Genocchi-Hermite formula

[Norlund24:p.16] mistakenly attributes the (univariate) Genocchi-Hermite formula

$$\Delta(x_0,\ldots,x_n)f = \int_0^1\int_0^{s_1}\cdots\int_0^{s_{n-1}} D^n f((1-s_1)x_0+\cdots+(s_{n-1}-s_n)x_{n-1}+s_n x_n)\,\mathrm{d}s_n\cdots\,\mathrm{d}s_1$$

$$=: \int_{[x_0,\ldots,x_n]} D^n f$$

to [Hermite59a], possibly because that paper carries the suggestive title "Sur l'interpolation".

At the end of a paper ([Hermite78a]) on polynomial interpolation to data at the n points $a_1,\ldots,a_n$ in the complex plane, Hermite does give a formula involving the righthand-side of the above, namely the formula

$$f(x) - Pf(x) = (x-a_1)\cdots(x-a_n)\int_{[a_n,\ldots,a_1,x]} D^n f.$$

Thus, it requires the observation that

$$f(x) - Pf(x) = (x-a_1)\cdots(x-a_n)\Delta(a_n,\ldots,a_1,x)f$$

to deduce the (Hermite-)Genocchi formula from [Hermite78a]. (He also gives the rather more complicated formula

$$f(x) - Pf(x) = (x-a_1)^\alpha\cdots(x-a_n)^\lambda\int_{[a_n,\ldots,a_1,x]} [\![s_n - s_{n-1}]\!]^{\alpha-1}\cdots[\![1-s_1]\!]^{\lambda-1}D^{\alpha+\cdots+\lambda}f$$

for the error in case of repeated interpolation.)

In contrast, Genocchi [Genocchi69a] is explicitly concerned with a representation formula for the divided difference. However, the 'divided difference' he represents is the following:

$$\Delta(x,x+h_1)\Delta(\cdot,\cdot+h_2)\cdots\Delta(\cdot,\cdot+h_n) = (\Delta_{h_1}/h_1)\cdots(\Delta_{h_n}/h_n)$$

and for it he gets the representation

$$\int_0^1\cdots\int_0^1 D^n f(x+h_1t_1+\cdots+h_nt_n)\,\mathrm{d}t_1\cdots\,\mathrm{d}t_n.$$

[Norlund24:p.16] cites [Genocchi78a,b] as places where formulations equivalent to the (Hermite-)Genocchi formula can be found. So far, I've been only able to find [Genocchi78b]. It is a letter to Hermite, in which Genocchi brings, among other things, the above representation formula to Hermite's attention, refers to a paper of his in [Archives de Grunert, t. XLIX, 3e cahier] as containing a corresponding error formula for Newton interpolation. He states that he, in continuing work, had obtained such a representation also for Ampére's fonctions interpolatoires (aka divided differences), and finishes with the formula

$$\int_0^1 \cdot\cdot \int_0^1 s_1^{n-1} s_2^{n-2} \cdots s_{n-1} D^n f(x_0 + s_1(x_1 - x_0) + \cdots + s_1 s_2 \cdots s_n(x_n - x_{n-1}))\, \mathrm{d}s_1 \cdots \mathrm{d}s_n.$$

for $\Delta(x_0, \ldots, x_n)f$, and says that it is equivalent to the formula

$$\Delta(x_0, \ldots, x_n)f = \int \cdots \int D^n f(s_0 x_0 + s_1 x_1 + \cdots s_n x_n)\, \mathrm{d}s_1 \cdots \mathrm{d}s_n$$

in which the conditions $s_0 + \cdots + s_n = 1$, $s_i \geq 0$, all i, are imposed.

[Steffensen27:17f] proves the Genocchi-Hermite formula but calls it Jensen's formula, because of Jensen94, and gives RungeWillers2x as a further source.

[Potra, F.-A., Ptak, V; Nondiscrete induction and iterative processes; Pitman (Boston); 1984;:p.190] use, without attribution but, as they say, in analogy with

$$\Delta(x, y)f = \int_0^1 Df(x + t(y - x))\, \mathrm{d}t,$$

the first of these last-mentioned two formulae. In fact, they use this for functions of many variables, in which case $D^n f$ is meant to be the n-linear map of n-fold differentiation.