B-net basics

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**0. Introduction; notation** This paper lists the essential facts about the representation of polynomials in $m$ variables as **Bernstein polynomials**. While the univariate situation is well studied - see, e.g., Lorentz' classical book [L53], - the multivariate version has only attracted attention sporadically. Lorentz' book devotes just one page to the two most direct generalizations: the tensor product or coordinate degree generalization and the total degree generalization which is the topic of the present paper.

Motivation for the paper comes from computer-aided design where, through the initiative of de Casteljau and Bézier, the Bernstein polynomials of mostly one variable have become the main tool for the representation and computational use of **pp** (:= piecewise polynomial) functions. Farin's work [F79], [F80] brought popularity and understanding to the use of bivariate Bernstein polynomials, and my own understanding starts from that work. My own interest has been started and repeatedly reinforced by work with smooth pp functions in two or more variables [BH82], [BH85] in which their representation in terms of Bernstein polynomials, i.e., their **B-net**, for short, plays an essential role, since it reflects so nicely, and far better than other standard representations, the interplay between the geometry of the underlying triangular partition and the smoothness requirements.

For the sake of brevity, and since there are several people and ideas responsible, I am proposing here the term **B-form** (and correspondingly, B-net) for what would, more properly, be called the **barycentric-Bernstein-de Casteljau-Bézier-Farin- $\cdots$ -form**. I apologize to de Casteljau and Farin and $\cdots$ for the slight they might feel.

While the bivariate and trivariate situation is of most practical interest, I have chosen here to record the facts in the general $m$-dimensional context. This forces careful consideration of notation and brings out the essential mathematical aspects and surprising beauty of the B-form. Specific illustrations will usually be in the practical, 2-dimensional context.

I will adhere to the following notational conventions: I won't bother with boldface, arrows, or underlines to distinguish points in $\mathbb{R}^m$ from other objects. The $j$-th component of a point $x \in \mathbb{R}^m$ I will denote by $x(j)$ (rather than $x_j$). I will use standard multi-index notation throughout. Thus

$$x^\alpha := x(1)^{\alpha(1)} x(2)^{\alpha(2)} \cdots x(m)^{\alpha(m)}$$

with $\alpha \in \mathbf{Z}^m$ , i.e., $\alpha$ an $m$-vector with integer entries. The **normalized monomial** is so handy a function that it deserves a special symbol:

$$[x]^\alpha := x^\alpha/\alpha! = \prod_{j=1}^{m} x(j)^{\alpha(j)}/\alpha(j)!, \qquad (0.1)$$

hence $[x]^\alpha = [x(1)]^{\alpha(1)}[x(2)]^{\alpha(2)} \cdots [x(m)]^{\alpha(m)}$, with the conventions

$$[\,]^n = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n < 0. \end{cases}$$

In these terms, the **multinomial theorem** takes the very simple form

$$[x+y+\cdots+z]^\alpha = \sum_{\xi+\upsilon+\cdots+\zeta=\alpha} [x]^\xi [y]^\upsilon \cdots [z]^\zeta,$$

whose proof by induction on the **length**

$$|\alpha| := \sum_{j=1}^m \alpha(j)$$

of $\alpha$ is immediate. It is possible (and a useful exercise) to build the entire discussion of the B-form on this identity.

The normalization of the monomials used here also makes differentiation neat. With

$$D^\alpha := D_1^{\alpha(1)} D_2^{\alpha(2)} \cdots D_m^{\alpha(m)}$$

and $D_j f$ the partial derivative of $f : \mathbb{R}^m \to \mathbf{R}$ with respect to its $j$-th argument, we have

$$D^\alpha[\ ]^\beta = [\ ]^{\beta-\alpha}.$$

Hence

$$D^\alpha[0]^\beta = \delta_{\alpha\beta}, \tag{0.2}$$

showing that $\{[\ ]^\alpha : |\alpha| \le k, \alpha \in \mathbf{Z}_+^m\}$ is linearly independent. Their span

$$\Pi_k := \operatorname{span}\{[\ ]^\alpha : |\alpha| \le k, \alpha \in \mathbf{Z}_+^m\}$$

is, by definition, the collection of all polynomials of degree $\le k$. We conclude that

$$\dim \Pi_k = \#\{[\ ]^\alpha : |\alpha| \le k, \alpha \in \mathbf{Z}_+^m\} = \binom{m+k}{k}. \tag{0.3}$$

While the **power form**

$$p = \sum_{|\alpha| \le k} [\ ]^\alpha c(\alpha)$$

(with $c(\alpha) = \big(D^\alpha p\big)(0)$, by (0.2)) is the standard mathematical representation for $p \in \Pi_k$, it is not suited for work with pp functions since it provides explicit information only about the behavior of $p$ near 0. By contrast, the B-form (with respect to some $(m+1)$-set $V \subset \mathbb{R}^m$) provides explicit information about the behavior of $p$ at all the faces of the **convex hull**

$$[V]$$

of $V$. This makes it handy for the representation of smooth multivariate pp functions over a triangular partition.

**1. Linear interpolation** A set $V$ of $m+1$ points in $\mathbb{R}^m$ is said to be **in general position** in case every linear polynomial on $\mathbb{R}^m$ can be written in terms of its values on $V$, i.e.,

$$\forall p \in \Pi_1 \qquad p \;=\; \sum_{v\in V} \xi_v p(v)\ . \tag{1.1}$$

There are equivalent definitions of this term. E.g., $V$ is in general position in case its affine hull is all of $\mathbb{R}^m$, or, in case the simplex $[V]$ is proper, or if the $m+1$ vectors $(v|1), v \in V$ in $\mathbf{R}^{m+1}$ are linearly independent, etc. But I stick with the above definition since it uses the property of immediate interest here. In this way we associate with each such $V$ $m+1$ linear polynomials $\xi_v, v \in V$, characterized by the fact that (1.1) holds. In particular, with $p$ the constant polynomial, we find that

$$1 \;=\; \sum_{v\in V} \xi_v, \tag{1.2}$$

while, with $p : x \mapsto x(j)$, we get

$$x(j) \;=\; \sum_{v\in V} \xi_v(x) v(j)$$

for $j = 1, \cdots, m$, hence

$$\forall x \qquad x \;=\; \sum_{v\in V} \xi_v(x) v. \tag{1.3}$$

We conclude that the vector

$$\xi(x) \;:=\; \big(\xi_v(x)\big)_{v\in V} \tag{1.4}$$

provides the **barycentric coordinates** for $x$ with respect to $V$. Note that I have chosen here to use the points in $V$ (rather than the integers from 0 to $m$, or from 1 to $m+1$) to index the components of the vector $\xi(x)$. This perhaps unorthodox notation seems more to the point since it does not impose some artificial order on the **vertices** $v$; it also simplifies notation.

Since dim $\Pi_1 \;=\; m+1$, we conclude from (1.1) that $\big(\xi_v\big)_{v\in V}$ is a basis for $\Pi_1$. In particular, the representation (1.1) is irredundant. Therefore

$$p \;=\; \sum_{v\in V} \xi_v a(v) \;\implies\; \forall v \in V \quad p(v) = a(v). \tag{1.5}$$

We conclude that

$$\xi_w(v) \;=\; \delta_{wv}, \tag{1.6}$$

hence

$$\mathrm{affine}(V\backslash w) \;=\; \ker \xi_w := \{x \in \mathbb{R}^m : \xi_w(x) = 0\}. \tag{1.7}$$

**2. Definition of the B-form** The B-form for $p \in \Pi_k$ is a somewhat unexpected generalization of the linear interpolation formula (1.1), viz.

$$p \;=\; (\xi E)^k c(0). \tag{2.1}$$

Here, $c$ is a **mesh function** and $\xi E$ is a **difference operator**, and the formula is to be read as an instruction: "Apply the difference operator $\xi E$ $k$ times, starting with the mesh function $c$, then evaluate the resulting mesh function at the mesh point 0."

