Sommerfeld's argument

In 1904, A. Sommerfeld published a paper in which he showed that the sequence of properly dilated cardinal B-splines converges to the Gauss function as the degree goes to infinity.

Explicitly, with B_n the centered cardinal B-spline of order n, Sommerfeld shows that, for large n and all x,

$$B_n(x/2h) \approx (k2h/\sqrt{\pi}) \exp(-k^2x^2),$$

with

$$kh\sqrt{n} = \sqrt{3/2}.$$

(Sommerfeld studies the error distribution of the sum of n random errors with uniform distribution $[-h..h]$. Thus the quantity y near the end of his paper equals $B_n(x/(2h))/(2h)$, since $\int B_n(x/a)dx = a$.)

The convergence is quite good. E.g., for $n = 10, 100, 400$, $B_n(0)$ is

$$0.43041776895944, \quad 0.13799020407550, \quad 0.06907291282945,$$

while the formula gives the approximations

$$0.43701937223683, \quad 0.13819765978853, \quad 0.06909882989427.$$

Sommerfeld's argument is deliberately cavalier near the end, with [Maurer, L.; Über die Mittelwerthe der Functionen einer reellen Variabeln; Math. Ann.; 47; 1896; 267;] cited as the place where the rigorous argument can be found. A rigorous argument has also beeen supplied by Lei (qv) Lei reports that Maurer's paper also handles the delicate estimate for the error in approximating B_n by a Gaussian. Roughly speaking, the approximation rate is, uniformly,

$$1/(n\sqrt{n}).$$

Sivakumar mentions that his colleague, Joel Zinn, pointed out to him that Sommerfeld's result is a special case of Theorem 1 on page 533 of Feller's book (I believe it is volume II); it appears in the section on 'Expansions for densities' in Chapter XVI (Expansions related to the Central Limit Theorem). He asked Professor Zinn to explain it to him in the special case of interest to us. If he has understood him correctly, this special case is:

Let $A_n := [-1/2, 1/2]/(\sqrt{n/12})$, $f_n(x) := \sqrt{n/12}g_n(x)$, where g_n = n-fold convolution of the function χ_{A_n} with itself. Then $f_n(x) - \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) = o(1/\sqrt{n})$ uniformly in x. Maurer's estimate would give $O(1/n)$ instead.