Newton form

For any $p \in \Pi_n$ and any scalar sequence $c_1, \ldots, c_n$, there exists exactly one choice of coefficients $a_0, \ldots, a_n$ so that

$$p = \sum_{k=0}^{n} a_k \prod_{j=1}^{k} (\cdot - c_j).$$

This is the Newton form with centers $c_1, \ldots, c_n$ for p, and its existence and uniqueness readily follows from the fact that the sequence

$$(w_k := \prod_{j=1}^{k} (\cdot - c_j) : k = 0, \ldots, n)$$

is linearly independent and in the $n+1$-dimensional linear space Π_n, hence a basis for it.

The efficient evaluation of the Newton form is by nested multiplication aka Horner's method. In this method, one takes advantage of the fact that $w_k = (\cdot - c_k) w_{k-1}$, all k, to write p in nested form,

$$p = a_0 + (\cdot - c_1)\left(a_1 + (\cdot - c_2)\left(\cdots + (\cdot - c_{n-1})\left(a_{n-1} + (\cdot - c_n) a_n\right) \cdots\right)\right),$$

and then, for any particular scalar z, obtains $p(z)$ by evaluating this nested expression from the inside out. This gives the following algorithm.

$$b_n := a_n;\ \textbf{for}\ k = n{-}1{:}{-}1{:}0,\ b_k := a_k + (z - c_{k+1}) b_{k+1};\ \textbf{endfor}$$

with b_0 equal to $p(z)$.

More than that, since, in this way, $a_k = b_k + (c_{k+1} - z) b_{k+1}$ for $k < n$ and $a_n = b_n$, it follows that

$$p = \sum_{k=0}^{n} a_k w_k = b_0 + (\cdot - z) \sum_{j=1}^{n} b_j w_{j-1}.$$

In other words, the sequence $b_0, \ldots, b_n$ generated in Horner's algorithm provides the coefficients in the Newton form for p with centers $z, c_1, \ldots, c_{n-1}$.

This also makes clear why Horner's method can be (and has been) thought of as an algorithm for dividing p by the linear polynomial $(\cdot - z)$.

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