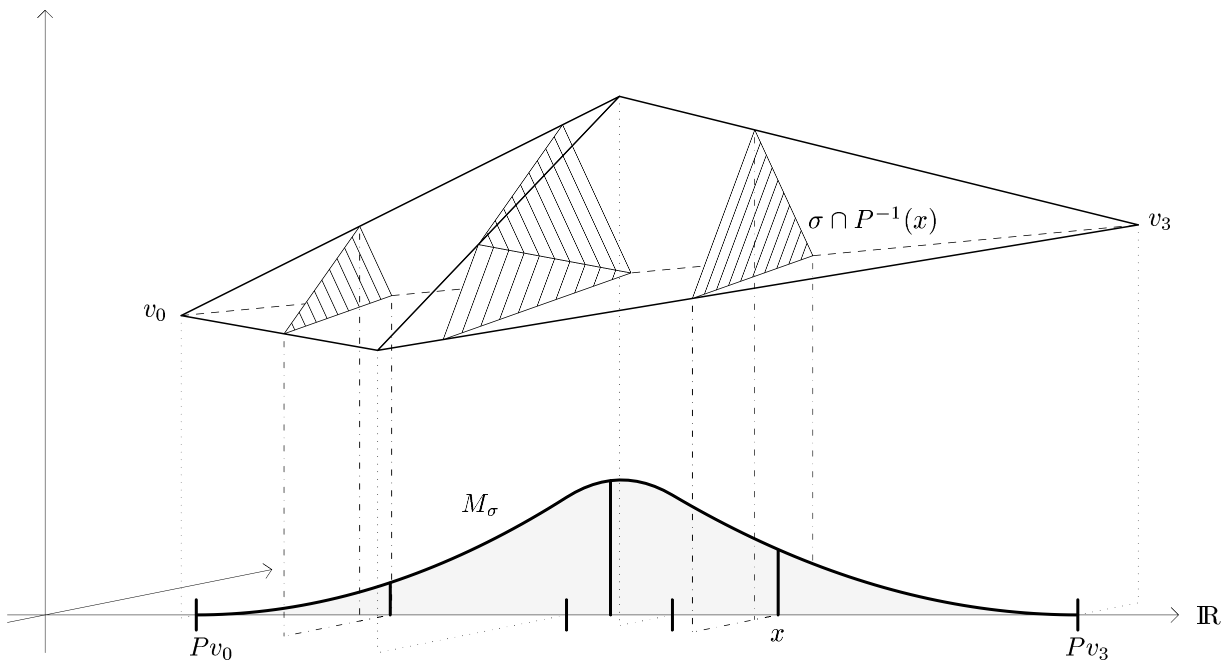
**Multivariate B-splines**

The definition of a multivariate B-spline as the $d$-dimensional 'shadow' of an $n$-dimensional polytope is based on Schoenberg's realization that a univariate B-spline is such a shadow, as is illustrated in (1)Figure.




(1)Figure. A quadratic B-spline as the shadow of a simplex

Here are the needed definitions. For any convex set $B$ in $\mathbb{R}^n$, and any function $\varphi$ defined at least on $B$, we define

$$\int_B \varphi := |\det V| \int_{V^{-1}B} \varphi V(a) \mathrm{d}a, \tag{2}$$

with $V : \mathbb{R}^r \to \mathbb{R}^n$ 1-1 affine and such that $\operatorname{ran} V = \flat(B)$, where

$$\flat(B)$$

denotes the **flat** spanned by $B$, i.e., the affine hull of $B$. Here, $|\det V|$ is, by definition, the $r$-dimensional volume of $V[0\,..\,1]^r$. Specifically, if

$$V : a \mapsto v_0 + [v_1, \ldots, v_r]a,$$

then

$$|\det V| = \sqrt{\det([v_1, \ldots, v_r]^*[v_1, \ldots, v_r])}.$$

The presence of the factor $|\det V|$ in (2) ensures that $\int_B \varphi$ is independent of the particular 1-1 affine map $V$ used in its definition (2); in particular, $\int_B ()^0 = \mathrm{vol}_r(B)$.

A **polytope** is the convex hull of a finite set.

