Birkhoff interpolation

Birkhoff interpolation (see BirkhoffGD06) is an extension of Hermite interpolation in that it involves matching of values and derivatives at certain points but does not insist that these be consecutive.

With this additional freedom, it is no longer automatic that such an interpolation problem is correct, i.e., has exactly one solution for every choice of data values. This is illustrated by the fact that the 1-1 data map $f \mapsto (f(a), Df((a+b)/2), f(b))$ (with $a \neq b$) is not 1-1 on Π_2.

Assume without loss that we are seeking to match, from $\Pi_{<n}$, the information

$$(\lambda_k f := D^{\nu_k} f(x_k) : k = 1, \ldots, n)$$

with

$$\nu_1 \le \cdots \le \nu_n \,, \quad \text{and} \ \ (x_i, \nu_i) = (x_j, \nu_j) \Longrightarrow i = j.$$

This interpolation problem is correct iff the Gramian

$$G := (\lambda_i ()^{j-1} : i, j = 1, \ldots, n)$$

is invertible. Yet if $\nu_k \ge k$ for some k, then the first k columns of G have nonzero entries only in the first $k-1$ rows, hence G is then not invertible.

Thus, the so-called Pólya condition, that $\nu_k < k$ for all k, is necessary for the correctness of Birkhoff interpolation. Further, if $\nu_k = k-1$ for all k, then G is upper triangular with nonzero diagonal entries, hence invertible.

That the Pólya condition is also sufficient in case there are just two distinct sites was proved by Polya31, using his generalization of Rolle's Theorem:

If the smooth function f satisfies

$$D^{\nu_i} f(x_i) = 0, \quad i = 1, \ldots, n,$$

with $\{x_1, \ldots, x_n\} = \{a, b\}$ and $a \neq b$, then $\nu_k < k$ for all k implies that, for $k = 1, \ldots, n$, $D^k f$ has at least $\#\{j : \nu_j \le k\} - k > 0$ distinct zeros in $[a \,..\, b]$.

To be sure,

$$\mu_k := \#\{j : \nu_j \le k\} > k$$

since, by assumption, $\nu_{k+1} \le k$. So, assume we know already that $D_k f$ has $\mu_k - k > 0$ distinct zeros in $[a \,..\, b]$ as we certainly do know for $k = 0$. Then, by Rolle's theorem, $D^{k+1} f$ has at least $\mu_k - k - 1$ distinct zeros in $(a \,..\, b)$, hence altogether at least $(\mu_{k+1} - \mu_k) + \mu_k - (k+1) = \mu_{k+1} - (k+1)$ distinct zeros in $[a \,..\, b]$.

It follows that any such smooth f has at least one zero in $[a \,..\, b]$ in each of its first $n-1$ derivatives. Hence, if also $f \in \Pi_{<n}$, then, by the correctness of Birkhoff interpolation when $\nu_k = k-1$ for all k, f must necessarily be zero. In other words, the Pólya condition, necessary in general, is also sufficient when there are only two distinct interpolation sites.

Note that the argument, hence the conclusion, still holds when there are more than just two sites, provided only that no x_j lies strictly between two earlier x_i's, i.e., provided $x_i < x_k < x_j$ implies $k < \max(i, j)$.

The book to read on Birkhoff interpolation is LorentzGJetterRiemenschneider83.

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