divided differences: basic formulas

formula for simple nodes:

$$\Delta(t_1, \ldots, t_n) = \sum_{j=1}^{n} \Delta(t_j)/(D\omega)(t_j),$$

with $\omega := \omega_t := \prod_k (\cdot - t_k)$, as the right side is easily computed as the leading coefficient from the Lagrange form. For multiple nodes, see Chakalov's formula below.

recurrence:

$$\Delta(t_1, \ldots, t_n) = \Delta(t_0, \ldots, t_{n-1}) + (t_n - t_0)\Delta(t_0, \ldots, t_n).$$

This is a direct consequence of the interpretation of Horner's method as a means to construct the coefficients in the Newton form with centers $z = t_0, t_1, t_2, \ldots$ from those for the Newton form with centers $t_1, t_2, \ldots$.

change of variable: For any affine polynomial $\ell : s \mapsto as + b$,

$$\Delta(\ell(t_1), \ldots, \ell(t_n))f = a^{1-n}\Delta(t_1, \ldots, t_n)(f \circ \ell),$$

as follows directly from the fact that $\Pi_{<n}$ is invariant under any such change of variables.

dvd of dvd:

$$\Delta(a, \ldots, z)\Delta(\cdot, \alpha, \ldots, \omega) = \Delta(a \ldots, z, \alpha, \ldots, \omega).$$

Indeed, by the symmetry and recurrence, $\Delta(a, b)\Delta(\cdot, \alpha, \ldots, \omega) = \Delta(a, b, \alpha, \ldots, \omega)$; now use induction.

mean value: By Rolle, for any $t_0 < \cdots < t_n$ and smooth enough f,

$$n\Delta(t_0, \ldots, t_n)f = \Delta(\xi_0, \ldots, \xi_{n-1})Df$$

for some $\xi_i \in (t_i \,..\, t_{i+1})$, all i. Hence, by induction,

$$n!\Delta(t_0, \ldots, t_n) = D^n f(\xi)$$

for some ξ in the smallest interval containing the t_j.

refinement (Popoviciu33=34a): For any $t_0, \ldots, t_n$ and ξ,

$$(t_n - t_0)\Delta(t_0, \ldots, t_n) = (\xi - t_0)\Delta(t_0, \ldots, t_{n-1}, \xi) + (t_n - \xi)\Delta(\xi, t_1, \ldots, t_n).$$

Indeed, by continuity of the divided difference, it is sufficient to consider the case where t_0, t_n, ξ are pairwise distinct. In that case and by the recurrence, the right side works out to

$$\Delta(t_1, \ldots, t_{n-1}, \xi) - \Delta(t_0, \ldots, t_{n-1}) + \Delta(t_1, \ldots, t_n) - \Delta(\xi, t_1, \ldots, t_{n-1}),$$

hence, after cancellation of the extreme terms, evidently equals the left side. Inductive application of this formula proves: *For any increasing refinement s of the sequence $t_0 \leq \cdots \leq t_n$,*

$$\Delta(t_0, \ldots, t_n) = \sum_j a_j(t, s)\Delta(s_j, \ldots, s_{j+n}),$$

with the $a_j(t, s)$ nonnegative and summing to 1.

Leibniz:

$$\Delta(t_0, \ldots, t_k)(fg) = \sum_{j=0}^{k} \Delta(t_0, \ldots, t_j)f \;\Delta(t_j, \ldots, t_k)g$$

General Leibniz:

For $f : \mathbb{R}^m \to \mathbb{R}$,

$$\Delta(t_0, \ldots, t_k)f(\cdot, \ldots, \cdot) = \sum_{0=i(0)\leq\cdots\leq i(m)=k} \left(\otimes_{j=1}^{m} \Delta(t_{i(j-1)}, \ldots, t_{i(j)})\right) f$$

Proof: By induction on m, with the case $m = 2$ done Kammler's way: The polynomial

$$\sum_{i=0}^{k}\sum_{j=0}^{k}\left(\prod_{r<i}(\cdot - t_r)\right)\left(\prod_{s>j}(\cdot - t_s)\right)\Delta(t_0, \ldots, t_i) \otimes \Delta(t_j, \ldots, t_k)f$$

agrees with $t \mapsto f(t, t)$ at the points $t_0, \ldots, t_k$, but all the terms with j less than i vanish at all the t_i, hence already the polynomial

$$\sum_{i=0}^{k}\sum_{j=i}^{k}\left(\prod_{r<i}(\cdot - t_r)\right)\left(\prod_{s>j}(\cdot - t_s)\right)\Delta(t_0, \ldots, t_i) \otimes \Delta(t_j, \ldots, t_k)f$$

agrees with $t \mapsto f(t, t)$ at the t_i, and is a polynomial of degree $\leq k$, hence its leading coefficient, $\sum_i \Delta(t_0, \ldots, t_i) \otimes \Delta(t_i, \ldots, t_k)f$, is the divided difference of interest. □