More explicitly, $c$ is defined on the mesh of nonnegative integer points

$$\alpha := \big(\alpha(v)\big)_{v\in V} \in \mathbb{Z}_+^V ,$$

and the difference operator $\xi E$ acts on mesh functions by the rule

$$(\xi E)c(\alpha) := \sum_{v\in V} \xi_v c(\alpha + e_v), \tag{2.2}$$

with $e_v$ the unit vector given by $e_v(w) := \delta_{vw}$.

If $k = 0$, then we are to apply the difference operator no times, i.e., then

$$p = c(0).$$

If $k = 1$, then we are to apply the difference operator one time, i.e, then

$$p = \sum_{v\in V} \xi_v c(e_v).$$

This is just (1.1) again, in slightly changed notation, i.e., $c(e_v) = p(v), v \in V$.

If $k = 2$, then we are to apply the difference operator two times, i.e., then

$$p = \sum_{v\in V} \sum_{w\in V} \xi_v \xi_w c(e_v + e_w).$$

In the general case, we obtain

$$p = \sum_{u\in V} \sum_{v\in V} \cdots \sum_{w\in V} \xi_u \xi_v \cdots \xi_w \;\; c(e_u + e_v + \cdots + e_w), \tag{2.3}$$

and this shows that *the function $p$ given by (2.1) is a polynomial of degree $\le k$* since it shows that $p$ is a linear combination of products of $k$ linear polynomials. This also shows that, for the purpose of the definition (2.1), we only need to know $c$ at meshpoints of the form

$$e_u + e_v + \cdots + e_w$$

involving exactly $k$ summands, i.e., at all mesh points $\alpha \in \mathbb{Z}_+^V$ with

$$|\alpha| := \sum_{v\in V} \alpha(v) = k.$$

But the whole point of the formulation (2.1) is to avoid having to deal with expressions like (2.3) and to operate, calculate and reason directly with the simple expression (2.1) according to the operations it prescribes. The next two paragraphs may make this clearer.

**3. Evaluation of the B-form** Evaluation of the B-form (2.1) at some point $x$ requires $k$-fold application of the difference operator $\xi(x)E$. Since we are only interested in $\big(\xi(x)E\big)^k c$ at the meshpoint 0, we only need to apply the difference operator "at" certain meshpoints. Precisely, we calculate

$$c_1(\alpha) := \big(\xi(x)E\big)c(\alpha) = \sum_{v\in V} \xi_v(x)c(\alpha+e_v) \quad \text{for} \quad |\alpha| = k-1$$

and this only requires knowledge of $c(\alpha)$ for $|\alpha| = k$. Then we calculate

$$c_2(\alpha) := \big(\xi(x)E\big)c_1(\alpha) = \sum_{v\in V} \xi_v(x)c_1(\alpha+e_v) \quad \text{for} \quad |\alpha| = k-2$$

and this only requires knowledge of $c_1(\alpha)$ for $|\alpha| = k-1$. In this way, we calculate

$$c_j(\alpha) := \big(\xi(x)E\big)c_{j-1}(\alpha) = \sum_{v\in V} \xi_v(x)c_{j-1}(\alpha+e_v) \quad \text{for} \quad |\alpha| = k-j$$

for $j = 1, \cdots, k$ (and with $c_0 = c$), and the final calculation gives

$$p(x) = c_k(0) = \big(\xi(x)E\big)c_{k-1}(0) = \sum_{v\in V} \xi_v(x)c_{k-1}(e_v).$$

In this description, I used different meshfunctions $c_j$ to avoid confusion. But, since $c_j$ is only considered and generated for $|\alpha| = k-j$, we might as well use the same letter $c$ for all of them. Thus, the evaluation amounts to generating the whole $(m+1)$-simplex

$$c(\alpha), |\alpha| \le k,$$

of numbers from its base

$$c(\alpha), |\alpha| = k,$$

by the calculation

$$\begin{aligned} &\text{for } j = 1, \cdots, k, \text{ do:} \\ &c(\alpha) = \big(\xi(x)E\big)c(\alpha) = \sum_{v\in V} \xi_v(x)c(\alpha+e_v), \ |\alpha| = k-j. \end{aligned} \tag{3.1}$$

While the base of this discrete simplex never changes, the layers built upon it do depend on $x$. Its apex, $c(0)$, provides the desired number $p(x)$. We will see later that many of the numbers $c(\alpha), |\alpha| < k$, generated here also provide useful information about $p$.

Figure 3 shows the meshpoints of interest for the case $m = 2$, i.e., the **bivariate** case. In this case, the mesh points have **three** components, corresponding to the fact that a two-dimensional simplex has three vertices. Correspondingly, each of the mesh point layers $|\alpha| = k-j$ of interest forms a triangle in this case, and the total set forms a tetrahedron. For $m = 1$, the meshpoints of interest would form a triangle. This is quite familiar from the evaluation of univariate polynomials, e.g., from their Newton form in which one also generates a triangular array of numbers.

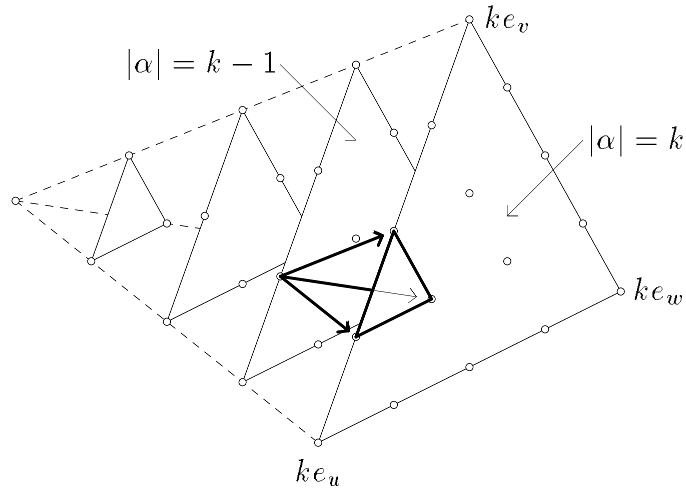


Figure 3. The meshpoint simplex for evaluation

Figure 3 also shows the typical stencil of the difference operator which, for the calculation of

$$\big(\xi(x)E\big)c(\beta) = \sum_{v\in V} \xi_v(x)c(\beta + e_v),$$

requires one to go from meshpoint $\beta$ into each of the $m+1$ coordinate directions, picking up the value of $c$ at the $m+1$ meshpoints $\beta + e_v, v \in V$ reached, multiplying that value with the number $\xi_v(x)$ and then summing these products over $v$ to obtain the value of $\big(\xi(x)E\big)c$ at $\beta$. Recall that $\sum_{v\in V} \xi_v(x) = 1$, hence application of the difference operator $\xi(x)E$ always amounts to **averaging**. If $x \in [V]$, then this average is proper since then $\xi(x) \geq 0$. Thus,

$$p(x) \textit{ is a convex combination of } \{c(\alpha), |\alpha| = k\} \textit{ in case } x \in [V]. \tag{3.2}$$

As an example, consider the calculation of $p(u)$. Since $\xi(u) = e_u$, the difference operator simplifies in this case to

$$\big(\xi(u)E\big)c(\beta) = c(\beta + e_u),$$

i.e., we merely pick up the value at the next meshpoint in the $u$-direction. We conclude that therefore

$$p(u) = c(e_u) \quad \text{for} \quad u \in V. \tag{3.3}$$

As another example, consider the calculation of $p(x)$ for some $x \in [V\backslash w]$. Now $\xi_w(x) = 0$, hence

$$\big(\xi(x)E\big)c(\beta) = \sum_{v\in(V\backslash w)} \xi_v(x)c(\beta + e_v).$$

We conclude that, in this case, $p(x)$ is a convex combination of just the coefficients $c(\alpha)$ with $|\alpha| = k$ and $\alpha(w) = 0$. More generally,

$$x \in [V \setminus W] \quad \Longrightarrow \quad p(x) \in [c(\alpha) : |\alpha| = k, \operatorname{supp} \alpha \subset (V \setminus W)] \tag{3.4}$$

To put it differently, *the coefficients for the B-form* (with respect to $V \setminus W$) *of the restriction of $p$ to the affine hull of $V \setminus W$ are provided by the restriction of $c$ to the corresponding mesh "simplex"* $\{\alpha \in \mathbb{Z}_+^V : |\alpha| = k, \ \operatorname{supp} \alpha \subset (V \setminus W)\}$. In these terms, (3.3) provides the extreme case $V \setminus W = \{u\}$.

