Nonuniqueness is free-knot spline approximation(C. de Boor, '62)

Here is a very simple example of nonuniqueness in free-knot spline approximation in smooth norms.

Consider best approximation on $[-1 \mathinner{..} 1]$ from piecewise constant functions with 1 break, ξ say, to an *even* function, f, in the 2-norm. Let s be a best approximation. Then also $x \mapsto s(-x)$ is a best approximation, hence, if best approximation were unique, s would have to be even. But any even piecewise constant with just one break is necessarily a constant, hence it would follow that $f - s$ is perpendicular to $x \mapsto (x - \xi)_+^0$ for all $\xi \in [-1 \mathinner{..} 1]$, leading to the conclusion that $f = s$ and, in particular, f is a constant. Conclusion: Any nonconstant even f has more than one best approximation by piecewise constants with just one break.

The argument is easily extended to best approximation from a spline space of arbitrary order with just one simple knot.

Since best approximation from a smooth manifold with respect to a smooth norm, such as L_2, is unique for any f close enough to the manifold, the above gives simple evidence for the non-smooth character of a free-knot spline space.