CS 880: Advanced Complexity Theory 1/27/06

Lecture 3: Spectral Gap and Implies Expansion

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In the last lecture we showed that most d-regular graphs are highly expanding. We also discussed some basic properties of the spectrum of the normalized adjacency matrix of a graph, and in particular we defined the notion of spectral gap, which is the difference between the largest and second largest eigenvalues of the normalized adjacency matrix in absolute value. In today's lecture, we discuss the connection between the expanding property and the spectral gap. In particular, we will prove the following two results for d-regular graphs. First, we show that if there is a spectral gap, then the graph is expanding. Second, we show that if the graph is sufficiently expanding, then the second largest eigenvalue is upper bounded by a certain constant (we don't get to proving this in today's next lecture). We then prove the Expander Mixing Lemma. We assume throughout today's lecture that all graphs are d-regular.

1 Spectral Gap implies Graph is Expanding

Theorem 1. *Let $G = (V, E)$, with $|V| = N$. Then, for any $\alpha < 1$, G is $\left(\alpha N, \frac{1}{(1-\alpha)\lambda_2^2+\alpha}\right)$- expanding.*

Observe that if $\lambda_2 < 1$, then $(1-\alpha)\lambda_2^2 + \alpha < 1$, and so the expanding constant

$$c = \frac{1}{(1-\alpha)\lambda_2^2 + \alpha} > 1.$$

1.1 Preliminaries

Before giving the proof, we establish a key property and define the notion of the Collision probability of a distribution. Recall that the normalized adjacency matrix A of the graph has row sum equal to 1, and since it is symmetric, has column sum also equal to 1. Thus, A transforms probability distributions to probability distributions, and A can be viewed as defining a Markov process with the uniform distribution u being invariant; i.e., $Au = u$. (Note: for G to be an expander graph, $\lambda = 1$ has multiplicity 1, so u is the only fixed point of A.) Now consider any probability distribution π. Since $\frac{<\pi,u>}{\|u\|_2^2} = 1$, we can write

$$\pi = u + \pi^{\perp},$$

where $\pi^{\perp} \perp u$. After one step of the random walk, the distribution π becomes $A\pi$. We first establish the following key property: after each step of the random walk, the distribution gets closer to the uniform distribution, and this rate of convergence depends on the second largest eigenvalue of A. More precisely,

$$\|A\pi - u\|_2 = \|A(u + \pi^{\perp}) - u\|_2 = \|A\pi^{\perp}\|_2 \leq \lambda_2 \|\pi^{\perp}\|_2 = \lambda_2 \|\pi - u\|_2.$$

Definition 1. *The Collision probability of a distribution π, denoted by $Coll(\pi)$, is the probability that two random variables X and Y, drawn independently from distribution π, take on the same value (Collide).*

Thus, we have that

$$Coll(\pi) = \Pr_{X \leftarrow \pi, Y \leftarrow \pi}[X = Y] = \sum_{i=1}^{N} \Pr[X = i] \Pr[Y = i] = \|\pi\|_2^2.$$

Lemma 1. *$Coll(\pi)$ satisfies the following properties:*
(1) $Coll(\pi) = \|\pi\|_2^2$.
(2) $Coll(u) = 1/N$.
(3) $Coll(\pi) = 1/N + \|\pi^{\perp}\|_2^2$.
(4) $Coll(\pi) \geq \frac{1}{|support(\pi)|}$.

Proof. The proof of (*1*) was given earlier, and (*2*) follows easily by computation. (*3*) follows from the two facts, $\pi = u + \pi^{\perp}$ and $\pi^{\perp} \perp u$, and lemma (*1*). To prove (*4*), observe that

$$Coll(\pi) = \sum_{i \in S} (\pi(i))^2,$$

where S is the support of π. We claim that, for a fixed set $|S|$, the probability distribution π minimizing the above sum is $\pi(i) = 1/|S|$ for $i \in S$ and 0 otherwise. To see this note that,

$$\begin{aligned}
Coll(\pi) &= \sum_{i=1}^{N} \pi(i)^2 \\
&= \sum_{i \in S} \pi(i)^2 \\
&= \Big(\sum_{i \in S} \big(\frac{1}{\sqrt{|S|}}\big)^2\Big)\Big(\sum_{i \in S} \pi(i)^2\Big) \\
&\geq \Big(\sum_{i \in S} \frac{\pi(i)}{\sqrt{|S|}}\Big)^2 \\
&\geq \frac{1}{|S|}\Big(\sum_{i \in S} \pi(i)\Big)^2 \\
&\geq \frac{1}{|S|} |\pi|_1^2 \\
&\geq \frac{1}{|S|}
\end{aligned}$$

