CS 880: Advanced Complexity Theory 1/30/05

Lecture 4: Expansion Implies Spectral Gap

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Today we will discuss the other direction of the spectral connection. We will prove Theorem 2 from last lecture. Recall, this said that if a graph is sufficiently expanding then the second eigenvalue is bounded away from one.

1 Recap

Remember,

$$\lambda_2 = \max_{\substack{x \perp u \\ \|x\|_2 = 1}} \|Ax\|_2 \tag{1}$$

There was some discussion last time about the appropriate norm to use in this equation. If we use any other norm λ_2 could increase, though if λ_2 is less than one using the L2 norm then it is bounded away from one under the other norms. We choose the L2 norm because it is invariant under our basis of orthogonal eigenvectors, though it is not necessarily invariant under others.

Last time we proved Theorem 1:

Theorem 1. *For any constant $\alpha < 1$, G is $(\alpha N, \frac{1}{(1-\alpha)\lambda_2^2+\alpha})$-expanding.*

This was shown by considering a random walk on the graph G. We started with a uniform distribution on some $S \subseteq V$. We evolved the Markov process using the normalized adjacency matrix A. This expanded the original distribution towards the uniform distribution on V.

2 Proof of Theorem 2

Today we will show the other side of the spectral connection, namely that if a graph is expanding its second eigenvalue is bounded away from one.

Theorem 2. *If G is $(\frac{N}{2}, 1+\gamma)$-expanding, then $\lambda_2 \leq \sqrt{1 - \frac{\gamma^2}{d^2(8+4\gamma^2)}}$.*

Notice that if $(\gamma > 0)$ then λ_2 must be strictly less than one. In order to prove Theorem 2 we must first prove a lemma:

2.1 Proving a Lemma

Lemma 1. *If G is $(\frac{N}{2}, 1+\gamma)$-expanding and the spectrum of G is nonnegative then $\lambda_2 \leq 1 - \frac{\gamma^2}{d(8+4\gamma^2)}$.*

Proof. Let $x \perp u$ such that $x \neq 0$ and $Ax = \lambda_2 x$.

Let $V_+ = \{i | x_i > 0\}$, this is the set of all components of x that are positive. At most half of the components of x must be positive (if not we can use $-x$ instead). Furthermore, $x \perp u$ and u is the uniform distribution so at least one of the components of x must be positive since $\langle x, u \rangle = 0$.

Let $y_i = x_i$ if $i \in V_+$ and zero otherwise.

$$\begin{aligned} Ax &= \lambda_2 x \\ y^T Ax &= \lambda_2 y^T x \\ &= \lambda_2 y^T y \\ &= \lambda_2 \|y\|_2^2 \end{aligned} \tag{2}$$

Define G' as an $N \times N$ bipartite graph with the adjacency matrix A: $G' = (V_L, V_R, E')$, such that $V_L = V_R = V$. If there are multi-edges in A, then multiple edges occur in E'. We use the bipartite representation here because it makes it easier to perform the calculations. Using this representation we can rewrite $y^T Ax$.

$$y^T Ax = \frac{1}{d} \sum_{(i,j)\in E'} y_i x_j \tag{3}$$

We want to now get rid of all occurrences of x in the equation and write it only in terms of y. Because of how we defined y_i, for all i, $x_i \leq y_i$.

$$\begin{aligned} \frac{1}{d} \sum_{(i,j)\in E'} y_i x_j &\leq \frac{1}{d} \sum_{(i,j)\in E'} y_i y_j \\ &= \|y\|_2^2 - \frac{1}{2d} \sum_{(i,j)\in E'} (y_i - y_j)^2 \end{aligned} \tag{4}$$

Substituting that back into the original equation we have:

$$\begin{aligned} \lambda_2 \|y\|_2^2 &\leq \|y\|_2^2 - \frac{1}{2d} \sum_{(i,j)\in E'} (y_i - y_j)^2 \\ \lambda_2 &\leq 1 - \frac{1}{2d\|y\|_2^2} \sum_{(i,j)\in E'} (y_i - y_j)^2 \end{aligned} \tag{5}$$