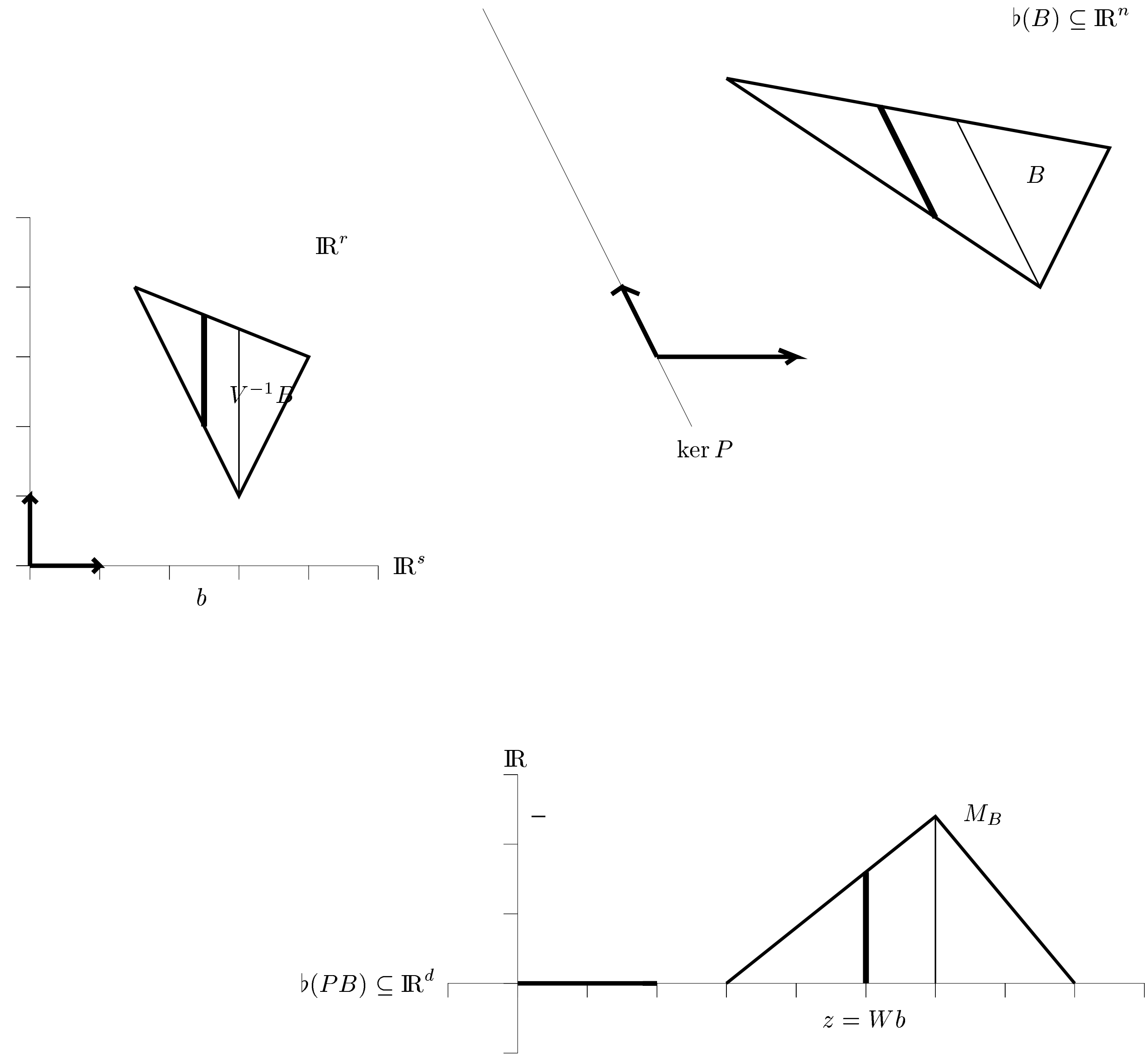
The $d$-dimensional B-spline determined from the polytope $B \subset \mathbb{R}^n$ by the affine map $P : \mathbb{R}^n \to \mathbb{R}^d$ is, by definition, the *distribution*

$$M_B : \varphi \mapsto \int_B \varphi P. \tag{3}$$

Equivalently, it is the *function* $M_B$ defined on $PB$ by the rule that

$$\int_B \varphi P = \int_{PB} M_B \varphi, \quad \forall \varphi \in C_0(\mathbb{R}^d).$$

Here is a formula for the function $M_B$.