Note that, to conclude,

$$\Big(\sum_{i \in S} \big(\frac{1}{\sqrt{|S|}}\big)^2\Big)\Big(\sum_{i \in S} \pi(i)^2\Big) \geq \Big(\sum_{i \in S} \frac{\pi(i)}{\sqrt{|S|}}\Big)^2,$$

we used the Cauchy-Schwartz inequality. We also know that, for any two non zero vectors A and B, $|A|_2^2 |B|_2^2 = |A \cdot .B|_2^2$, if and only if the vectors are parallel to each other. This claim easily follows from the aforesaid observation. Note that, (*4*) follows from the above claim. In fact, it followed just from the Cauchy-Schwartz inequality. We delved into the question of when the equality occurs to elucidate that a flat distribution has the lowest collision probability and hence highest entropy. □

1.2 Proof of Theorem 1

Proof. Let V denote the vertex set of the graph G. Let $N = |V|$ and S be a subset of V such that $|S| \leq \alpha|V|$. Let π denote the uniform distribution over the vertices in the set S. The following claims follow from the previous definitions and lemmas related to Collision probability.

$$\begin{aligned} Support(\pi) &= S, & (1)\\ Coll(\pi) &= \frac{1}{|S|}. & (2) \end{aligned}$$

Now consider the distribution $A\pi$ where A is the normalized adjacency matrix of the graph G. We have already seen that this is the distribution obtained after one step of the Markov process defined by A. Hence, we will end up in the distribution obtained by taking one step random walk starting from the uniform distribution over the vertices in set S. Therefore, a vertex will have a non zero probability in the distribution $A\pi$ if and only if it lies in the set $\Gamma(S)$. The following relations are immediate,

$$\begin{aligned} Support(A\pi) &= \Gamma(S),\\ Coll(A\pi) &\geq \frac{1}{|\Gamma(S)|}. \end{aligned}$$

We can decompose π in terms of its component along u and the subspace perpendicular to u. The component of π along u is u itself. Hence we can write π as follows :

$$\pi = u + (\pi - u),$$

And $(\pi - u)$ is perpendicular to u. We also have that,

$$A\pi = u + A(\pi - u).$$

Since $Coll(\pi) = \|\pi\|_2^2$, we have the following relations

$$\begin{aligned} Coll(A\pi) &= Coll(u + A(\pi - u))\\ &= \|u + A(\pi - u)\|_2^2\\ &= \|u\|_2^2 + \|A(\pi - u)\|_2^2\\ &= \frac{1}{N} + \|A(\pi - u)\|_2^2. \end{aligned}$$

Now consider the quantity $\|A(\pi - u)\|_2^2$. We had that the vector $(\pi - u)$ is perpendicular to u. Hence it is spanned by the eigen vectors of A apart from u. Since λ_2 is the magnitude of the second largest eigen value of A, we have that,

$$\begin{aligned} Coll(A\pi) &= \frac{1}{N} + \|A(\pi - u)\|_2^2 & (3)\\ &\leq \frac{1}{N} + \lambda_2^2\|(\pi - u)\|^2 & (4) \end{aligned}$$

It follows from the Pythagoras theorem that $\|(\pi - u)\|^2 = \|\pi\|_2^2 - \|u\|_2^2 = \frac{1}{|S|} - \frac{1}{N}$. Using (4), we have that

$$\frac{1}{|\Gamma(S)|} \leq Coll(A\pi) \leq \frac{1}{N} + \lambda_2^2 \left(\frac{1}{|S|} - \frac{1}{N} \right)$$

Since $|S| \leq \alpha N$, we have that $\frac{1}{N} \leq \frac{\alpha}{|S|}$. Using this we get that,

$$\begin{aligned}
\frac{1}{|\Gamma(S)|} &\leq (1-\lambda_2^2)\frac{1}{|N|} + \lambda_2^2 \frac{1}{|S|}, \\
&\leq (1-\lambda_2^2)\frac{\alpha}{|S|} + \lambda_2^2 \frac{1}{|S|}, \\
&\leq \frac{1}{|S|}[\alpha(1-\lambda_2^2) + \lambda_2^2].
\end{aligned}$$

Therefore we have that,

$$\frac{|\Gamma(S)|}{|S|} \geq \frac{1}{\alpha(1-\lambda_2^2) + \lambda_2^2}.$$

This completes the proof.

□

2 Expander Mixing Lemma

Lemma 2. *Let $G = (V, E)$. For all $S, T \subseteq V$, we have that*

$$\left| E(S,T) - \frac{d\ |S|\ |T|}{N} \right| \leq \lambda_2\ d\ \sqrt{|S|\ |T|\ \left(1 - \frac{|S|}{N}\right)\ \left(1 - \frac{|T|}{N}\right)},$$

where $E(S,T)$ is the number of edges between the sets S and T.