**4. Differentiation of the B-form** The **directional derivative** $D_y f$ of the function $f$ on $\mathbb{R}^m$ in the direction $y$ is, by definition, given by the rule

$$(D_y f)(x) := \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t}.$$

Hence, in terms of the partial derivatives,

$$D_y = \sum_{j=1}^{m} y(j) D_j.$$

(This would suggest the alternative notation $yD$ for the operator $D_y$ or of $E_\xi$ for $\xi E$.)

It requires nothing more than the chain rule to differentiate the B-form:

$$D_y p = D_y \big(\xi E\big)^k c(0) = \big(\xi E\big)^{k-1} k \big(D_y \xi E\big) c(0). \tag{4.1}$$

Since $\xi(x)$ is the unique solution of the linear system

$$\sum_v \xi_v(x) = 1, \qquad \sum_v \xi_v(x) v = x,$$

we have $\xi(x + ty) - \xi(x) = t\eta(y)$, with $\eta(y)$ the unique solution of

$$\sum_v \eta_v(y) = 0, \qquad \sum_v \eta_v(y) v = y.$$

E.g.,

$$\eta(w - v) = e_w - e_v, \quad w \in V \backslash v \tag{4.2}$$

Thus, *with*

$$c_y := \big(\eta(y) E\big) c, \tag{4.3a}$$

*we obtain explicitly the B-form*

$$D_y p = \big(\xi E\big)^{k-1} k c_y(0) \tag{4.3b}$$

*for the polynomial $D_y p$.* Therefore, at a vertex,

$$D_y p(v) = k c_y\big((k-1)e_v\big) = k\big(\eta(y)E\big) c\big((k-1)e_v\big), \quad v \in V.$$

In particular, from (4.2),

$$D_{w-v}p(v) = \frac{c\big((k-1)e_v + e_w\big) - c(ke_v)}{1/k}, \quad w \in V\backslash v,$$

hence *the* $m+1$ *distinct points*

$$\{\big(v_\alpha, c(\alpha)\big) \in \mathbf{R}^{m+1} : \alpha = (k-1)e_v + e_w, w \in V\}$$

with

$$v_\alpha := \sum_{v\in V} v\alpha(v)/|\alpha| \tag{4.4}$$

*all lie on the tangent plane to* $p$ *at* $v$ and therefore determine that plane. This is a first indication of the usefulness of the B-net for $p$, which, by definition, is the collection of points

$$\{\big(v_\alpha, c(\alpha)\big) \in \mathbf{R}^{m+1} : |\alpha| = k, \alpha \in \mathbb{Z}_+^V\}.$$

Higher derivatives are obtained by iteration of this process. If $Y \subset \mathbb{R}^m \setminus 0$ contains $r$ points, then

$$D_Y p := \big(\prod_{y\in Y} D_y\big)p = \frac{k!}{(k-r)!}\big(\xi E\big)^{k-r} c_Y(0) \tag{4.5a}$$

with

$$c_Y(\alpha) := \prod_{y\in Y}\big(\eta(y)E\big)c(\alpha), \quad |\alpha| = k-r. \tag{4.5b}$$

The evaluation of such a derivative at some point proceeds just as the evaluation of $p$ itself, except that different difference operators are to be used at different stages. The order in which these are applied is immaterial since all (constant coefficient) difference operators commute (see Appendix X).

**5. A Taylor formula** By the binomial theorem and the fact that $\xi(x+y) = \xi(x) + \eta(y)$,

$$p(x+y) = \big(\xi(x+y)E\big)^k c(0) = \sum_{r=0}^{k}\binom{k}{r}\big(\xi(x)E\big)^{k-r}\big(\eta(y)E\big)^r c(0). \tag{5.1}$$

Since

$$\binom{k}{r}\big(\xi(x)E\big)^{k-r}\big(\eta(y)E\big)^r c(0) = \frac{1}{r!}\big({D_y}^r p\big)(x),$$

by (4.5), we obtain

$$p(x+y) = p(x) + \big(D_y p\big)p(x) + \frac{1}{2}\big({D_y}^2\big)(x) + \cdots$$

Of course, this is obtainable directly from the univariate Taylor formula and the fact that

$$\big(D_y{}^r p\big)(x) = f^{(r)}(0)$$

for the *univariate* function $f : t \mapsto p(x+ty)$.

**6. Invariance under an affine change of variables** Any affine change of variables leaves the B-form unchanged. More precisely, if also $W$ is an $(m+1)$-point set in $\mathbb{R}^m$ in general position, and $f$ is an affine map so that $f(W)=V$, then the B-form for the polynomial $q := p\circ f$ with respect to $W$ has again $c$ as its coefficient sequence, in the following sense. With $\xi'(x)$ the barycentric coordinates of $x$ with respect to $W$,

$$q(x) = p(f(x)) = \big(\xi(x)E\big)^k c(0) = \big(\xi'(x)E\big)^k c^f(0),$$

where

$$c^f(\alpha) := c(\alpha\circ f^{-1}).$$

This looks a bit confusing until one recalls that, after all, the meshpoints of concern in the B-form with respect to $V$ are indexed by, i.e. defined on, $V$, thus those of concern in a B-form with respect to $W$ are defined on $W$, and $f^{-1}$ carries $V$ to $W$. Since an affine change of variables also leaves $\Pi_k$ invariant, it is at times very useful to use such an affine change to simplify the situation. This can be done in many ways and requires an ordering of $V$,

$$V \;=\; \{v_j : j=0,\cdots,m\}$$

say. The corresponding affine map

$$s : \mathbb{R}^m \to\ \mathbb{R}^m : x\mapsto v_0 + (v_1-v_0|\cdots|v_m-v_0)x$$

(with $(a_1|\cdots|a_m)$ the $m\times m$ matrix whose $j$-th column is $a_j$, all $j$) carries the **standard m-simplex**

$$S_m \;:=\; [e_0 := 0, e_1, e_2, \cdots, e_m]$$

affinely to $[V]$.

