cardinal B-splines

The (centered) cardinal B-spline of order k was introduced in Schoenberg46a(page 68) as the inverse Fourier transform

$$M_k(x) := \int_{-\infty}^{\infty} \left(\frac{\sin u/2}{u/2}\right)^k e^{\mathrm{i}ux}\, \mathrm{d}u/(2\pi)$$

of a certain function having zeros of order k at all points in $2\pi\mathbb{Z}\backslash 0$ and shown directly to equal the kth order central difference of the (normalized) truncated power of order k, i.e.,

$$M_k(x) = \delta^k x_+^{k-1}/(k-1)! = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (x + \frac{k}{2} - j)_+^{k-1}/(k-1)!.$$

as well as the convolution product

$$M_k = M_1 * \cdots * M_1$$

of k copies of the characteristic function of the interval $[-1/2 \,..\, 1/2]$ (see loc.cit. page 69). This latter fact gives at once the positivity of M_k on its support, $(-k/2 \,..\, k/2)$, as well as the fact that

$$\sum_{j\in\mathbb{Z}} M_k(\cdot - j) = 1.$$

Schoenberg later (e.g., in his sequence of papers on cardinal spline interpolation, summarized in his book Schoenberg73a) also uses the forward B-spline

$$Q_k(x) = M_k(x + k/2) = M(x|0, \ldots, k)$$

and observes, on page 61 of that book, that

$$\int_{-\infty}^{\infty} Q_m(x-j) Q_m(x-k)\, \mathrm{d}x = M_{2m}(j-k).$$