Moments and Fourier transform of a B-spline

Since

$$\Delta(t_0,\ldots,t_k)f = \int M(\cdot|t_0,\ldots,t_k)D^k f/k!,$$

with M the B-spline normalized to have integral 1, some integrals of the form

$$\int M(\cdot|t_0,\ldots,t_k)g$$

can (and certainly have been) computed from knowledge of divided differences, as long as it is easy to obtain $D^{-k}g$ and compute its divided difference.

This is certainly the case for the power function,

$$g = ()^n : x \mapsto x^n,$$

for which

$$D^{-k}g = ()^{n+k} n!/(n+k)!$$

while (see pagep101)

$$\Delta(t_0,\ldots,t_k)()^{n+k} = \sum(t^\alpha : |\alpha| = n, \alpha \ge 0).$$

Therefore, the nth moment of the B-spline is

$$\int ()^n M(\cdot|t_0,\ldots,t_k) = \Delta(t_0,\ldots,t_k)()^{n+k}/\binom{n+k}{n} = \sum(t^\alpha : |\alpha| = n, \alpha \ge 0)/\binom{n+k}{n}.$$

In the same way, since $D^k \exp(-\mathrm{i}(\cdot)\xi) = (-\mathrm{i}\xi)^k \exp(-\mathrm{i}(\cdot)\xi)$, the Fourier transform of the B-spline is

$$\int e^{-\mathrm{i}(\cdot)\xi} M(\cdot|t_0,\ldots,t_k) = (k!/(-\mathrm{i}\xi)^k)\,\Delta(t_0,\ldots,t_k)e^{-\mathrm{i}(\cdot)\xi}.$$

An early reference (other than Schoenberg's use of this connection throughout his papers on B-splines, including the very first one, in '46), is

Neuman, E.; Moments and Fourier transforms of B-splines; J.Comput.Applied Math.; 7; 1981; 51–62;