The barycentric coordinates $\xi'(x)$ of $x$ with respect to $\{e_0,\cdots,e_m\}$ are particularly simple:

$$\xi'_{e_j}(x) = x(j), j=1,\cdots,m;\ \xi'_{e_0}(x) = 1-|x|.$$

**7. The basis property** While I have already made clear in paragraph 2 that the B-form

$$p = \big(\xi E\big)^k c(0) \tag{2.1}$$

describes a polynomial of degree $\le k$, I still owe the verification that *every* $p\in\Pi_k$ *can be written in that way*. For this, recall that (2.1) can be written out more explicitly

$$p = \sum_{u\in V}\sum_{v\in V}\cdots\sum_{w\in V} \xi_u\xi_v\cdots\xi_w\ c(e_u+e_v+\cdots+e_w). \tag{2.3}$$

To see that every $p \in \Pi_k$ can be written in this form for some choice of the coefficients $c(e_u + e_v + \cdots + e_w)$, observe that every $p \in \Pi_k$ can be written as a sum of products of $k$ linear polynomials, – e.g.,

$$\forall |\beta| \le k \quad x^\beta = \Big[ \prod_{\beta(j)>0} \big(x(j)\big)^{\beta(j)} \Big] (1)^{k-|\beta|},$$

– and each linear polynomial can be written as a linear combination of the $\xi_v$ as in (1.1). Thus, for every $p \in \Pi_k$, there is a collection of $k$-tuples $(q, r, \cdots, s)$ of linear polynomials so that

$$\begin{aligned} p &= \sum_{(q,r,\cdots,s)} q \cdot r \cdots s \\ &= \sum_{(q,r,\cdots,s)} \Big(\sum_{u\in V} \xi_u q(u)\Big)\Big(\sum_{v\in V} \xi_v r(v)\Big) \cdots \Big(\sum_{w\in V} \xi_w s(w)\Big) \\ &= \sum_{u,v,\cdots,w} \xi_u \xi_v \cdots \xi_w \sum_{(q,r,\cdots,s)} q(u) r(v) \cdots s(w) \end{aligned}$$

which shows our claim.

The sum in (2.3) involves $(m+1)^k$ summands, but many of these summands coincide. Writing (2.3) in terms of distinct meshpoints $\alpha \in \mathbb{Z}_+^V$, we get

$$p = \sum_{|\alpha|=k} B_\alpha c(\alpha), \tag{7.1}$$

with

$$B_\alpha := |\alpha|! [\xi^\alpha] = \binom{|\alpha|}{\alpha} \xi^\alpha$$

and

$$[\xi]^\alpha := \prod_{v\in V} [\xi_v]^{\alpha(v)} = \prod_{v \in V} {\xi_v}^{\alpha(v)} / \alpha(v)!.$$

The **multinomial coefficient** $\binom{|\alpha|}{\alpha} = |\alpha|! / \prod_{v\in V} \alpha(v)!$ appears here in accordance with the multinomial theorem, but its precise value is not important here. The only thing that matters is that there are exactly

$$\#\{\alpha \in \mathbb{Z}_+^V : |\alpha| = k\} = \#\{\beta \in \mathbf{Z}_+^m : |\beta| \le k\} = \dim \Pi_k$$

summands in (7.1) . Hence,

$\big(B_\alpha\big)_{|\alpha|=k}$ and $\big(\xi^\alpha\big)_{|\alpha|=k}$ *are both bases for* $\Pi_k$.

In particular, the representation (7.1) is irredundant, i.e., for each $p \in \Pi_k$, there is exactly one choice for $c(\alpha), |\alpha| = k$, so that (2.1) (or, equivalently (7.1)) holds.

Since $\sum_{|\alpha|=k} B_\alpha = \big(\sum_{v\in V} \xi_v\big)^k = 1$, the basis $\big(B_\alpha\big)_{|\alpha|=k}$ forms a **partition of unity**, and this partition of unity is nonnegative on $[V]$ (cf. (3.1)).

Further, $B_\alpha$ *is unimodular on* $[V]$, *and the coefficient* $c(\alpha)$ *has its biggest influence on* $p$ *in* $[V]$ *at the point* $v_\alpha = \sum_{v \in V} v\alpha(v)/|\alpha|$ (already mentioned in (4.4)) since $B_\alpha$, and equivalently $\xi^\alpha$, *takes its maximum over* $[V]$ *at* $v_\alpha$. To see this, observe that the **gradient** $D\xi^\alpha := \bigl(D_j \xi^\alpha\bigr)_{j=1}^m$ satisfies

$$D\xi^\alpha = \sum_{v \in V} \alpha(v) \xi^{\alpha - e_v} D\xi_v = \sum_{v \in V} d_v D\xi_v$$

with

$$d_v(x) := \begin{cases} \alpha(v)\xi^{\alpha - e_v}(x), & \text{if } \alpha(v) > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$D\xi^\alpha(x) = 0 \quad \iff \quad d_v(x) \in G := \{a \in \mathbf{R}^V : \sum a(v) D\xi_v(x) = 0\}.$$

Since

$$\operatorname{span}\{D\xi_v(x) : v \in V\} = \{Dp : p \in \Pi_1\} = \mathrm{I\!R}^m,$$

we have $\dim G = 1$, and, since $\sum D\xi_v(x) = 0$ (remember that $\sum \xi_v = 1$), we know that $(1, 1, \cdots, 1) \in G$. Therefore, altogether,

$$D\xi^\alpha(x) = 0 \quad \iff \quad \forall v \in V \quad \alpha(v)\xi^{\alpha - e_v}(x) = \text{const}.$$

Thus, for $x$ in the interior of $[V]$, $D\xi^\alpha(x) = 0 \quad \iff \quad \alpha(v) = \text{const}\,\xi_v(x), v \in V$, with const necessarily equal to $k$ since $\sum \alpha(v) = k$ while $\sum \xi_v(x) = 1$. In short, necessarily $x = v_\alpha$. Thus $\xi^\alpha$ has a critical point in the interior of $[V]$ iff $\operatorname{supp}\alpha = V$, and, in that case, the critical point is necessarily the point $v_\alpha$, and, since $\xi^\alpha$ is nonnegative on $[V]$ and vanishes on the boundary of $[V]$, it is necessarily the maximum point for $\xi^\alpha$ on $[V]$.

This also says that, for $\operatorname{supp}\alpha \neq V$, $\xi^\alpha$ takes on its maximum over $[V]$ on the boundary, necessarily on the face of largest dimension on which $\xi^\alpha$ does not vanish identically, i.e., the face

$$F_\alpha := [v \in V : \alpha(v) \neq 0].$$

On that face,

$$D\bigl(\xi^\alpha{}_{|F_\alpha}\bigr) = \sum_{\alpha(v) \neq 0} d_v D\xi_v$$

vanishes at $x \quad \iff \quad \forall \alpha(v) \neq 0 \quad \alpha(v) = k\xi_v(x)$. Thus, also in this case does $\xi^\alpha$ take on its maximum over $[V]$ at $v_\alpha$.

**8. Product of two B-forms** The B-form of the product of two polynomials is obtainable from their B-forms with the aid of a few factorials:

$$\bigl(\xi E\bigr)^k c(0) \bigl(\xi E\bigr)^h d(0) = \bigl(\xi E\bigr)^{k+h} c * d(0)$$

with

$$c * d(\gamma) := \sum_{\alpha + \beta = \gamma} c(\alpha) d(\beta) C_{\alpha,\beta} \tag{8.1}$$

and

$$C_{\alpha,\beta} := \frac{\binom{|\alpha|}{\alpha}\binom{|\beta|}{\beta}}{\binom{|\alpha+\beta|}{\alpha+\beta}},$$

since

$$B_\alpha B_\beta = \binom{|\alpha|}{\alpha}\xi^\alpha \binom{|\beta|}{\beta}\xi^\beta = C_{\alpha,\beta} B_{\alpha+\beta}.$$

The special choice $d(\beta) = 1, |\beta| = h$, gives the formula

$$\big(\xi E\big)^k c(0) = \big(\xi E\big)^{k+h} c'(0)$$

with

$$c'(\gamma) := \sum_{\alpha+\beta=\gamma} c(\alpha) C_{\alpha,\beta}, \quad |\gamma| = k + h.$$

The particular case $h = 1$ of such **degree raising** is discussed in Section 10.