Actually, this is a special case of the fact that polynomial interpolation is an ideal projector (meaning that its kernel is an ideal or, equivalently, $P(fg) = P(fPg)$; see Boor03), hence one has

Opitz' formula (see Opitz64): *For any polynomial p (hence also for any p that is the limit of polynomials),*

$$p(A_t) = (\Delta(t_i, \ldots, t_j)p : i, j = 1{:}n)$$

with $\Delta(t_i, \ldots, t_j) = 0$ in case $i > j$ and

$$A_t := (\Delta(t_i, \ldots, t_j)()^1 : i, j = 1{:}n).$$

In particular, $(pq)(A_t) = p(A_t)q(A_t)$, whence Leibniz.

powers (see, e.g., Steffensen27:pp19ff):

$$\Delta(t_0, \ldots, t_k)()^n = \begin{cases} \sum(t^\alpha : |\alpha| = n - k, \alpha \ge 0), & n \ge 0; \\ (-)^n \sum((-t)^\alpha : |\alpha| = n - k, \alpha \le -e), & n < 0; \end{cases}$$

Here, $|\alpha| := e * \alpha = \sum_j \alpha_j$, $e := (1, \ldots, 1)$, and $t^\alpha := \prod_j t_j^{\alpha_j}$.

Proof: By Leibniz and induction,

$$\Delta(t_0, \ldots, t_k)()^{n+1} = (\Delta(t_0, \ldots, t_k)()^n)t_0 + \Delta(t_1, \ldots, t_k)()^n =$$

$$= \sum(t^\alpha : |\alpha| = n + 1 - k; \alpha_0 > 0) + \sum(t^\alpha : |\alpha| = n + 1 - k; \alpha_0 = 0).$$

$0 = \Delta(t_0, \ldots, t_k)()^{-1}()^1 = \Delta(t_0, \ldots, t_{k-1})()^{-1} + \Delta(t_0, \ldots, t_k)()^{-1}t_k$, hence

$$\Delta(t_0, \ldots, t_k)()^{-1} = \Delta(t_0, \ldots, t_{k-1})()^{-1}/(-t_k) = \Delta(t_0)()^{-1}/((-t_1)\cdots(-t_k)) = -(-t)^{-e}.$$

$$\Delta(t_0, \ldots, t_k)()^{-n-1} = \sum_j \Delta(t_0, \ldots, t_j)()^{-n}\,\Delta(t_j, \ldots, t_k)()^{-1} =$$

$$= \sum_j (-1)^n \sum((-t)^\alpha : |\alpha| = -n-j; \alpha_i < 0, i \le j; \alpha_i = 0, i > j)(-1)(-t_j)^{-1}\cdots(-t_k)^{-1} =$$

$$= (-1)^{n+1} \sum_j \sum((-t)^\alpha : |\alpha| = -n - 1 - k; \alpha \le -e; \alpha_j < -1; \alpha_i = -1, i > j)$$

This implies

Chakalov (see Chakalov38a,b):

$$\Delta(t_0,\ldots,t_k)(z-\cdot)^{-1} = 1/\omega_t(z) := 1/\prod_j (z-t_j).$$

Therefore, with $\#_t\xi := \#\{j : \xi = t_j\}$ the multiplicity with which ξ occurs in the sequence t, and

$$1/\omega_t(z) =: \sum_{\xi\in t}\sum_{0\le\mu<\#_t\xi} \frac{\mu! A_{\xi\mu}}{(z-\xi)^{\mu+1}}$$

the partial fraction expansion of $1/\omega_t$, we obtain Chakalov's expansion

$$\Delta(t_0,\ldots,t_k)f = \sum_{\xi\in t}\sum_{0\le\mu<\#_t\xi} A_{\xi\mu} D^\mu f(\xi),$$

directly for $f := 1/(z-\cdot)$ for arbitrary z since $D^\mu 1/(z-\cdot) = \mu!/(z-\cdot)^{\mu+1}$, hence, by the density of $\{1/(z-\cdot) : z\}$, for any smooth enough f.

This is an illustration of the peculiar effectiveness of the formula for the divided difference of $1/(z-\cdot)$ for deriving and verifying divided difference identities.