Now we need to lower bound the second term by the expression we are trying to prove:

$$\frac{1}{2d\|y\|_2^2} \sum_{(i,j)\in E'} (y_i - y_j)^2 \geq \frac{\gamma^2}{d(8 + 4\gamma^2)} \tag{6}$$

Using Cauchy-Schwartz

$$\left| \underbrace{\sum_{e=(i,j)\in E'} f(e)(y_i^2 - y_j^2)}_{\alpha} \right|^2 \leq \left(\sum_{e=(i,j)\in E'} (y_i - y_j)^2 \right) \underbrace{\left(\sum_{e=(i,j)\in E'} f(e)^2 (y_i + y_j)^2 \right)}_{\beta},$$

for any function $f : E' \to \mathbb{R}$

We want to choose f so that $\frac{\alpha}{\beta}$ will be large compared to $\|y\|_2^2$. This will give us the lower bound we are looking for.

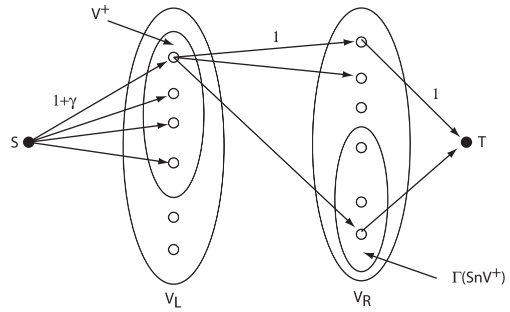

Figure 1: G' seen as a network flow problem

To construct $f(e)$ consider the flow problem on G', where links from s to V_+ each have capacity $1+\gamma$, links from s to $V_L - V_+$ have capacity 0, links from V_+ to V_R have capacity 1, and links from V_R to t have capacity 1.

Claim 1. $MinCut(G') = (1+\gamma)|V_+|$

Proof. Consider any cut (S,T)

$$\begin{aligned} Capacity(S,T) &\geq (1+\gamma)|V_+ \setminus S| + \underbrace{|\Gamma(V_+ \cap S)|}_{\geq (1+\gamma)|V_+ \cap S|} \\ &\geq (1+\gamma)|V_+| \end{aligned}$$

□

By the Max-flow Min Cut Theorem $\exists f : E' \to [0,1]$, this will be the value of the flow on each edge of the bipartite graph. It has the following properties:

$$\begin{aligned} (\forall i \in V_+) \textstyle\sum_{e=(i,j)\in E'} f(e) &= 1+\gamma \\ (\forall j \in V) \textstyle\sum_{e=(i,j)\in E'} f(e) &\leq 1 \end{aligned}$$

(the difference between these two will give us the lower bound.) The first property holds because to realize the max flow each edge from s to V_+ must be used to full capacity. The second property holds because the amount that passes though the nodes in V_R is limited by the capacity of the edges from V_R to t.

Then apply the constructed f back to α and β:

$$\begin{aligned}
\alpha &= \textstyle\sum_{e=(i,j)\in E'} f(e)(y_i^2 - y_j^2) \\
&\geq (1+\gamma)\textstyle\sum_{i\in V} y_i^2 - \sum_{j\in V} y_j^2 \\
&= \gamma\|y\|_2^2
\end{aligned}$$

$$\begin{aligned}
\beta &= \textstyle\sum_{(i,j)\in E'}(f(e))^2 \underbrace{(y_i+y_j)^2}_{\leq 2(y_i^2+y_j^2)} \\
&\leq 2\textstyle\sum_{i\in V} \overbrace{\left(\displaystyle\sum_{e\in(i,j)\in E'} (f(e))^2\right)}^{\leq 1+\gamma^2} y_i^2 + 2\textstyle\sum_{j\in V} \overbrace{\left(\displaystyle\sum_{e\in(i,j)\in E'} (f(e))^2\right)}^{\leq 1} y_j^2 \\
&\leq ((4+2\gamma^2)\|y\|_2^2
\end{aligned}$$

