attempt on blossoms

An n-**affine form** on some flat F in $\mathbb{F}^d$ is any element of

$$M_n := M_n(F) := \underbrace{\Pi_1(F) \otimes \cdots \otimes \Pi_1(F)}_{n \text{ terms}}.$$

M_n is a linear space, with **standard (power) basis**

$$(e_\gamma : u \mapsto \prod_{i=1}^n u_i(\gamma(i)) : \gamma : \{1,\ldots,n\} \to \{0,\ldots,d\}).$$

Here and below, the jth entry of $x \in \mathbb{F}^d$ is denoted by $x(j)$, $j = 1,\ldots,n$, while

$$x(0) := 1.$$

We use $\mathbb{F}$ here to stand for either $\mathbb{R}$ or $\mathbb{C}$.

The linear map diag, given by the rule

$$\operatorname{diag}(f) : F \to \mathbb{F} : x \mapsto f(x,\ldots,x),$$

associates each $f \in M_n(F)$ with a polynomial of degree $\le n$ on F. This map carries $M_n(F)$ *onto* $\Pi_n(F)$ since, e.g.,

$$\operatorname{diag}(e_\gamma) = ()^{(\#\gamma^{-1}(i)\,:\,i=1,\ldots,d)}.$$

However,

$$\operatorname{diag}(e_\gamma) = \operatorname{diag}(e_{\gamma'}) \quad \Longleftrightarrow \quad \exists\{\pi \in \$_n\}\ \gamma' = \gamma \circ \pi,$$

with

$$\$_n$$

the symmetric group of order n. Hence, diag is 1-1 on

$$SM_n := \{\sum_{\pi \in \$_n} f \circ \pi / n! : f \in M_n\},$$

the linear space of **symmetric** n-affine forms. We set

$$e_\alpha := \sum_{\pi \in \$_n} \gamma_\alpha \circ \pi,$$

with γ_α the unique monotone γ with $\operatorname{diag}(e_\gamma) = ()^\alpha$, to be specific, and note that it is possible for specific α to give a description involving fewer summands. To give an extreme example, $e_0 = e_0$, with the suffix on the left the d-sequence $(0,\ldots,0)$, and the suffix on the right the map $j \mapsto 0$. In general, there is a 1-1 correspondence between

$$I(n,\alpha) := \{\gamma : \operatorname{diag}(e_\gamma) = ()^\alpha\}$$

and the set

$$\binom{\{1,\ldots,n\}}{\alpha} := \{(S_i : i=1,\ldots,d) : S_i \subseteq \{1,\ldots,n\}, \#S_i = \alpha(i), i \neq j \Longrightarrow S_i \cap S_j = \emptyset\}$$

of all subpartitions of $\{1,\ldots,n\}$ with cardinalities as specified. Consequently,

$$e_\alpha = \sum_{\gamma \in I(n,\alpha)} e_\gamma / \#I(n,\alpha).$$

Definition. *The unique pre-image in SM_n of $p \in \Pi_n$ under* diag *is called the* **blossom** *or* **polar form** *(or, more precisely, the n-***blossom** *or n-***polar form***) for p and is denoted by*

$$p^\omega.$$

If f is any affine form on the flat, F, then the difference, $f(x) - f(y)$, depends only on the *vector* $\vec{v} := x - y$. Correspondingly, it is customary to define the corresponding *linear* form on the corresponding *vector space*, $F{-}F$, by the rule

$$f : F{-}F \to \mathbb{F} : \vec{v} \mapsto f(x+\vec{v}) - f(x),$$

with x an arbitrary element of F.

Lemma. *For any $\vec{v} \in F{-}F$ and any $p \in \Pi_n(F)$,*

$$(D_{\vec{v}}p)^\omega = np^\omega(\vec{v},\cdot).$$

Proof: For any $f \in M_n$ and any $\vec{v} := y - x \in F{-}F$,

$$\begin{aligned} f(y,\ldots,y) - f(x,\ldots,x) = \quad & f(y,\ldots,y,\vec{v}) \\ & + f(y,\ldots,y,\vec{v},x) \\ & \cdots \\ & + f(\vec{v},x,\ldots,x). \end{aligned}$$

Hence, for any $p \in \Pi_n(F)$, and using the symmetry of p^ω,

$$D_{\vec{v}}p(x) = \lim_{t\to 0} \frac{p(x+t\vec{v}) - p(x)}{t} = \lim_{t\to 0} \frac{p^\omega(x+t\vec{v},\ldots,x+t\vec{v}) - p^\omega(x,\ldots,x)}{t} = np^\omega(\vec{v},x\ldots,x).$$

Since $p^\omega(\vec{v},\cdot) \in SM_{n-1}$ and $D_{\vec{v}}p \in \Pi_{n-1}(F)$, the former must be the $(n-1)$-blossom of the latter. □

More generally, with

$$\vec{V} := (\vec{v}_j : j = 1, \ldots, r)$$

any sequence in $F{-}F$, set

$$D_{\vec{V}} := D_{\vec{v}_1} \cdots D_{\vec{v}_r}.$$

Corollary. *For any sequence* $\vec{V} := (\vec{v}_j : j = 1, \ldots, r)$ *in* $F{-}F$ *with* $r \le n$*, and any* $p \in \Pi_n(F)$*,*

$$(D_{\vec{V}} p)^\omega = \frac{n!}{(n-r)!} \, p^\omega(\vec{V}, \cdot).$$

This leads to simple necessary and sufficient conditions for a polynomial on a flat containing F to vanish r-fold on F. Such conditions are essential when joining polynomial patches across a piece of a flat, e.g., across some face of some simplex.

blossoms and BB-form

The BB-form is based on the observation that, for any maximal affinely independent set V in a given flat F, the map

$$\Pi_1(F) \to \mathbb{F}^V : p \mapsto (p(v) : v \in V)$$

is invertible. Its inverse is a column map,

$$[\ell_v : v \in V]$$

say. In other words, ℓ_v is the unique linear polynomial on F satisfying

$$\ell_v(w) = \delta_{vw}, \qquad w \in V.$$

Now, we may use $[\ell_v : v \in V]$ as a basis for $\Pi_1(F)$ instead of the power basis and, in this way, make explicit the beautiful connection between the BB-form and the blossom of a polynomial.

Here is the basic fact.

Proposition. *For any* $p \in \Pi_n$ *and any* $0 \le r \le n$*,*

$$p = \sum_{|\beta| = n-r} B_\beta \, p^\omega(V_\beta, \cdot, \ldots, \cdot),$$

with

$$B_\alpha := \binom{|\alpha|}{\alpha} \ell^\alpha$$

and

$$V_\beta := (\underbrace{v, \ldots, v}_{\beta(v) \text{ terms}} : v \in V).$$

Corollary. *For any* r*-sequence* $\vec{W} := (\vec{w}_1, \ldots, \vec{w}_r)$ *in* $F{-}F$*,*

$$D_{\vec{W}} p = \sum_{|\beta| = n-r} B_\beta \, p^\omega(V_\beta, \vec{W}).$$