CS 880: Advanced Complexity Theory	02/22/06

Lecture 13: Super concentrators

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This lecture deals with linear size super concentrators. Super concentrators of linear size have their main application in telephone switching networks. The existence of linear size super concentrators also debunked the notion that certain explicit functions which could be expressed as graphs that were concentrators (namely the Fourier transform) had super linear lower bounds.

1 Definition of Super concentrator

Definition 1 (Super concentrator)**.** *A super concentrator of capacity N is a graph $G = (V, E)$ with two disjoint sets of distinguished vertices (I, O), with $|I| = |O| = N$ such that for all subsets $I' \subseteq I$ and $O' \subseteq O$ with $|I'| = |O'|$, there exist vertex-disjoint paths connecting I' and O'.*

The size *of a super concentrator is the cardinality of its edge set E.*

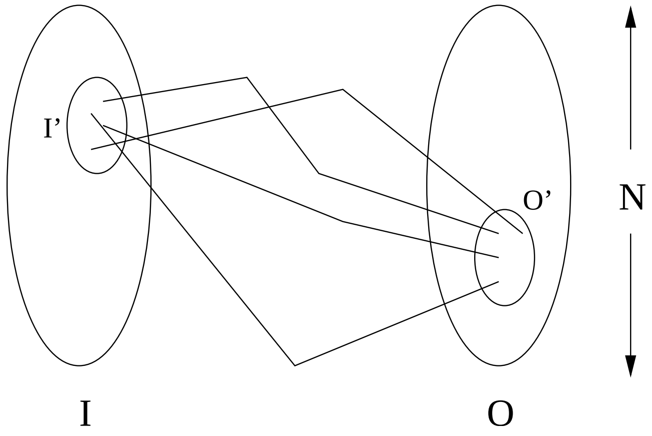

Figure 1: Subsets I' and O' of a Super Concentrator

Note that the complete bipartite graph is a trivial super concentrator. Many applications require the size of the super concentrator to be linear in its capacity N.

2 Construction of Super Concentrators

Our construction of linear size super concentrators will make use of a related type of graph, a *concentrator*.

Definition 2 (Concentrator)**.** *A graph $G = (V, E)$ is a (K, M) concentrator with N inputs if it has two sets of distinguished vertices, I and O, with $|I| = N$ and $|O| = M$ such that for all subsets $I' \subseteq I$ with $|I'| \leq K$ there are vertex-disjoint paths from every vertex in I' to O'.*

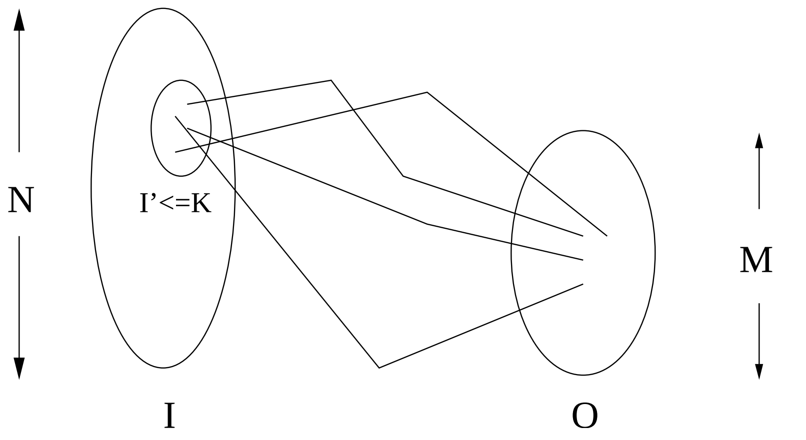

Figure 2: A $(K,\, M)$ Concentrator with N Inputs

2.1 Notes

- For a typical concentrator, $M < N$.
- By definition, a super concentrator is a $(K,\, K)$ concentrator for all $1 \leq K \leq N$.
- For bipartite graphs, being a $(K,\, M)$ concentrator is equivalent to being a $(K,\, 1)$ expander. That a $(K,\, M)$ bipartite concentrator is a $(K,\, 1)$ expander follows from the definition of concentrator, the reverse implication follows from Hall's theorem. Note that a bipartite concentrator $G = (L, R)$ is usually unbalanced with $|L| > |R|$.