So

$$\sum_{(i,j)\in E'} (y_i - y_j)^2 \geq \frac{\alpha^2}{\beta} \geq \frac{(\gamma\|y\|_2^2)^2}{(4+2\gamma^2)\|y\|_2^2} = \frac{\gamma^2\|y\|_2^2}{(4+2\gamma^2)}$$

Therefore

$$\lambda_2 \leq 1 - \frac{1}{2d}\left(\frac{\sum_{(i,j)\in E'}(y_i-y_j)^2}{\|y\|_2^2}\right) \leq 1 - \frac{\gamma^2}{d(8+4\gamma^2)}$$

□

2.2 Generalized Proof

In the previous proof we were limited to graphs whose normalized adjacency matrices contained only non-negative eigenvalues. We will now generalize our proof to include all graphs by analyzing the square of the normalized adjacency matrix of a graph G. This is a very straightforward proof.

We will use G^2, which is defined to be the graph with the same vertex set as G but replaces the adjacency matrix of G with its square. This implies that the edge set E for G^2 is equavalent to the set of all paths of length 2 in G.

Proof. Apply Lemma 1 to G^2.

Properties of G^2:

- E^2 contains all paths of length 2 in G.
- $degree(G^2) = d^2$.
- $spectrum(G^2) = (spectrum(G))^2 \geq 0$.
- $\lambda_2(G^2) = (\lambda_2(G))^2$
- G^2 is at least $(\frac{N}{2}, 1+\gamma)$-expanding, though it likely has a stronger expansion property. *exercise left to the reader*

Applying the Lemma to G^2 we get

$$\lambda_2(G^2) \leq 1 - \frac{\gamma^2}{d^2(8+4\gamma^2)}$$

After taking the square root of each side we get

$$\lambda_2(G) \leq \sqrt{1 - \frac{\gamma^2}{d^2(8+4\gamma^2)}}$$

$\square$

3 Minimum Value of the Spectral Gap

Theorem 3. $\lambda_2 \geq \frac{2}{\sqrt{d}}\left(\sqrt{1-\frac{1}{d}} - o(1)\right)$ *where* $o(1) \rightarrow 0$ *as* $|V| \rightarrow \infty$

This is the tight result. We will prove a slightly weaker statement, but one that is easier to analyze. Namely, we shall prove $\lambda_2 \geq \frac{1}{\sqrt{d}}(1-o(1))$.

Proof. $(A^2)_{jj} \geq \frac{1}{d}$. Because $(A^2)_{jj}$ is the sum of the squares of the entries in row j, and the sum of the entries in row j must equal one, the lowest value is achieved when every element in row j has the same value. In other words the weights of the elements in row j are uniformly distributed. So,

$$(A^2)_{jj} \geq \frac{1}{d} \Rightarrow Trace(A^2) \geq \frac{N}{d}$$

and we know that

$$Trace(A^2) = \sum_{i=1}^{N} (\lambda_i(A))^2 \leq 1 + (N-1)\lambda_2^2$$

So, this gives us

$$\lambda_2^2 \geq \frac{N-d}{N-1}\frac{1}{d}$$

after some re-arrangement we get

$$\lambda_2 \geq \frac{1}{\sqrt{d}}(1-o(1))$$

$\square$

There is a simpler proof of Theorem 3 based on Theorem 1. We know that a d-regular graph G is $(\alpha N, \frac{1}{(1-\alpha)\lambda_2^2+\alpha})$-expanding. Since the expansion ratio $c \leq d$, we get λ_2 to be inversely proportional to $\sqrt{d}$.