**9. Integration of the B-form** We obtain the B-form

$$\big(\xi E\big)^{k+1} c^y(0)/(k+1)$$

of a polynomial $I_y p$ with

$$D_y(I_y p) = p = \big(\xi E\big)^k c(0)$$

by inverting the differentiation process (4.1). For this, choose $u \in V$ with $\eta_u(y) \neq 0$ and choose $c^y(\alpha)$ for $|\alpha| = k+1$ and $\alpha(u) = 0$ quite arbitrarily. After that,

$$\begin{aligned}&\text{for } |\alpha| = k \text{ and } \alpha(u) = 0, 1, ..., k, \text{ do:}\\ &\qquad \text{compute } c^y(\alpha + e_u) = \big(c(\alpha) - \sum_{v \neq u} \eta_v(y) c^y(\alpha + e_v)\big)/\eta_u(y).\end{aligned}$$

For numerical stability, $u$ should be chosen, more precisely, so that

$$|\eta_u(y)| = \max_v |\eta_v(y)|.$$

In principle, such an anti-derivative can be used to compute

$$\int_{[V]} p$$

i.e., the integral of $p$ over the simplex $[V]$, since, by Stokes' Theorem,

$$\int_{[V]} p \ = \ \int_{[V]} D_y(I_y p) \ = \ \int_{\partial[V]} (yn) I_y p,$$

with $(yn)(x)$ the scalar product of $y$ with the unit outward normal $n(x)$ to $[V]$ at $x$. This reduces the integration of $p$ over the $m$-simplex $[V]$ to the integration of $D_y p$ over the

$m+1$ $(m-1)$-simplices $[V\backslash v], v \in V$. These integrations, in turn, can be related to the integration of anti-derivatives over the $(m-2)$-faces of $[V]$, etc. arriving, finally, at a linear combination of vertex values of certain higher-degree polynomials as the desired value of the integral. It would not be difficult to write down the details of such an algorithm for low $m$, say $m=2$ or $m=3$.

On the other hand, it is possible to express the integral of $p$ over $[V]$ quite simply in terms of the B-form:

$$\int_{[V]} p = \frac{\mathrm{vol}_m[V]}{\binom{k+m}{k}} \sum_{|\alpha|=k} c(\alpha). \tag{9.1}$$

To see this, use an affine change of variables (see Section 6) to reduce $V$ to the standard simplex $S_m = [e_0 = 0, e_1, \cdots, e_m]$ in $\mathrm{IR}^m$, i.e.,

$$\int_{[V]} p \;=\; \det(v_1 - v_0|\cdots|v_m - v_0) \int_{S_m} q,$$

with

$$q(x) = p(s(x)) = p\big(v_0 + \sum_{j=1}^{m} (v_j - v_0) x(j)\big)$$

and $\left(v_j\right)_{j=0}^{m}$ an ordering of $V$. Denote the meshfunction $c^s$ for $q$ again simply by $c$ and let $B_\alpha$ be the basis functions for the B-form with respect to the standard simplex. Then

$$\int_{[S_m]} B_\alpha = \int_0^1 \int_0^{1-x(1)} \cdots \int_0^{1-\sum_{j<m} x(j)} B_\alpha(x) dx(m) \cdots dx(1)$$

with

$$B_\alpha(x) = |\alpha|! [1-|x|]^{\alpha(e_0)} \prod_{j=1}^{m} [x(j)]^{\alpha(e_j)}.$$

Now use the abbreviation $s := 1 - \sum_{j<m} x(j)$ and carry out the integration with respect to $x(m)$ (using integration by parts),

$$\int_0^s [s - x(m)]^{\alpha(e_0)} [x(m)]^{\alpha(e_m)} dx(m) = [s]^{\alpha(e_m)+\alpha(e_0)+1},$$

to conclude that

$$\int_{[S_m]} B_\alpha = \int_{[S_{m-1}]} B_\beta / |\beta|,$$

with $\int_{[S_0]} B_\beta = 1$ and

$$\beta(e_j) = \begin{cases} \alpha(e_j), & \text{for } j = 1, \cdots, m-1; \\ \alpha(e_0) + \alpha(e_m) + 1, & \text{for } j = 0, \end{cases}$$

hence $|\beta| = |\alpha| + 1$. Therefore, by induction,

$$\int_{[S_m]} B_\alpha = |\alpha|!/(|\alpha| + m)!.$$

**10. Degree raising** Observe that

$$[\xi]^\alpha = \big(\sum_v \xi_v\big)[\xi]^\alpha = \sum_v \big(\alpha(v) + 1\big)[\xi]^{\alpha+e_v}.$$

Hence

$$\begin{aligned}\sum_{|\alpha|=k} B_\alpha c(\alpha) &= k! \sum_{|\alpha|=k} \sum_v \big(\alpha(v) + 1\big)[\xi]^{\alpha+e_v} c(\alpha) \\ &= k! \sum_{|\alpha|=k+1} [\xi]^\alpha \sum_v \alpha(v) c(\alpha - e_v).\end{aligned}$$

Conclude that

$$\sum_{|\alpha|=k} B_\alpha c(\alpha) \;=\; \sum_{|\alpha|=k+1} B_\alpha \big(Rc\big)(\alpha) \tag{10.1}$$

with

$$\big(Rc\big)(\alpha) \;:=\; \sum_{v\in V} c(\alpha - e_v)\alpha(v)/|\alpha|. \tag{10.2}$$

Note that (10.1-2) requires knowledge of $c(\beta)$ for $\beta$ with $|\beta| = k$ and, possibly, a negative entry. This presents no difficulty, though, since all such values are multiplied by zero, hence are not really needed.

Formula (10.2) can be interpreted as **linear interpolation** at the point $v_\alpha$ of the plane or linear polynomial through the points

$$\big(v_\beta, c(\beta)\big) \quad \text{with} \quad \beta = \alpha - e_v, v \in V.$$

Indeed, the barycentric coordinates $\Big(\alpha(v)/|\alpha|\Big)_{v\in V}$ for $v_\alpha$ with respect to $V$ are also the barycentric coordinates for $v_\alpha$ with respect to $\{v_{\alpha-e_v} : v \in V\}$, as can be seen from the following calculation:

$$\sum_{v\in V} v_{\alpha-e_v}\alpha(v)/|\alpha| \;=\; \sum_{v\in V}\sum_{u\in V} u\frac{\alpha(u)-\delta_{uv}}{|\alpha|-1}\alpha(v)/|\alpha| = \sum_{u\in V}\sum_{v\in V} \frac{\alpha(u)-\delta_{uv}}{|\alpha|-1}\alpha(v)/|\alpha|$$

and

$$\sum_{v\in V} \frac{\alpha(u)-\delta_{uv}}{|\alpha|-1}\alpha(v) = \frac{\alpha(u)}{|\alpha|-1}|\alpha| - \frac{\alpha(u)}{|\alpha|-1} = \alpha(u).$$

This draws further attention to the control polytopes for $p$, i.e., the piecewise linear functions obtained from the B-net

$$\{\big(v_\alpha, c(\alpha)\big) : |\alpha| = k\}$$

by local linear interpolation.

**11. The Bernstein polynomial** The **Bernstein polynomial for $f$ of order $k$ with respect to $V$** is, by definition, the particular polynomial

$$B_k f := \sum_{|\alpha|=k} B_\alpha f(v_\alpha). \tag{11.1}$$

The Bernstein polynomial provides an approximation to $f$ which, on $[V]$, converges uniformly to $f$ as $k \to \infty$ in case $f$ is continuous (cf. [L53; p.51]). The convergence is monotone in case $f$ is **V-convex** in the sense that each of the univariate functions $t \mapsto f(x+t(v-w))$ is convex ([Be76]). Moreover, in this case, $B_k f$ is also $V$-convex. But $B_k f$ need not be convex even if $f$ is ([St59], [Be76], [CD84]).