2.2 Step 1: Construction of Concentrators

We will construct a bipartite, constant degree $(\frac{N}{2},\, \frac{3N}{4})$ concentrator $C = (L', R')$ using a $(\frac{M}{2},\, \frac{3}{2})$ constant degree bipartite expander $G = (L, R)$ on M by M vertices. Note that we can efficiently make such an expander using constructions we have already seen.

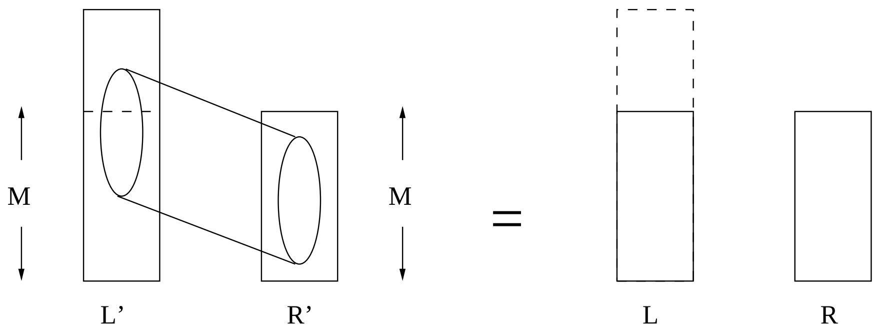

Figure 3: Creating a $(\frac{N}{2},\, \frac{3N}{4})$ Concentrator $C = (L', R')$ using a $(\frac{M}{2},\, \frac{3}{2})$ expander $G = (L, R)$

In this construction, note that any subset $S \subseteq L'$ of the left vertices of the concentrator C that has a large enough intersection with the left vertices L of the expander will be $(K,\, 1)$ expanding.

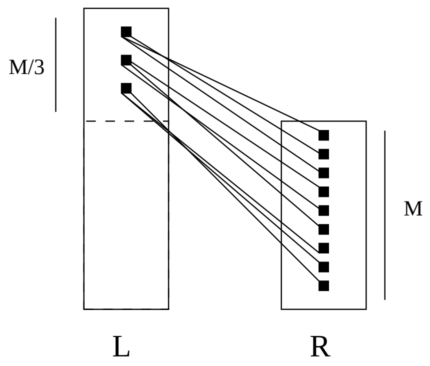

Figure 4: Mapping L to R when $|S' \cap L| < \frac{2|L|}{3}$

Specifically, this construction guarantees $(K,\ 1)$ expansion for any subset $S \subseteq L'$ with $\frac{M}{2} \geq |S \cap L| \geq \frac{2}{3}|S|$, since $|\Gamma(S \cap L)| \geq \frac{3}{2}\,|\Gamma(S \cap L)| \geq \frac{3}{2}\frac{2}{3}|S| \geq |S|$. If $|S \cap L| > \frac{M}{2}$, simply take a subset $S' \subset (S \cap L)$ with $|S'| = \frac{M}{2}$ and

$$\left|\Gamma(S')\right| \geq \frac{3}{2}\frac{M}{2} = \frac{3M}{4} = \frac{9N}{16} \geq \frac{N}{2}$$

now for those subsets with $|S' \cap L| < \frac{2}{3}\,|L'|$ we create the following constructions. For every point outside of L we include an edge to three points in R, and do this in sequential order. This guarantees $(K, 1)$ expansion for $S' \subseteq L'$ with $|S' \cap L'| < \frac{2}{3}\,|L|$.

2.3 Degree of the graphs

This construction produces a left side with the same degree as the expander used to construct it, and is at least of degree three. The degree of the right side will always be one more than that of the left side. Further, the construction of these concentrators is explicit because the construction of the expander family used is explicit.

