The perfect B-spline

The perfect B-splines, i.e., B-splines whose piecewise constant derivative is absolutely constant, make their first appearance in Schoenberg71d, as the solution to an optimal control problem.

These are, up to a scalar multiple, the B-splines

$$M(\cdot|t_0,\ldots,t_k) : x \mapsto k\Delta(t_0,\ldots,t_k)(\cdot - x)_+^{k-1}$$

whose knots $t_j := \cos((k-j)\pi/k)$, $j = 0{:}k$, are the extrema of the Chebyshev polynomial of degree k, hence (e.g., Rivlin74b or [http://mathworld.wolfram.com/ChebyshevPolynomialoftheSecondKi $\omega(x) := \prod_j (x - t_j) = (x^2-1)U_{k-1}(x)/2^{k-1}$, with $U_{k-1}(\cos\theta) = \sin(k\theta)/\sin(\theta)$, i.e., U_n is a Chebyshev polynomial of the second kind, while, by [divided difference: basic formula],

$$\Delta(t_0,\ldots,t_k) = \sum_j \Delta(t_j)/(D\omega)(t_j).$$

Now observe that $D\omega(x) = 2xU_{k-1}(x) + (x^2-1)DU_{k-1}(x)$. So, for $0 < j < k$, $2^{k-1}D\omega(t_j) = (t_j^2 - 1)DU_{k-1}(t_j)$, hence, with $x := t_j =: \cos\theta$ and $y := \sin\theta$,

$$2^{k-1}D\omega(t_j) = -y^2(k\cos(k\theta)/y) * 1/(-y) = (-)^j,$$

while, for $j = 0$ or k, $2^{k-1}D\omega(t_j) = 2t_jU_{k-1}(t_j) = 2(-)^j$.

Hence,

$$2^{k-1}M(x|t_0,\ldots,t_k)/k = (-1-x)_+^{k-1}/2 + \sum_{j=1}^{k-1}(-)^j(t_j - x)_+^{k-1} + (-1)^k(1-x)_+^{k-1}/2,$$

showing $|D^{k-1}M| = k/2^k$ on its support.

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