CS 880: Advanced Complexity Theory 2/3/06

Lecture 5: Explicit Constructions

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This lecture deals with explicit construction of expanders. Specifically, we discuss Gabber-Galil graphs and define Ramanujan graphs as designed by Margulis and by Lubotzky, Phillips, and Sarnak. In both of these graph families, it is difficult to intuit why the construction leads to good expansion properties. In the next lecture, we shall talk about another explicit construction, zig-zag products, in which the connection between the construction and its expansion properties is more transparent.

Recall that some applications might require the graph G to be of exponential size, so it is desirable that given a vertex $v \in V(G)$ and an integer i, our construction is able to find the *ith* neighbor of v efficiently, so that we can construct the graph on the fly. Both graph constructions we will discuss today have this property.

Note that the argument at the end of the previous lecture can establish a lower bound on the expansion ratio of at most $\frac{d}{4}$, but today we see an explicit construction (Ramanujan graphs) with expansion ratio $\frac{d}{2}$.

1 Gabber-Galil Graphs

There are a number of versions of the Gabber-Galil graph, both bipartite and non-bipartite constructions. For now, we discuss the standard non-bipartite case. The explicit construction is as follows:

The vertex set V is defined as $\mathbb{Z}_n \times \mathbb{Z}_n$ where $n \geq 1$ is an integer. The set of edges E is defined using transformations

$T_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $T_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

Thus for each $z = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{Z}_n \times \mathbb{Z}_n$ we include the edges from z to each of $T_1 z$,$T_1 z + e_1$, $T_2 z$ and $T_2 z + e_2$, where $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The degree of each vertex v is 8, with 4 edges corresponding to the given transformations on v, and one corresponding to the inverse of each transformation. The resulting family of graphs is expanding. In fact, the same holds for most choices of T_1 as a unimodular 2×2 matrix and $T_2 = T_1^t$. Later in this lecture we show the explicit bound $\lambda_2 \leq \frac{5\sqrt{2}}{8} < 1$ for $T_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = T_2^t$.

In the bipartite construction of a Gabber-Galil graph both the left vertex set L and the right vertex set R are $\mathbb{Z}_n \times \mathbb{Z}_n$, with edges corresponding to the transformations given above and the identity transformation. The degree of the bipartite construction is 5.

2 Ramanujan Graphs

The Lubotzky-Phillips-Sarnak construction defines a family of graphs as follows: Let p, q be two different primes that are congruent to 1 mod 4 and let $i^2 \equiv -1 \bmod q$. The set of vertices $V = \mathbb{Z}_q \cup \{\infty\}$ and $E = \left\{\left(z, \frac{(a_0+ia_1)z+(a_2+ia_3)}{(ia_3-a_2)z+(a_0-ia_1)}\right) \mid z \in V\right\}$, for all odd $a_0 \in \mathbb{N}$ and even $a_1, a_2, a_3 \in \mathbb{Z}$ such that ${a_0}^2 + {a_1}^2 + {a_2}^2 + {a_3}^2 = p$.

The properties of the graphs are stated as facts; proofs and technical details are not dealt with. The degree of the graph is $p+1$ and for Ramanujan graphs in general we have $\lambda_2 \leq \frac{2}{\sqrt{d}}\sqrt{1-\frac{1}{d}}$ which is asymptotically tight.

3 Proof of Gabber-Galil Graph Expansion

We now proceed to prove the expansion properties of the Gabber-Galil construction for the special case $T_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = T_2^t$. The proof uses the Fourier transform on functions $f : \mathbb{Z}_n^2 \to \mathbb{C}$ to establish a lower bound on the second-largest eigenvalue of the normalized adjacency matrix of the graph.

3.1 Fourier Transform

Given a group $G = \langle \mathbb{Z}_n^k, + \rangle$ of k length vectors we consider the set of all functions $f : G \to \mathbb{C}$ as a vector space of dimension $N = n^k$. We take the inner product of two functions f and g to be $\langle f, g \rangle = \sum_{z \in G} \overline{f(z)} g(z)$.

