Monosplines

Monosplines occur as representers of the error in quadrature rules (see, e.g., BirkhoffGD06, Peano14, Chakalov38a).

Let

$$I := \sum_{i=1}^{n} \sum_{j=1}^{\mu_i} c_{ij} \delta_{x_i} D^{j-1}$$

be a **quadrature rule of order** k **for** $f \mapsto \int_a^b f(t)\,\mathrm{d}t$, i.e.,

$$\max_i \mu_i < k, \quad a < x_1 < \cdots < x_n < b,$$

and

$$\int_a^b = I \quad \text{on} \quad \Pi_{<k}.$$

Then, for any function f with k derivatives on $[a\,..\,b]$,

$$f = \sum_{r<k} (\cdot - a)^r D^r f(a)/r! + \int_a^b (\cdot - s)^{k-1} D^k f(s)\,\mathrm{d}s/(k-1)!,$$

hence

$$\int_a^b f(t)\,\mathrm{d}t - If = \int_a^b M_I(s) D^k f(s)\,\mathrm{d}s,$$

with

$$M_I(s) := (b-s)^k/k! - \sum_{i=1}^{n} \sum_{j=1}^{\mu_i} c_{ij} (x_i - s)_+^{k-j}/(k-j)!$$

a spline of *order* k with knot sequence $(x_i^{[\mu_i]} : i = 1, \ldots, n)$ to which is added a polynomial of exact *degree* k. Any such is called a **monospline of degree** k **with knot sequence** $(x_i^{[\mu_i]} : i = 1, \ldots, n)$.

The monospline M_I has the additional property that

$$D^j M_i(t) = 0, \quad j < k, \quad t = a, b,$$

due to the fact that the quadrature rule I does not involve the endpoints a and b. If, more generally,

$$I := \sum_{t=a,b} \sum_{j \in J_t} c_{tj} \delta_t D^j + \sum_{i=1}^{n} \sum_{j=1}^{\mu_i} c_{ij} \delta_{x_i} D^{j-1},$$

then the corresponding monospline M_I is

$$M_I(s) := (b-s)^k/k! - p(s) - \sum_{i=1}^{n} \sum_{j=1}^{\mu_i} c_{ij} (x_i - s)_+^{k-j}/(k-j)!$$

with the polynomial p of order k so chosen that

$$D^j M_I(t) = 0, \quad j \in \{0, \ldots, k-1\} \backslash J_t, \quad t = a, b.$$

Source Chapter 7 of BojanovHakopianSahakian93

06mar04