0) = 1$ *and* $\widehat{\theta} \in L_1(\mathbb{R}^d)$. *If*

$$\lim_{T \to \infty,\, T \in \mathbb{R}^d_\tau} \sigma_T^\theta f(x) = f(x)$$

for all Lebesgue points of $f \in L_1(\mathbb{R}^d)$, *then* $\widehat{\theta} \in E_\infty(\mathbb{R}^d)$.

Corollary 18.5 *Suppose that* $\theta \in L_1(\mathbb{R}^d)$, $\theta(0) = 1$ *and* $\widehat{\theta} \in L_1(\mathbb{R}^d)$. *Then*

$$\lim_{T \to \infty, T \in \mathbb{R}^d_\tau} \sigma_T^\theta f(x) = f(x)$$

for all Lebesgue points of $f \in L_1(\mathbb{R}^d)$ *if and only if* $\widehat{\theta} \in E_\infty(\mathbb{R}^d)$.

With the help of (17.15), these results can be proved as were the corresponding results in Subsection 17.4.

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Index of Notations

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