(4)Figure. The three spaces involved in the formula for $M_B$

After translation of the origin in $\mathbb{R}^n$, we may assume that $\flat(B)$ is a linear subspace, of dimension $r$, hence $V = [v_1, \ldots, v_r]$. Also, after translation of the origin in $\mathbb{R}^d$, we may assume that $P$ is linear. We also may choose $V$ so that $[v_{s+1}, \ldots, v_r]$ is a basis for $\ker P \cap \flat(B)$. Then, $W := P[v_1, \ldots, v_s]$ is a basis for $\flat(PB)$, and, with $a = (b, c) \in \mathbb{R}^s \times \mathbb{R}^{r-s} = \mathbb{R}^r$, hence $P(Va) = Wb$,

$$\begin{aligned}\int_B \varphi P &= |\det V| \int_{V^{-1}B} \varphi(P(Va))\,\mathrm{d}a = |\det V| \int_{V^{-1}B} \varphi(Wb)\,\mathrm{d}c\,\mathrm{d}b \\ &= |\det V| \int_{W^{-1}(PB)} \varphi(Wb)\,\mathrm{vol}_{r-s}(V^{-1}B \cap \{(b,c) \in \mathbb{R}^r : c \in \mathbb{R}^{r-s}\})\,\mathrm{d}b \\ &= \left|\frac{\det V}{\det W}\right| \int_{PB} \varphi g,\end{aligned}$$

with

$$\begin{aligned} g(z) &= \operatorname{vol}_{r-s}(V^{-1}B \cap \{x \in \mathbb{R}^r : W(x_1,\dots,x_s) = z\}) \\ &= \operatorname{vol}_{r-s} V^{-1}(B \cap P^{-1}\{z\}) \\ &= \operatorname{vol}_{r-s}(B \cap P^{-1}\{z\}) / |\det[v_{s+1},\dots,v_r]|, \end{aligned}$$

the last equality since $\operatorname{ran}[v_{s+1},\dots,v_r] = \ker P$. Therefore, with the additional requirement that

$$\operatorname{ran}[v_1,\dots,v_s] \perp \ker P$$

(which implies that $|\det V| = |\det[v_1,\dots,v_s]\det[v_{s+1},\dots,v_r]|$), we have, altogether,

(5) $$M_B(z) = \left|\frac{\det[v_1,\dots,v_s]}{\det[Pv_1,\dots,Pv_s]}\right| \operatorname{vol}_{r-s}(B \cap P^{-1}\{z\}).$$

It follows that $M_B$ is in $L_\infty(\mathbb{R}^d)$ exactly when $PB$ has interior.

While this formula tells us that $M_B$ can be considered a function on $PB$ (and this was, indeed, how multivariate B-splines were first defined), the formulation (3) as a distribution provides mathematical efficiency.

As an illustration, consider the simplex spline

$$M(\cdot|\Theta) : \varphi \mapsto n! \int_\Theta \varphi$$

with

$$\int_\Theta : \varphi \mapsto := \int_0^1 \int_0^{s_1} \cdots \int_0^{s_{n-1}} \varphi(\theta_0 + s_1\nabla\theta_1 + \cdots + s_n\nabla\theta_n)\mathrm{d}s_n \cdots \mathrm{d}s_1$$

the **Genocchi functional** for the sequence $\Theta =: (\theta_0,\dots,\theta_n)$ in $\mathbb{R}^d$. Set

$$P : \mathbb{R}^n \to \mathbb{R}^d : x \mapsto \theta_0 + [\theta_i - \theta_{i-1} : i = 1,\dots,n]x =: \theta_0 + P_0 x.$$

Then

$$\int_\Theta \varphi = \int_\sigma \varphi P,$$

with the simplex $\sigma := \{x \in \mathbb{R}^n : 1 \ge x_1 \ge \cdots \ge x_n \ge 0\}$ mapped by $P$ onto $\operatorname{conv}(\Theta)$. Therefore, for any basis $[v_1,\dots,v_s]$ for $(\ker P)\perp$,

$$M(z|\Theta) = n! \left|\frac{\det[v_1,\dots,v_s]}{\det[P_0v_1,\dots,P_0v_s]}\right| \operatorname{vol}_{r-s}(\sigma \cap P^{-1}\{z\}).$$

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