a multivariate Newton form

Here are the facts. Given a coefficient vector a, 'centers' $c_1, \ldots, c_k \in \mathbb{R}^d$, and some $x \in \mathbb{R}^d$, the algorithm

$$
\begin{aligned}
&b(\alpha) \leftarrow a(\alpha), \quad |\alpha| = k \\
&\texttt{for } j = k{:}-1{:}1 \\
&\quad b(\alpha) \leftarrow a(\alpha) + \textstyle\sum_{i=1}^d b(\alpha + \mathbf{i}_i)(x - c_j)(i), \quad |\alpha| = j - 1 \\
&\texttt{end}
\end{aligned}
$$

returns, in $b(0)$, the value of a certain polynomial. Since each coefficient $a(\alpha)$ enters the calculation exactly once, we can trace its contribution to $b(0)$:

$$b(0) = \sum_{|\alpha| \le k} a(\alpha) \sum_{i \in \{1,\ldots,d\}^{|\alpha|}} \prod_{j=1}^{|\alpha|} (x - c_j)(i(j)).$$

In particular, if all the c_j are equal, to c, say, then

$$b(0) = \sum_{|\alpha| \le k} a(\alpha) \binom{|\alpha|}{\alpha} (x - c)^\alpha =: p(x).$$

In other words, in this case, the algorithm produces the value at x, of the polynomial of degree $\le k$ whose coefficients in its normalized power form, centered at c, are provided by a.

More than that, the various intermediate results $b(\alpha)$ form the coefficients for the same polynomial, but with the center sequence $x, c_1, \ldots, c_{k-1}$. Also, if already $c_1 = \cdots = c_r = x$, then the steps $j = 1{:}r$ can be omitted since then all the factors $(x - c_j)(i)$ for $j = 1{;}r$ are zero. Hence, the following will produce the coefficients of this polynomial in the normalized power form centered at x:

$$
\begin{aligned}
&\texttt{for } m = 1{:}k \\
&\quad \texttt{for } j = k{:}-1{:}m \\
&\qquad a(\alpha) \leftarrow a(\alpha) + \textstyle\sum_{i=1}^d a(\alpha + \mathbf{i}_i)(x - c_j)(i), \quad |\alpha| = j - 1, \\
&\quad \texttt{end} \\
&\texttt{end}
\end{aligned}
$$

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