CS 880: Advanced Complexity Theory 2/10/06

Lecture 7: Expansion Property of Zig-Zag Products

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In this lecture, we will prove a bound on the second largest eigenvalue of the zig-zag product of two graphs. While this is not quite enough to claim that the zig-zag product of two expanders is also an expander, similar technique with some tighter analysis can be used to get that result. We urge the reader to have a look at the intuition provided in the last lecture as that will be useful in understanding the proof.

1 The Expansion Property

Theorem 1. $\lambda_2(G ⓩ H) \leq \lambda_2(G) + \lambda_2(H) + (\lambda_2(H))^2$.

Proof. We will denote $\lambda_2(G ⓩ H)$ by just λ_2. Let A, $A(G)$ and $A(H)$ denote the normalized adjacency matrices of $G ⓩ H$, G and H respectively. We can write A as

$$A = (I \otimes A(H))P(I \otimes A(H)),$$

where $\otimes$ is the tensor product and P is a permutation matrix. To see that P is a permutation, note that $I \otimes A(H)$ represents a one-step random walk in each *cloud* of $G ⓡ H$ and P represents the *edges of* G in $G ⓩ H$. Since every vertex of $G ⓩ H$ has exactly one *edge of* G incident to it, P is a permutation matrix. We can now write λ_2 as

$$\lambda_2 = \max_{x \perp u} \frac{x^T A x}{x^T x}, \tag{1}$$

Given any $x \perp u$, let's write $x = \sum_{v \in V(G)} e_v \otimes x_v$, where e_v is a unit vector and x_v is the restriction of x to the cloud of v. Then, x_v can be broken into components parallel and perpendicular to the uniform distribution as $x_v = x_v^{\parallel} + x_v^{\perp}$. So we have

$$x = \sum_{v \in V(G)} e_v \otimes x_v^{\parallel} + \sum_{v \in V(G)} e_v \otimes x^{\perp},$$

where $x^{\parallel} = \sum_{v \in V(G)} e_v \otimes x_v^{\parallel}$ is not necessarily parallel to the uniform distribution but $x^{\perp} = \sum_{v \in V(G)} e_v \otimes x_v^{\perp}$ is perpendicular to it.

Let $y_v = \sum_{i \in V(H)} (x_v)_i$. Then $x^{\parallel} = y \otimes u$ and $y \perp u$. Note that the u in $y \otimes u$ is different from the u in $y \perp u$ even though they both denote the uniform distribution. This is because they have different dimensions. The dimension of u in $y \otimes u$ is $V(H)$ and in $y \perp u$, it is $V(G)$. Now $x^T A x$ can be written as

$$\begin{aligned} x^T A x &= (x^{\parallel})^T A x^{\parallel} + (x^{\parallel})^T A x^{\perp} + (x^{\perp})^T A x^{\parallel} + (x^{\perp})^T A x^{\perp} \\ &= (x^{\parallel})^T A x^{\parallel} + 2(x^{\parallel})^T A x^{\perp} + (x^{\perp})^T A x^{\perp}. \end{aligned}$$

Let $a = (x^{\|})^T A x^{\|}$, $b = (x^{\|})^T A x^{\perp}$ and $c = (x^{\perp})^T A x^{\perp}$. Let's first look at c.

$$\begin{aligned}
|c| &= |((I \otimes A(H))x^{\perp})^T P((I \otimes A(H))x^{\perp})| \\
&\leq \|((I \otimes A(H))x^{\perp})\|^2 \qquad \text{(since } P \text{ is a permutation matrix)} \\
&= \sum_{v \in V(G)} \|A(H)x_v^{\perp}\|^2 \\
&\leq \sum_{v \in V(G)} \lambda_2(H)^2 \|x_v^{\perp}\|^2 \\
&= \lambda_2(H)^2 \|x^{\perp}\|^2
\end{aligned}$$

The last equation above is true since the action of $I \otimes A(H)$ can be considered separately on each cloud v where the bound $\|A(H)x_v^{\perp}\| \leq \lambda_2(H)\|x_v^{\perp}\|$ applies.

