definition of divided difference

By `pagep111.pdf`: "Newton form", for every scalar sequence $t_0, t_1, \ldots$, every polynomial p can be written in exactly one way in the form

$$p = \sum_k a_k(p,t) \prod_{0\le j<k} (\cdot - t_j).$$

Further, by `pagep112.pdf`: "Polynomial interpolation: existence and uniqueness",

$$P_n p = \sum_{k=0}^{n} a_k(p,t) \prod_{0\le j<k} (\cdot - t_j)$$

is the unique polynomial of degree $\le n$ for which $p - P_n p$ is divisible by $\prod_{j=0}^{n}(\cdot - t_j)$. This implies, in particular, that the scalar $a_n(p,t)$ depends only on p and $t_0, \ldots, t_n$, and this is reflected in any of the many notations for it:

$$a_n(p,t) =: p(t_0,\ldots,t_n) =: p[t_0,\ldots,t_n] =: [t_0,\ldots,t_n]p =: \delta_{t_0,\ldots,t_n} p =: \ldots.$$

I prefer W. Kahan's literal symbol *) (a cap Delta with a vertical bar dividing it), and so set

$$a_n(p,t) =: \Delta\!\!\!|\,(t_0,\ldots,t_n)p$$

since $a_n(p,t)$ is, by definition, the divided difference of p at $t_0, \ldots, t_n$.

Thus

$$p =: \sum_{k=0}^{\infty} \Delta\!\!\!|\,(t_0,\ldots,t_k) \prod_{0\le j<k} (\cdot - t_j).$$

For more general functions f, $\Delta\!\!\!|\,(t_0,\ldots,t_n)f$ is defined analogously as the leading coefficient, i.e., the coefficient of $()^n$ in the power form, for the polynomial $P_n f$ of degree $\le n$ that agrees with f at $t_0, \ldots, t_n$ in the sense that $f - P_n f = (\cdot - t_0)\cdots(\cdot - t_n)q$ for some well-defined function q.

An early good reference (but missing the Leibniz formula and misattributing the Genocchi-Hermite formula) is Steffensen27a.

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*) `\def\dvd#1{\mathord{\kern.43em\vrule width.6pt height5.95pt depth.28pt\kern-.43em\Delta(#1)}}`