The B-form $(\xi E)^k c(0)$ for $p \in \Pi_k$ (with respect to $V$) provides the essential information about any $f$ for which $p = B_k f$. We have

$$p = B_k f \quad \Longleftrightarrow \quad \forall |\alpha| = k \quad f(v_\alpha) = c(\alpha). \tag{11.2}$$

The simplest such functions $f$ are the **control polytopes** for $p$, i.e., any linear interpolant to the data

$$\big(v_\alpha, c(\alpha)\big), |\alpha| = k. \tag{11.3}$$

For this reason, we denote any such control polytope by

$$B_k^{-1} p. \tag{11.4}$$

I have used the plural here advisedly since, for $m > 2$, there are several equally reasonable piecewise linear interpolants, as has been rightfully stressed and detailed by Dahmen and Micchelli in [DM86]. Different interpolants differ in how the points $\{v_\alpha : |\alpha| = k\}$ are connected to produce a **triangulation**, i.e., a partition into simplices, for the simplex $[V]$. The typical triangulation is obtained by choosing an ordering $v_0, v_1, \ldots, v_m$ of the vertex set $V$, thus obtaining the directions $d_i := v_i - v_{i-1}$, $i = 1, \ldots, m$. The corresponding triangulation with meshpoints $\{v_\alpha : |\alpha| = k\}$ for $V$ consists of all simplices in $[V]$ of the form

$$\sigma_{\alpha,q} := v_\alpha + [0, d_{q(1)}, d_{q(1)} + d_{q(2)}, \ldots, d_{q(1)} + \cdots + d_{q(m)}]/k$$

with $|\alpha| = k$ and $q$ a permutation of the first $m$ integers. Thus,

$$\sigma_{\alpha,q} = [v_{\alpha_0}, \ldots, v_{\alpha_m}],$$

with

$$\begin{aligned} \alpha_0 &:= \alpha, \\ \alpha_j &:= \alpha_{j-1} + e_{v_{q(j)}} - e_{v_{q(j)-1}}, \quad j = 1, \ldots, m. \end{aligned}$$

A simplex may appear in the triangulation only for certain orderings of $V$ and not for others. To see this, observe that two points $v_\alpha$ and $v_\beta$ will be vertices for the same simplex if and only if their difference can be written as a sum of some of the vectors $d_j/k$. If we order the entries of $\alpha$ and $\beta$ to correspond to the ordering of the vertex set used, writing, e.g., $\alpha(j)$ instead of $\alpha(v_j)$, this means that either $\beta-\alpha$ or else $\alpha-\beta$ must be writable as a sum of distinct vectors of the form $e_j - e_{j-1}$. For example, for $m=3$ and $k=2$, the two vertices $v_{(1,0,0,1)}$ and $v_{(0,1,1,0)}$ are not connected by a meshline (in the triangulation corresponding to the ordering used), while the two vertices $v_{(0,1,0,1)}$ and $v_{(1,0,1,0)}$ are. This shows that the reordering $v_1, v_0, v_2, v_3$ would connect the former and disconnect the latter.

On the other hand, the simplices

$$[v_{\beta-e_v} : v \in V] \quad \text{with } |\beta| = k+1$$

involved in degree raising are part of any such triangulation since, in terms of the particular ordering used,

$$[v_{\beta-e_v} : v \in V] = \sigma_{\alpha,q}$$

with $\alpha = \beta - e_{v_m}$ and $q(j) = m+1-j,\ j=1,\ldots,m$.

Thus, regardless of the particular ordering of the vertex set $V$ used, the resulting piecewise linear interpolant $B_k^{-1}p$ to the data (11.3) will agree with $B_{k+1}^{-1}p$ at the base-points $v_\beta, |\beta| = k+1$, as we saw in Section 10. This implies, by induction, that the Lipschitz constant (over $[V]$) for any $B_{k+n}^{-1}p, n>0$, is no bigger than that for $B_k^{-1}p$, hence the sequence $\left(B_{k+n}^{-1}p\right)$ has uniform limit points. It implies further that $B_{k+n}^{-1}p$ converges pointwise to some function $f$ as $n \to \infty$, hence $f$ is the uniform limit of $B_{k+n}^{-1}p$. But this limit function is necessarily $p$ since

$$p = B_{k+n}B_{k+n}^{-1}p = B_{k+n}f + B_{k+n}(B_{k+n}^{-1}p - f) \longrightarrow f, \text{ as } n \to \infty$$

using the facts that $B_{k+n}f$ converges to $f$ and $\|B_{k+n}(B_{k+n}^{-1}p - f)\| \le \|B_{k+n}^{-1}p - f\| \to 0$.

Since local linear interpolation preserves convexity, we conclude that $p$ is convex in case its control polytope $B_k^{-1}p$ is.

**12. The subpolynomials** The evaluation of $p \in \Pi_k$ and its derivatives from the B-form proceeds by repeated **differencing**. It is a remarkable fact that this differencing is **uniform**. Regardless of the meshpoint at which it is applied, the difference operator is the same. This implies that, *during the calculation of some information about $p$, we are simultaneously computing the same information for* a whole host of polynomials, viz. *all polynomials whose B-form coefficients* (with respect to $V$) *form a subsimplex of those for $p$*. These are the polynomials

$$p_\alpha := \left(\xi E\right)^{k-|\alpha|} c(\alpha), \quad |\alpha| \le k. \tag{12.1}$$

For $|\alpha| = k$, $p_\alpha$ is the constant $c(\alpha)$, while, at the other extreme, $p_0 = p$.

The coefficient simplex for $p_\alpha$ has $\alpha + (k - |\alpha|)e_v, v \in V$, as its extreme meshpoints, hence

$$p_\alpha(v) = c(\alpha + (k-|\alpha|)e_v), \quad v \in V. \tag{12.2}$$

More generally, from (4.5), with $Y \subset \mathbb{R}^m \backslash 0$ and $r := \#Y$, and $|\alpha| \le k - r$,

$$D_Y p_\alpha \;=\; \frac{(k-|\alpha|)!}{(k-|\alpha|-r)!}(\xi E)^{k-|\alpha|-r}\Big(\prod_{y\in Y}\eta(y)E\Big)c(\alpha).$$

Hence

$$D_Y p_\alpha(v) \;=\; \mathrm{const}_{k,|\alpha|,r}\, c_Y\big(\alpha + (k-|\alpha|-r)e_v\big), \quad v \in V, \tag{12.3}$$

with

$$c_Y := \prod_{y\in Y}(\eta(y)E)c$$

the essential part of the B-form coefficients for $D_Y p$. Specifically, since

$$|\alpha + (k-|\alpha|-r)e_v| = k - r,$$

*$D_Y p_\alpha(v)$ is, up to a constant factor, equal to one of the B-form coefficients for $D_Y p$.*

Among the **subpolynomials** $p_\alpha$, we single out those with $\alpha(W) = 0$, i.e., $\alpha(w) = 0, w \in W$, for some $W \subset V$. We call these the **W-face polynomials** since, as it turns out, these describe completely the behavior of $p$ and its derivatives on the $W$-face of $[V]$, i.e., on $[V \setminus W]$, as I will now explain.

Let $\alpha(W) = 0$ and $v \in V\backslash W$. Then $v$ lies in the $W$-face, and the meshpoint $\beta := \alpha + (k-|\alpha|-r)e_v$ mentioned in (12.3) also has $\beta(W) = 0$. Therefore, $D_Y p_\alpha(v)$ equals, up to a constant factor, a B-form coefficient for $D_Y p$ associated with the $W$-face, hence is determined by $D_Y p$ on the $W$-face.