2.4 Step 2: Constructing Super concentrators using Concentrators

To construct the super concentrator out of the concentrators we just created, we will first try to make use of as many connections as possible from one perfect matching. After doing this, at most $\frac{N}{2}$ connections will still need to be made. This is because if $|I'| = |O'| > \frac{N}{2}$ than there will be at least one connection between I' and O' from the perfect matching. Next, use the $(\frac{N}{2}, \frac{3N}{4})$ concentrator we just constructed to concentrate the inputs I into $\frac{3N}{4}$ new vertices M_1 between I and O. Similarly, use the same construction (oriented in reverse) to concentrate the $\frac{N}{2}$ outputs O into $\frac{3N}{4}$ new vertices M_2 between M_1 and the outputs. We have now reduced the problem into creating a super concentrator of capacity $\frac{3N}{4}$ between the vertices of M_1 and M_2. We do this by recursively applying our construction until we get to the base case, which will take $O(\log N)$ steps. The construction is shown in Figure 5.

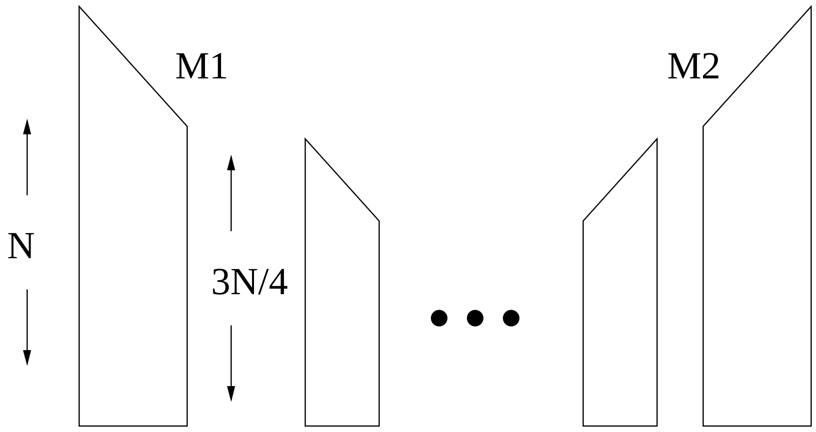

Figure 5: Iterative Construction of Linear-Size Super Concentrator of Capacity N.

2.5 Properties of the Construction

This leaves the degree of the super concentrator and the size of the vertex set required to construct it. The degree of the input of the super concentrator is the same as the degree of the left side of the concentrator used plus one, and is a constant. The degree of the output of the super concentrator is equal to the degree of the input of the super concentrator. Finally the degree of the connection vertices is equal to the degree of the output of the concentrator on their left plus the degree of the input of the concentrator on their right, which are also constant.

The size of the vertex set is also limited. Consider that there are N input vertices connected to $\frac{3N}{4}$ middle vertices (via the first concentrator), which are connected to $\frac{9N}{16}$ vertices (via the second concentrator), yielding the geometric sum:

$$N + \frac{3N}{4} + \frac{9N}{16} + \cdots \leq N \sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^i$$

and since, the right side of the construction mirrors the left side, there are fewer than twice this number of vertices so that $|V| = O(N)$.

3 Remarks

A Fourier transform's graph can be seen as a super concentrator, so in the 70's this was seen as evidence that the Fourier transform had a super-linear lower bound. This was disproved by the discovery of linear size super concentrators.

In our construction, the number of levels of recursion required is $O(logN)$. Depth 2 super concentrators of capacity N and size $O\left(N\frac{(logN)^2}{loglogN}\right)$ can be constructed through use of dispersers, as we proved in homework 2.

Briefly, a *disperser* can be seen as a bipartite graph $G = (L, R, E)$ in which any subset $S \subseteq L$ of the left vertices that is sufficiently large is connected to all but a small fraction of the right vertices. A disperser can be seen as a weak form of *extractor*, where an extractor is a bipartite graph $G = (L, R, E)$ in which, for any sufficiently large subset $S \subseteq L$ of the left vertices, the fraction of edges connecting S to any subset $T \subseteq R$ of the right vertices is proportional to the measure of T in R.