Using the same method as above, we can get a bound for $|b|$.

$$\begin{aligned}
|b| &= |((I \otimes A(H))x^{\|})^T P((I \otimes A(H))x^{\perp})| \\
&\leq \|((I \otimes A(H))x^{\|})\| \|((I \otimes A(H))x^{\perp})\| \\
&= \left(\sum_{v \in V(G)} \|A(H)x_v^{\|}\|\right)\left(\sum_{v \in V(G)} \|A(H)x_v^{\perp}\|\right) \\
&\leq \left(\sum_{v \in V(G)} \|x_v^{\|}\|\right)\left(\sum_{v \in V(G)} \lambda_2(H)\|x_v^{\perp}\|\right) \\
&= \lambda_2(H) \parallel x^{\|} \parallel \parallel x^{\perp} \parallel
\end{aligned}$$

Finally, let's look at a.

$$\begin{aligned}
a &= (x^{\|})^T P x^{\|} \\
&= (y \otimes u)^T P (y \otimes u) \\
&= (y \otimes u)^T (A(G) \otimes I)(y \otimes u)
\end{aligned}$$

Note that $P(y \otimes u)$ need not be equal to $(A(G) \otimes I)(y \otimes u)$ in the above. Intuitively, here's why the last equation is true. Consider y as assigning a mass to every cloud. Then, $y \otimes u$ distributes the mass of each cloud equally among its own vertices. And $P(y \otimes u)$ means that the masses are permuted according to the edges G. The term $(A(G) \otimes I)(y \otimes u)$ on the other hand, can be written as $((A(G)y) \otimes (Iu)$ and represents the following: consider y a *distribution* over G and then take one step of random walk in G i.e. each vertex gives its mass equally to all its neighbors. Then distribute the new mass of each cloud equally among its vertices. Multiplying by $(y \otimes u)^T$ simply takes a weighted sum of the mass of each cloud. The last equality above then states that the total mass of each cloud after the two processes is the same.

Now,

$$\begin{aligned}
|a| &\leq \|(y \otimes u)^T\| \cdot \lambda_2(G)\|(y \otimes u)\| \\
&\leq \lambda_2(G)\|y \otimes u\|^2 \\
&= \lambda_2(G)\|x^{\|}\|^2
\end{aligned}$$

and therefore we can bound $x^T Ax$ as

$$|x^T Ax| \leq \lambda_2(G)\|x^{\|}\|^2 + 2\lambda_2(H)\|x^{\|}\|\|x^{\perp}\| + \lambda_2(H)^2\|x^{\perp}\|^2.$$

The maximum value of the expression above is $(\lambda_2(G) + \lambda_2(H) + \lambda_2(H)^2)\|x\|^2$ because $\|x^{\|}\|$ and $\|x^{\perp}\|$ are both at most $\|x\|$ and $\|x^{\|}\|\|x^{\perp}\|$ reaches the maximum when both are $\frac{\|x\|}{\sqrt{2}}$. $\square$

As noted earlier, the bound obtained here is not enough to prove that the zig-zag product of two expanders is also an expander. This can be proved using similar techniques.

2 Constructing expanders using zig-zag products

As stated in last lecture, the goal of zig-zag products is to decrease the degree of the graphs in iterative construction of expanders without deteriorating the expansion property too much. One way to construct expanders by using zig-zag products is by combining it with squaring.

Suppose H is a constant-degree expander with number of vertices, $N(H) = d(H)^4$. Let $G_1 = H^2$. Define G_{i+1} as $G_i^2 ⓩ H$. Then, G_is have the following properties:

- $N(G_i) = N(H)^i$.
- $d(G_i) = d(H)^2$.
- $\lambda_2(G_{i+1}) \leq \lambda_2(G_i)^2 + \lambda_2(H) + \lambda_2(H)^2 \leq \frac{2}{5}$, if $\lambda_2(H) \leq \frac{1}{5}$.

This gives us a series of constant-degree expanders. We will see more about explicit construction of expanders using zig-zag products in the next lecture.