Conversely, if we know $p_\alpha$ for all $\alpha$ with $\alpha(W) = 0$ and $|\alpha| = k - r$, then we can compute all the B-form coefficients for $D_Y p$ on $[V\backslash W]$ via (12.3). This gives the following theorem.

**Theorem** *Let $p, q \in \Pi_k$, $W \subset V$. Then*

$$\forall (Y \subset \mathbb{R}^m\backslash 0, \#Y \le r) \quad D_Y p = D_Y q \text{ on } [V\backslash W] \iff \forall(|\alpha| = k-r, \alpha(W) = 0) \quad p_\alpha = q_\alpha.$$

**Corollary** *Each $p_\alpha$ depends linearly on $p$ and its derivatives of order $\le k - |\alpha|$ on* $[\mathrm{supp}\ \alpha]$.

**13. Change of $V$** The subpolynomials $p_\alpha$ introduced in the preceding section depend on $V$. This is reflected in the notation since, after all, $\alpha$ is defined on $V$. But, by the corollary to Theorem 12, $p_\alpha$ depends, more precisely, only on the points in supp $\alpha$. To say it differently:

**Lemma** *If also $V'$ is an $(m+1)$-set in $\mathbb{R}^m$ in general position, and $\alpha \in \mathbb{Z}_+^V$ has its support in $V \cap V'$, then*

$$p_\alpha = p_{\alpha'},$$

*with*

$$\alpha' : V' \to \mathbb{Z}_+ : v \mapsto \begin{cases} \alpha(v), & \text{if } v \in V \cap V'; \\ 0, & \text{otherwise.} \end{cases}$$

This is so because, by the corollary, $p_\alpha$ only depends on $p$ and its derivatives on $[\text{supp } \alpha]$. This suggests the identification of any two $\alpha$, $\alpha'$ which agree on their support, and we will follow this suggestion from now on. In effect, we think of $\alpha$ as defined at all the vertices that might enter the discussion, but to be zero on all but at most $m+1$ of them.

As a first application of the subpolynomial notion, consider now the change of $V$, i.e., the derivation of the B-form $p =: (\xi' E)c'(0)$ for $p$ with respect to $V'$ from the B-form with respect to $V$. It is sufficient to consider the case

$$V' = (V \backslash w) \cup w',$$

since an arbitrary $V'$ can be reached from $V$ as the $(m+1)$-st in a chain of $(m+1)$-sets whose neighbors only differ by one point. Now note that, by (12.2),

$$c'(\alpha) = p_{\alpha - \alpha(w')e_{w'}}(w'), \quad \text{for } |\alpha| = k, \tag{13.1}$$

and $\beta := \alpha - \alpha(w')e_{w'}$ vanishes at $w'$. This implies that $\text{supp}\beta \subset V \backslash w$, hence $p_\beta$ is a subpolynomial associated with $V$ and therefore evaluated during the course of evaluation of $p$ from its B-form with respect to $V$. Specifically, we find $c'(\alpha) = p_\beta(w')$ at position $\beta$ in the $(m+1)$-simplex $c(\beta)$, $|\beta| \le k$, generated during the evaluation of $p$ at $w'$, i.e.,

$$c'(\alpha) = c(\alpha - \alpha(w')e_{w'}), \quad \text{for } |\alpha| = k. \tag{13.2}$$

In fact, since the evaluation of $p$ at $w'$ from the B-form with respect to $V$ proceeds without any special attention paid to the vertex $w$, it follows that we are generating simultaneously the B-form coefficients for $p$ with respect to every one of the $(m+1)$ sets $V'$ obtainable from $V$ by an exchange of some $w \in V$ for $w'$. This provides a **subdivision algorithm**. Choosing $w'$ somewhere in $[V]$, we obtain a triangulation of $[V]$ into at most $m+1$ nontrivial simplices $[V_w]$, with $V_w := (V \backslash w) \cup w'$, and the coefficient simplex for the B-form of $p$ with respect to $V_w$ is to be found in the $w$-facet of the $(m+1)$-simplex $c(\beta), |\beta| \le k$ generated during the evaluation of $p$ at $w'$.

This generation of B-forms with respect to a slightly changed pointset during evaluation is reminiscent of what happens during the evaluation of the Newton form of a univariate polynomial by Horner's scheme.

The meshpoint $\alpha = 0$ plays a special role since its support is empty. But that is quite alright, since, in this case, the Lemma reduces to the tautology $p = p$ in which $V$ does not enter explicitly.

I mention in passing that $\xi$ and $\xi'$ (with $V' = (V \backslash w) \cup w'$) are closely related: $\xi - \xi'$ is linear and vanishes on $[V \backslash w]$, while $\xi_w$ is also linear, vanishes on $[V \backslash w]$, and equals 1 on $w$. Hence

$$\xi - \xi' = (\xi - \xi')(w)\xi_w = (\xi - \xi')(w')\xi'_w.$$

**14. Smoothness across an interface** The matching of derivatives of polynomial pieces across an interface between two simplices is easily described in terms of the subpolynomials associated with that interface, since these describe completely the behavior of a polynomial and its derivatives on that interface, by Theorem 12. The precise statement of the smoothness conditions is made quite simple by our agreement to think of meshpoints $\alpha$ as defined on all vertices that might appear in the discussion, with its value usually zero, with at most $m+1$ exceptions.

**Theorem** *Let $p, q \in \Pi_k$ , let $r \le k$, and let $V, V'$ be the vertex sets of two simplices in a triangulation. Then the pp function*

$$f : [V] \cup [V'] \to \mathbf{R} : x \mapsto \begin{cases} p(x), & \text{if } x \in [V]; \\ q(x), & \text{if } x \in [V'], \end{cases}$$

*is in $C^r$ if and only if*

$$\forall(\mathrm{supp}\alpha \subset V \cap V', |\alpha| = k - r) \quad p_\alpha = q_\alpha. \tag{14.1}$$

If $V$ and $V'$ differ by just one point,

$$V' = (V \backslash w) \cup w',$$

say, and $q = \bigl(\xi' E\bigr)^k c'(0)$ is the B-form for $q$ with respect to $V'$, then the condition (14.1) reads more explicitly

$$\forall(\mathrm{supp}\ \alpha \subset (V \backslash w), k - r \le |\alpha| \le k) \quad p_\alpha(w') = c'(\alpha + (k - |\alpha|)e_{w'}). \tag{14.2}$$

Note that *these conditions are independent of $k$*. Specifically, were we to write down the conditions (14.1) or (14.2) explicitly in terms of the coefficients $c$ and $c'$, we would obtain linear relations whose weights only depend on $r$ or $k - |\alpha|$, i.e., on the order of the derivatives being constrained to be continuous. This means that, in studying a linear system of such conditions across one or more neighboring facets, we can choose $k$ at will, e.g., $k = r$.

For $r = 0$ (and with supp $\alpha \subset V$), this condition reduces to

$$\forall(\alpha(w) = 0, |\alpha| = k) \quad c(\alpha) = c'(\alpha). \tag{14.3}$$

This illustrates the "superficiality" of polynomials as the dimension increases. For, if $m \ge k$, then each $\alpha$ with $|\alpha| = k$ and supp $\alpha \subset V$ must have $\alpha(w) = 0$ for some $w \in V$, hence any $p \in \Pi_k$ is completely determined by its behavior on the boundary of $[V]$, specifically by its behavior on the $k$-dimensional faces of $[V]$.