Next we show that the characters of G (i.e. homomorphisms from G to $\langle \mathbb{C}, \cdot \rangle$) form a basis for this vector space. Specifically all the characters are of the form $\chi_\zeta(z) = \omega^{\zeta \cdot z}$ where $\zeta \in \mathbb{Z}_n$. The product in the exponent is the inner product over $\mathbb{Z}_n^k$ and ω is the primitive n^{th} root of unity. We note that

$$\langle \chi_{\zeta_1}, \chi_{\zeta_2} \rangle = \sum_{z \in G} \overline{\chi_{\zeta_1}(z)} \chi_{\zeta_2}(z) = \sum_{z \in G} \omega^{(\zeta_2 - \zeta_1) z}$$

Hence if $\zeta_1 \neq \zeta_2$ the inner product is 0, as the roots lie on the vertices of a regular polygon in the complex plane and sum to zero, else when $\zeta_1 = \zeta_2$ it is equal to N. Note that $\left(\frac{1}{\sqrt{N}}\chi_\zeta\right)_{\zeta \in G}$ forms an orthonormal basis since the dimension is "full" for this vector space with:

$$f(z) = \frac{1}{\sqrt{N}} \sum_{\zeta \in G} \hat{f}(\zeta) \chi_\zeta(z)$$

for any $f(z)$ where $\hat{f}(\zeta) = \langle f, \frac{\chi_\zeta}{\sqrt{N}} \rangle$ is the coefficient of the projection of f onto the basis vector.

Some properties of the Fourier transform we will need to prove an upper bound for the second eigenvalue of a Gabber-Galil graph include the following:

1. We have $\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$, particularly, if $f = g$, we have Parseval's equality:

$$\begin{aligned} \|\hat{f}\|_2^2 &= \|f\|_2^2 \text{ or} \\ \sum_{\zeta \in G} \left|\hat{f}(\zeta)\right|^2 &= \sum_{z \in G} |f(z)|^2 . \end{aligned}$$

2. Let T be a non-singular $k \times k$ matrix over $\mathbb{Z}_n$, $s \in \mathbb{Z}_n^k$, and $g(z) = f(Tz + s)$. Then

$$\hat{g}(\zeta) = \omega^{(T^{-1}s)\cdot\zeta} \hat{f}((T^{-1})^{\dagger}\zeta).$$

3. $\sum_{z \in G} f(z) = 0 \Longleftrightarrow \hat{f}(0) = 0$.

3.2 Proof of Gabber-Galil Graph Theorem

When we build a graph using the Gabber-Galil construction, we claim the following:

Theorem 1. *A Gabber-Galil graph based on* $T_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = T_2^t$ *has second largest eigenvalue* $\lambda_2 \leq \frac{5\sqrt{2}}{8}$.

Proof. Recall that, in the Gabber-Galil construction there is en edge from each vertex $z \in \mathbb{Z}_n^2$ to vertices $T_1 z$, $T_1 z + e_1$, $T_2 z$, $T_2 z + e_2$ with

$$T_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, T_2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, T_1^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, (T_1^{-1})^{\dagger} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = T_2^{-1},$$
$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

To prove this lower bound for the second eigenvalue of a Gabber-Galil graph G with normalized adjacency matrix A, we need to show that for any $f : \mathbb{Z}_n^2 \to \mathbb{R}$ with $\sum_{z \in G} f(z) = 0$ that $|f^{\dagger} A f| \leq \frac{5\sqrt{2}}{8} |f^{\dagger} f|$ (corresponding to $max_{x \perp u} \|Ax\|$).

Consider a bipartite graph G' with left and right vertex sets L and R, each containing vertices $z \in \mathbb{Z}_n^2$ and each $z \in L$ having an edge to $T_1 z$, $T_1 z + e_1$, $T_2 z$, $T_2 z + e_2$ in R. Let A' be the normalized adjacency matrix for this bipartite graph.

