Polynomial interpolation: existence, uniqueness

Let $t_0, \ldots, t_n$ be an arbitrary scalar sequence and extend it, in any manner whatsoever, to an infinite scalar sequence. Then, by `pagep111.pdf`: "Newton form", every $f \in \Pi$ can be written in exactly one way in the form

$$f = \sum_{k=0}^{\infty} a_k(f,t) \prod_{j<k} (\cdot - t_j)$$

with the sum actually finite since $a_k(f,t) = 0$ for all $k > \deg p$.

It follows that, with

$$w_k := \prod_{j=0}^{k-1} (\cdot - t_j),$$

the map

$$P_n : f \mapsto \sum_{k=0}^{n} a_k(f,t) w_k$$

is well-defined on Π, linear, and maps Π into Π_n. Further,

$$f - P_n f = w_{n+1} q$$

for some polynomial q. If also $p - g = w_{n+1} r$ for some $g \in \Pi_n$ and some $r \in \Pi$, then $P_n f - g$ is a polynomial of degree $\le n$ and divisible by w_{n+1}, hence must be zero.

It follows that P_n is a linear projector with range Π_n, and $P_n f$ is the unique polynomial of degree $\le n$ for which $f - P_n f$ is divisible by $w_{n+1} = (\cdot - t_0) \cdots (\cdot - t_n)$. But such divisibility is equivalent to the requirement that

$$D^r (f - P_n f)(z) = 0, \quad 0 \le r < \#\{0 \le i \le n : t_i = z\}, \tag{1}$$

i.e., to interpolation, in particular to repeated, or osculatory, or Hermite, interpolation in case of coincidences among the t_i.

This suggests the standard extension of P_n to all sufficiently smooth f, namely defining $P_n f$ to be the unique polynomial of degree $\le n$ that satisfies the interpolation conditions (1).

23-24mar03