The Kowalewski error formula

Apply the linear projector $P_n : f \mapsto \sum_{j=0}^{n} \ell_j f(t_j)$, of polynomial interpolation at the distinct sites $t_0, \dots, t_n$, to the Taylor identity

$$f(y) = \sum_{j\le k} (y-x)^j D^j f(x)/j! + \int_x^y (y-t)^k D^{k+1} f(t)\,\mathrm{d}t/k!$$

as a function of y, getting

$$P_n f = \sum_{j\le k} P_n(\cdot - x)^j D^j f(x)/j! + P_n F_k(x, \cdot, D^{k+1} f),$$

with

$$F_k(x,y,g) := \int_x^y (y-t)^k g(t)\,\mathrm{d}t/k!.$$

If now $k \le n$, then $P_n(\cdot - x)^j = (\cdot - x)^j$ for all $j \le k$, hence then

(1) $$P_n f(x) = f(x) + P_n F_k(x, \cdot, D^{k+1} f)(x).$$

To be sure, Kowalewski32a (see pp21–24) only considers the case $k = n$, and then gets

$$f(x) = P_n f(x) + \sum_{j=0}^{n} \ell_j(x) \int_{t_j}^{x} (t_j - t)^n D^{n+1} f(t)\,\mathrm{d}t/n!.$$

But if $k < n$, then, for any polynomial p of degree $< n-k$, we can find a polynomial f of degree $\le n$ for which $D^{k+1} f = p$, and for such f, $P_n f(x) = f(x)$, hence, by (1), $P_n F_k(x, \cdot, p)(x) = 0$. Thus,

$$f(x) = P_n f(x) + \sum_{j=0}^{n} \ell_j(x) \int_{t_j}^{x} (t_j - t)^k (D^{k+1} f - p)(t)\,\mathrm{d}t/k!, \quad \forall p \in \Pi_{<n-k}.$$

With $w_n := (\cdot - t_0)\cdots(\cdot - t_n)$, this error term can also be written

$$w_n(x) \sum_{j=0}^{n} \int_{t_j}^{x} \frac{(t_j - t)^k}{D w_n(t_j)(x - t_j)} (D^{k+1} f - p)(t)\,\mathrm{d}t/k!, \quad \forall p \in \Pi_{<n-k}.$$

Since

$$DF_k(x, \cdot, g) = F_{k-1}(x, \cdot, g), \quad k \ge 0,$$

it seems consistent to define

$$F_{-1}(x, y, g) := g(y).$$

This also makes it easy to treat in the same way the linear map

$$P_n^{(r)} : f \mapsto D^r P_n D^{-r} f$$

which reproduces Π_{n-r}. For it,

$$f(x) = P_n^{(r)} f(x) + \sum_{j=0}^{n} D^r \ell_j(x) \int_{t_j}^{x} (t_j - t)^n D^{n+1-r} f(t)\,\mathrm{d}t/n!.$$

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