Then $A = \frac{A' + (A')^{\dagger}}{2}$ and $|f^{\dagger} A f| = |f^{\dagger} A' f|$. We would like to show that $|f^{\dagger} A' f| \leq \frac{5\sqrt{2}}{8} \|f\|_2^2$:

$$\begin{aligned} |f^{\dagger} A' f| = \quad & \tfrac{1}{4} \left| \textstyle\sum_{z \in G} \overline{f(z)} [f(T_1 z) + f(T_1 z + e_1) + f(T_2 z) + f(T_2 z + e_2)] \right| & \leq \frac{5\sqrt{2}}{8} \|f\|_2^2 \\ & \left| \textstyle\sum_{z \in G} \overline{f(z)} [f(T_1 z) + f(T_1 z + e_1) + f(T_2 z) + f(T_2 z + e_2)] \right| & \leq \frac{5\sqrt{2}}{2} \|f\|_2^2 \end{aligned}$$

We can use properties (1) and (2) of the Fourier transform to simplify each of the terms:

$$\begin{array}{lcll} \langle f(z), f(T_1 z) \rangle & = & \langle \hat{f}(\zeta), \hat{f}(T_2^{-1}\zeta) \rangle & (*) \\ \langle f(z), f(T_1 z + e_1) \rangle & = & \langle \hat{f}(\zeta), \omega^{(T_1^{-1}e_1)\cdot\zeta} \hat{f}(T_2^{-1}\zeta) \rangle & (*) \\ \langle f(z), f(T_2 z) \rangle & = & \langle \hat{f}(\zeta), \hat{f}(T_1^{-1}\zeta) \rangle & \\ \langle f(z), f(T_2 z + e_2) \rangle & = & \langle \hat{f}(\zeta), \omega^{(T_2^{-1}e_2)\cdot\zeta} \hat{f}(T_1^{-1}\zeta) \rangle & \end{array}$$

Since $T_1^{-1} e_1 = e_1$ and $\sum_z f(z) = 0 \Rightarrow \hat{f}(0) = 0$ (by property 3), the first two terms (*) collapse to: $\sum_{\zeta \neq 0} \left| \hat{f}(\zeta) \right| (1 + \omega^{\zeta_1}) \left| \hat{f}(T_2^{-1}\zeta) \right|$, and the second two terms collapse similarly, so it is sufficient to show that:

$$\sum_{\zeta\neq 0}\left[\left|\hat{f}(\zeta)\right|\cdot\left|1+\omega^{\zeta_1}\right|\cdot\left|\hat{f}(T_2^{-1}\zeta)\right|+\left|\hat{f}(\zeta)\right|\cdot\left|1+\omega^{\zeta_2}\right|\cdot\left|\hat{f}(T_1^{-1}\zeta)\right|\right]\leq\frac{5\sqrt{2}}{2}\left\|\hat{f}\right\|_2^2.$$

We can use the following identity to simplify the expression:

$$\begin{aligned}1+\omega^t &= 1+\cos\left(\frac{2\pi t}{n}\right)+\sqrt{-1}\sin\left(\frac{2\pi t}{n}\right)\\ \left|1+\omega^t\right| &= 2\left|\cos\left(\frac{\pi t}{n}\right)\right|\end{aligned}$$

yielding the following inequality:

$$\sum_{\zeta\neq 0}\left[2\cdot\left|\hat{f}(\zeta)\right|\cdot\left|\cos\left(\frac{\pi\zeta_1}{n}\right)\right|\cdot\left|\hat{f}(T_2^{-1}\zeta)\right|+2\cdot\left|\hat{f}(\zeta)\right|\cdot\left|\cos\left(\frac{\pi\zeta_2}{n}\right)\right|\cdot\left|\hat{f}(T_1^{-1}\zeta)\right|\right]\leq\frac{5\sqrt{2}}{2}\left\|\hat{f}\right\|_2^2.$$

With careful analysis of the cosine values, this inequality will yield $\lambda_2\leq\frac{5\sqrt{2}}{8}$ (details omitted). $\square$