Before proving the lemma let us have a look at the implication of the lemma. Note that in randomly chosen $d-$regular graphs the expected number of edges in between any two sets of vertices, say S and T, is equal to $\frac{d|S||T|}{N}$. In a randomly chosen $d-$regular graph the i^{th} edge from a vertex, say u, has the other end point inside a particular set of vertices, say T, with probability exactly equal to $\frac{|T|}{N}$. The claim now follows from the linearity of expectation. Therefore, in an expander graph the number of edges between any two subsets of vertices is not too far away from what it should be for random $d-$regular graphs. In a way this means that instead using perfect source of randomness to generate $d-$regular graphs, we can use expander graphs and still ensure that we do not deviate much from the expectation. However, it is important to note that this lemma does not mean that expander graphs are identical to random $d-$regular graphs in all aspects. Adjacency is just one of many statistical tests and in that sense these two classes are very close. As we will see later in the course, it turns out that even this property has a lot of uses; especially for derandomization purposes and for saving random bits needed for probabilistic simulations.

Proof. Now we will prove the main theorem. Let S, T be subsets of V. Let χ_S, χ_T denote the characteristic vector of S and T respectively. Consider the following inner product expression:

$\chi_S^T dA\chi_T$. We claim that this quantity is exactly equal to the number of edges with one end in S and the other one in T. To see this note that,

$$\begin{aligned}\chi_S^T dA\chi_T &= \sum_{v\in V}\sum_{v'\in V} dA_{v,v'}\chi_{S_v}\chi_{T_{v'}}\\ &= \sum_{v\in S}\sum_{v'\in T} dA_{v,v'}\end{aligned}$$

Since $dA_{v,v'}$ the number of edges between the vertices v and v', the above sum is equal to the number of edges between the sets S and T. We need to relate the magnitude of this quantity with the spectrum of the matrix A. Hence, we will again decompose the vectors χ_S and χ_T in terms of the vector u and the subspace perpendicular to it. We claim that the inner product of χ_S and u is equal to $\frac{|S|}{N}$, while that with χ_T is equal to $\frac{|T|}{N}$. To see this, note that the inner product of a vector with u is just the one norm of that vector divided by N. Since the one norm of χ_S is $|S|$, the claim follows. Hence we have the following two relations:

$$\begin{aligned}\chi_S &= |S|u + u_1^{\perp},\\ \chi_T &= |T|u + u_2^{\perp}.\end{aligned}$$

Using the above relations, we have that:

$$\begin{aligned}\chi_S^T dA\chi_T &= (|S|u + u_1^{\perp})^T dA(|T|u + u_2^{\perp})\\ &= (|S|u + u_1^{\perp})^T(d|T|u + dAu_2^{\perp})\\ &= d|S||T|\|u\|_2^2 + du_1^{\perp T}Au_2^{\perp}\end{aligned}$$

Note that,

$$\begin{aligned}|\chi_S|_2^2 &= |S|^2|u|^2 + |u_1^{\perp}|_2^2\\ \Rightarrow |S| &= \frac{|S|^2}{N} + |u_1^{\perp}|_2^2\\ \Rightarrow |u_1^{\perp}|_2^2 &= |S|\big(1 - \frac{|S|}{N}\big)\end{aligned}$$

Similarly, we get that,

$$|u_2^{\perp}|_2^2 = |T|(1 - \frac{|T|}{N}).$$

Using the fact that $E(S,T) = \chi_S^T dA\chi_T$, we have that,

$$\begin{aligned} E(S,T) - \frac{d|S||T|}{N} &= d{u_1^\perp}^T A u_2^\perp \\ \Longrightarrow \left| E(S,T) - \frac{d|S||T|}{N} \right| &= |{u_1^\perp}^T A u_2^\perp| \\ &\leq d\|u_1^\perp\|_2 \|Au_2^\perp\|_2 \\ &\leq d\|\chi_S\|_2 \lambda_2 \|u_2^\perp\|_2 \\ &\leq d\sqrt{|S|\big(1 - \frac{|S|}{N}\big)}\lambda_2\sqrt{|T|\big(1 - \frac{|T|}{N}\big)} \\ &= d\lambda_2\sqrt{|S||T|\big(1 - \frac{|S|}{N}\big)\big(1 - \frac{|T|}{N}\big)} \end{aligned}$$

The theorem ((follows.

$\square$

3 Theorem 2

We will prove the following theorem in class next time.

Theorem 2. *If G is $(N/2, 1+\gamma)$-expanding, then*

$$\lambda_2 \leq \sqrt{1 - \frac{\gamma^2}{d^2(8 + 4\gamma^2)}}.$$

Thus, if the graph is expanding, by this theorem we have a lower bound on the spectral gap. In Theorem 1, we showed that if there is a spectral gap, then the graph is expanding. Observe that the constants given in the two theorems are not just inverses of each other; there is a square-root factor missing, and due to this gap the two theorems are only roughly converses of each other.