divided difference: basic properties

Since the divided difference is defined as a particular coefficient in a particular Newton form, it is a linear functional.

More explicitly, since $\Delta(t_0,\ldots,t_n)f$ is the leading coefficient in the polynomial of degree $\le n$ that matches f at $t_0,\ldots,t_n$, it is linear in f and symmetric in the t_j's.

For the same reason, $\Delta(t_0,\ldots,t_n)$ is a particular linear combination of the linear functionals

$$(\delta_z D^r : 0 \le r < \#\{0 \le j \le n : t_j = z\}, z \in \mathbb{F}).$$

Let

$$w_k := (\cdot - t_0)\cdots(\cdot - t_{k-1}), \quad k = 0, 1, 2, \ldots,$$

and let $\mathbb{F}_0^{\mathbb{N}}$ be the collection of all infinite scalar sequences with all but finitely many entries nonzero. Then the map

$$W_t : \mathbb{F}_0^{\mathbb{N}} \to \Pi : a \mapsto \sum_{k=0}^{\infty} a_k w_k$$

is linear, 1-1 and onto, hence invertible, and

$$W_t^{-1} : p \mapsto (\Delta(t_0,\ldots,t_k)p : k = 0, 1, \ldots), \quad p \in \Pi.$$

Since $t \mapsto W_t$ is infinitely differentiable, so is $t \mapsto W_t^{-1}$. In particular, $(a,\ldots,z) \mapsto \Delta(a,\ldots,z)$ is infinitely differentiable, at least as a map into the linear functionals on Π. By the recurrence (see `pagep101.pdf`: "divided differences: basic formulas"),

$$\Delta(a, a+h)\Delta(\cdot, b, \ldots, z) = \Delta(a, a+h, b, \ldots, z) \xrightarrow[h\to 0]{} \Delta(a, a, b, \ldots, z),$$

thus providing a ready formula for differentiating the divided difference as a function of one of its nodes.

23-24mar03