As an example, one may verify, e.g., by induction on m and n and using the fact that

$$\sum_{j=k}^{m} \binom{m+r-1-j}{r-1} = \binom{m+r-k}{r},$$

that, for any $m, n \in \mathbb{N}$ and any z,

$$\frac{1}{z^m(1-z)^n} = \sum_{k=1}^{m} \binom{n+m-1-k}{m-k}/z^k + \sum_{j=1}^{n} \binom{m+n-1-j}{n-j}/(1-z)^j.$$

By Chakalov's formula, this implies that

$$(-1)^n \Delta(0^{[m]}, 1^{[n]}) = \sum_{k=1}^{m} \binom{n+m-1-k}{m-k} \Delta(0^{[k]}) + \sum_{j=1}^{n} \binom{m+n-1-j}{n-j} (-1)^j \Delta(1^{[j]}).$$

By the change of variables $t = (y-x)s + x$, we have therefore, more generally, for any $x \ne y$,

$$(-1)^n (y-x)^{m+n-1} \Delta(x^{[m]}, y^{[n]}) = \sum_{k=1}^{m} \binom{n+m-1-k}{m-k} (y-x)^{k-1} \Delta(x^{[k]}) - \sum_{j=1}^{n} \binom{m+n-1-j}{n-j} (x-y)^{j-1} \Delta(y^{[j]}).$$

integral representation (Peano kernel): Application of the divided difference to the Taylor identity

$$f(x) = \sum_{j=0}^{n-1} D^j f(a)(x-a)^j/j! + \int_a^x (x-s)^{n-1} D^n f(s)\,\mathrm{d}s/(n-1)!$$

gives

$$\Delta(t_0,\ldots,t_n)f = \int M(\cdot|t_0,\ldots,t_n)D^n f/n!,$$

with $M(s|t_0,\ldots,n) := n\Delta(t_0,\ldots,t_n)(\cdot - s)_+^{n-1}$ the B-spline with knots $t_0,\ldots,t_n$ that is normalized to integrate to 1.

lower bound on the norm: For $-1 \le t_0 < \cdots < t_n \le 1$,

$$\|\Delta(t_0,\ldots,t_n) : C([-1\,..\,1]) \to \mathbb{F}\| = \sum_j 1/|D\omega(t_j)| \ge 2^{n-1},$$

with equality iff $\omega = (()^2 - 1)U_{n-1}$, where U_{n-1} is the second-kind Chebyshev polynomial. This appears in the proof of Lemma I of ErdosTuran40. Correspondingly, the minimizing site sequence also provides the knots for the perfect B-spline (qv).

Here is a quick proof: for any such $t = (t_0,\ldots,t_n)$, the restriction λ of $\Delta(t)$ to Π_n is the unique linear functional on Π_n that vanishes on $\Pi_{<n}$ and takes the value 1 at $()^n$, hence takes its norm on the error of the best (uniform) approximation to $()^n$ from $\Pi_{<n}$, i.e., on the Chebyshev polynomial of degree n. Each such $\Delta(t)$ is an extension of this λ, hence has norm $\ge \|\lambda\| = 1/\operatorname{dist}(()^n, \Pi_{<n}) = 2^{n-1}$, with equality iff $\Delta(t)$ takes on its norm on that Chebyshev polynomial, i.e., iff t is the sequence of extreme points of that Chebyshev polynomial.

divided difference expansion:

By applying $\Delta(t_1,\ldots,t_k)$ to both sides of the Newton polynomial with remainder

$$\Delta(y) = \sum_{j=1}^{m} \psi_{j-1}(y)\Delta(s_1,\ldots,s_j) + \psi_m(y)\Delta(s_1,\ldots,s_m,y)$$

with

$$\psi_n(y) := (y-s_1)\cdots(y-s_n),$$

one obtains the expansion

$$\Delta(t_1,\ldots,t_k) = \sum_{j=k}^{m} \Delta(t_1,\ldots,t_k)\psi_{j-1}\Delta(s_1,\ldots,s_j) + E(t,s),$$

where, by Leibniz,

$$E(t,s) := \Delta(t_1,\ldots,t_k)(\psi_m \Delta(s_1,\ldots,s_m,\cdot))$$
$$= \sum_{i=1}^{k} \Delta(t_i,\ldots,t_k)\psi_m \Delta(t_1,\ldots,t_k,s_1,\ldots,s_m).$$

But, following Floater03 and with $p := m-k$, one gets the better formula

$$E(t,s) := \sum_{i=1}^{k}(t_i - s_{i+p})(\Delta(t_1,\ldots,t_i)\psi_{i+p})\Delta(s_1,\ldots,s_{i+p},t_i,\ldots,t_k)$$

in which all the divided differences on the right side are of the same order, m. The proof, by induction on m, uses the easy consequence of Leibniz that

$$(t_i - y)\Delta(t_i,\ldots,t_k)f = \Delta(t_i,\ldots,t_k)(\cdot - y)f - \Delta(t_{i+1},\ldots,t_k)f$$

whose special case $y = t_i$ was put to good use in Hakopian82a and in LeeETY89a. The induction is anchored at $m = k$ for which the formula

$$\Delta(t_1,\ldots,t_k) - \Delta(s_1,\ldots,s_k) = \sum_{i=1}^{k}(t_i - s_i)\Delta(t_1,\ldots,t_i,s_i,\ldots,s_k)$$

can already be found in Hopf26.

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