For $r = 1$, this condition forces the linear polynomials $p_\alpha$ and $q_\alpha$ with $\alpha(w) = 0$ and $|\alpha| = k - 1$ to agree. Since, in this case, (the graph of) $p_\alpha$ is the plane through the points

$$\bigl(v_\beta, c(\beta)\bigr), \beta = \alpha + e_v, v \in V,$$

the condition reads that the point $\bigl(v_{\beta'}, c'(\beta')\bigr)$, with $\beta' := \alpha + e_{w'}$ should also lie on the same plane.

In general, $C^r$-continuity across $[V\backslash w]$ imposes conditions which connect $c(\alpha)$ for $\alpha(w) \le r$ with $c'(\alpha)$ for $\alpha(w') \le r$. The form (14.2) makes explicit that this involves exactly

$$\#\{\alpha \in \mathbb{Z}_+^V : \alpha(w) = 0, k - r \le |\alpha| \le k\}$$

linearly independent conditions, i.e., exactly as many conditions as there are degrees of freedom in $p$ and its directional derivatives of order $\le r$ on $[V\backslash w]$ in some fixed direction transversal to $[V\backslash w]$.

**15. The B-net** Let $\Delta$ be a triangulation of some domain in $\mathbb{R}^m$. This means that $\Delta$ consists of simplices $\delta$, with the intersection $\delta \cap \delta'$ of any two always a face (possibly the empty face) of both of them.

I denote by $V_\delta$ the vertex set of the simplex $\delta \in \Delta$, and by

$$V := \cup_{\delta\in\Delta} V_\delta$$

the totality of the vertices of simplices of $\Delta$. Denote by

$$A = A_{k,\Delta} := \{\alpha \in \mathbb{Z}_+^V : |\alpha| = k, \exists\delta \in \Delta \quad \mathrm{supp}\ \alpha \subset V_\delta\}$$

the corresponding set of index meshpoints of interest. In words, these are elements of $\mathbb{Z}_+^V$, i.e., defined on $V$ and with nonnegative integer entries. In addition, each $\alpha \in A$ has support only on some $V_\delta$ and has length $|\alpha| = k$.

Consider now the space

$$S := \Pi^\rho_{k,\Delta}$$

of pp functions of degree $\le k$ on the triangulation $\Delta$ and in $C^\rho$. This means that each $f \in S$ agrees on each simplex in $\Delta$ with some polynomial of degree $\le k$, and these polynomial pieces fit together to form a function with $r$ continuous derivatives.

Consider specifically

$$S_0 := \Pi^0_{k,\Delta},$$

the space of continuous pp functions (of degree $\le k$) on $\Delta$. Since two of its polynomial pieces on neighboring simplices fit together continuously exactly when their B-form coefficients associated with the common face coincide, it is possible to describe an element $f$ of $S_0$ by the meshfunction $c$ defined on the mesh $A$ and providing in

$$c(\alpha),\ \mathrm{supp}\ \alpha \subset V_\delta,$$

the B-form coefficients for the polynomial piece $f_{|\delta}$ with respect to the vertex set $V_\delta$ of $\delta$.

The **B-net** for such $f$ is, by definition, the collection of points

$$\big(v_\alpha, c(\alpha)\big), \quad \alpha \in A,$$

with

$$v_\alpha := \sum_{v\in V} v\alpha(v)/|\alpha|, \quad \alpha \in A.$$

While it is satisfactory to deal with the meshfunction $c$, the B-net reflects more explicitly the geometry of the situation. We think of the B-net as the function

$$b_f : V_A \to \mathbf{R} : v_\alpha \mapsto c(\alpha)$$

on the discrete set

$$V_A := \{v_\alpha : \alpha \in A\},$$

which is a subset of the domain of $f \in S_0$. The values of this discrete function at all the points in some face of some $\delta$ determine $f$ on that face. In particular,

$$f(v) = c(ke_v) = b_f(v), \quad v \in V.$$

Further, $C^r$-continuity of $f$ is equivalent to certain linear relations involving $b_f$ on points at most $r$ layers away from the facets of the $\delta$. For example, $C^1$-continuity is equivalent to having each $(m+2)$-tuple

$$\big(v_\beta, b_f(v_\beta)\big), \ \beta = \alpha + e_v, \ v \in W$$

lie on a plane, with $W$ the vertices of any two simplices having a facet in common, and $|\beta| = k-1$ with support only on the vertices common to both simplices. This localizes the effect of such continuity conditions as much as possible.

**16. Example** Here is just one example of the usefulness of the B-form and the B-net in the handling of smooth pp functions of many variables. Consider the problem of determining an $f \in \Pi^\rho_{k,\Delta}$ which matches given information at the vertices of the partition and, perhaps, at other points. If $\rho = 0$, this is easy to accomplish (for standard information such as function values) since the smoothness requirements only involve B-net points at simplex interfaces. For $\rho > 0$, though, there are difficulties because the smoothness conditions involve B-net points in the interior of the simplices and one has to untangle the effect of smoothness conditions from different facets involving the same B-net point. This is a hopeless task in general. But if the polynomial degree, $k$, is large enough compared to $\rho$, then this can be accomplished.

The key to such an understanding is the **ring** $R_v$ of smoothness conditions associated with a vertex $v$ in the following manner. Recall from (14.2) that the typical smoothness condition of order $r$ is associated with a facet $F := [V_\delta \cap V_{\delta'}]$ with $V_{\delta'} = \big(V_\delta \backslash w\big) \cup w'$, and is of the form

$$\big(\xi E\big)^{k-|\alpha|} c(\alpha) = c'\big(\alpha + (k - |\alpha|)e_{w'}\big) \tag{16.1}$$

with supp $\alpha \subset F$ and $|\alpha| = k - r$. We associate this condition with the vertex $v$ provided (i) $r \le \rho$, (ii) $v \in F$, and (iii) some smoothness condition of order $\le \rho$ across a different facet, $F'$, containing $v$, involves some of the same B-net points as (16.1) does.

One has to study these rings since not all conditions belonging to such a ring are linearly independent. On the other hand, in principle, one can select a maximally linearly independent subset of conditions from each such ring. These, together with the conditions not belonging to any ring provide a maximally linearly independent subset of all the smoothness conditions provided rings belonging to different vertices to not interfere with one another, i.e., provided no B-net point is involved in smoothness conditions from more than one ring.

If some do, then it is possible to connect any two rings by a chain of rings in which neighboring rings contain conditions involving the same B-net point. This implies that the linear system of smoothness conditions is global, meaning that its solution may well change everywhere when we make a change somewhere.

On the other hand, if no two rings interfere, then we can deal with the smoothness conditions in an entirely local manner. We first deal with the conditions in any one ring, augmenting the given information at the vertex if necessary in order to determine uniquely the B-net points involved in the conditions belonging to that ring. The remaining conditions fall into groups according to the facet to which they belong, and different such facet groups involve different B-net points. Since the smoothness conditions associated with any one facet (whether or not they also belong to some ring) are linearly independent, it is possible in many ways to choose the B-net points not yet determined so as to satisfy the smoothness conditions in the facet groups, typically by prescribing mid-facet normal derivative information.

It is possible to prove that such a smooth interpolant can be constructed in such a manner that it reduces to a polynomial of degree $\le k$ in case all the information comes from such a polynomial. Since the scheme is local, this implies good approximation properties for the scheme.

This makes it important to know just when the vertex rings of smoothness conditions of order $\le \rho$ are disjoint. For the condition (16.1) to involve B-net points (in $V_\delta$) which are also involved in some smoothness condition of order $\le \rho$, say of the $v$-facet, of $V_\delta$ requires the meshpoint $\beta := \alpha + (k - |\alpha|)e_w$ to be within $\rho$ steps of the corresponding meshpoint facet, i.e., to satisfy $\beta(v) \le \rho$.