CS/Math 240: Intro to Discrete Math 2/25/2011

Solutions to the First Midterm Exam

Instructor: Dieter van Melkebeek

Problem 1

(a) *Poets and artists do not go mad but some mathematicians do.*

We will separate this sentence into two statements; "Poets and artists do not go mad" and "Some mathematicians go mad." For our first statement, we can see that an equivalent statement is "For any person, if that person is a poet or an artist, they do not go mad," i.e., $(\forall x)[(P(x) \vee A(x)) \Rightarrow \neg G(x)]$. The second statement has a clear translation as $(\exists x)(M(x) \wedge G(x))$. So combined, we have

$$(\forall x)[(P(x) \vee A(x)) \Rightarrow \neg G(x)] \wedge (\exists x)(M(x) \wedge G(x)).$$

Note that we could have also separated our original sentence into three statements: "Poets do not go mad" and "Artists do not go mad" and "Some mathematicians go mad." In that case we would have come to the slightly longer but logically equivalent

$$(\forall x)[P(x) \Rightarrow \neg G(x)] \wedge (\forall x)[A(x) \Rightarrow \neg G(x)] \wedge (\exists x)(M(x) \wedge G(x)).$$

(b) $(\forall x)(P(x) \Rightarrow A(x))$

This directly translates as "For any person, if that person is a poet, then he/she is an artist," or more succinctly, "Poets are artists."

(c) *Leonardo da Vinci is an artist and a mathematician, not a poet, but he is the friend of a poet.*

We will again separate the sentence into two statements. "Leonardo da Vinci is an artist and a mathematician, not a poet" has a direct translation, $A(d) \wedge M(d) \wedge \neg P(d)$ [recall from the problem statement that d is used to denote da Vinci]; likewise, "Leonardo da Vinci is the friend of a poet" can be stated as $(\exists x)(P(x) \wedge F(d, x))$. So we have:

$$A(d) \wedge M(d) \wedge \neg P(d) \wedge (\exists x)(P(x) \wedge F(d, x))$$

(d) $(\forall x, y)(\neg M(x) \vee \neg F(y, x))$

It is useful to rewrite this statement into an equivalent form before translating; to do so, we take one of two approaches. Recalling that $(P \Rightarrow Q)$ if equivalent to $(\neg P \vee Q)$, we can substitute $M(x)$ for P and $\neg F(y, x)$ for Q to get $(\forall x, y)(M(x) \Rightarrow \neg F(y, x))$. This reads "All mathematicians are not friends of anyone," or "Mathematicians have no friends."

Alternately, by DeMorgan's Law, $\neg(P \wedge Q)$ is equivalent to $(\neg P \vee \neg Q)$. Aapplying this to our statement gives $(\forall x, y)[\neg((M(x) \wedge F(y, x))]$. We can push this negation past the quantifier to give $\neg(\exists x, y)[(M(x) \wedge F(y, x)]$, "No mathematician has a friend."

(e) *Without friends one goes mad.*

An alternate statement of this sentence is "For any person x, if there is no person that is friends with x, x will go mad." This directly translates as

$$(\forall x)([\neg(\exists y)(F(y,x)] \Rightarrow G(x))$$

Problem 2

Let a and b be arbitrary positive integers. We will show the equivalence $M_a \cap M_b = M_{\gcd(a,b)} \Leftrightarrow a = b$ by proving both implications separately.

Let's first prove that $M_a \cap M_b = M_{\gcd(a,b)} \Rightarrow a = b$. Any natural number is multiple of itself, therefore $\gcd(a,b) \in M_{\gcd(a,b)}$. Since $M_a \cap M_b = M_{\gcd(a,b)}$, therefore $\gcd(a,b) \in M_a$ and $\gcd(a,b) \in M_b$. This means that $\gcd(a,b)$ is multiple of a and b both. Since any multiple of a natural number is at least as large as the number itself, we have $\gcd(a,b) \geq a$ and $\gcd(a,b) \geq b$. But $\gcd(a,b)$ divides both a and b which means that $\gcd(a,b) \leq a$ and $\gcd(a,b) \leq b$. Hence, $\gcd(a,b) = a = b$.

Let's now prove that $a = b \Rightarrow M_a \cap M_b = M_{\gcd(a,b)}$. Since $a = b$, therefore $M_a = M_b$ and $M_a \cap M_b = M_a$. Also, since $a = b$, $\gcd(a,b) = \gcd(a,a) = a$. Hence, $M_a \cap M_b = M_{\gcd(a,b)}$.

Problem 3

We use induction to prove that $(\forall n \geq 1)\, P(n)$, where $P(n)$ denotes

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1} \tag{1}$$

For the base case we have $n = 1$, so the LHS of (1) is

$$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} = \frac{1}{2} = \frac{1}{1+1},$$

which is same as the RHS. Thus, $P(0)$ holds.

Now suppose $P(n)$ holds for $n \geq 1$. We want to show that $P(n+1)$ holds, i.e., that

$$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \frac{n+1}{n+2} \tag{2}$$

We start with the LHS of (2), $\displaystyle\sum_{i=1}^{n+1} \frac{1}{i(i+1)}$, and split the sum into $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)}$ and $\displaystyle\frac{1}{(n+1)(n+2)}$.

The former equals $\dfrac{n}{n+1}$ by the induction hypothesis, so we have

$$\begin{aligned}
\sum_{i=1}^{n+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\
&= \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} \\
&= \frac{n+1}{n+2}
\end{aligned}$$

This completes the induction step of the proof, and the proof by induction that (1) holds for all natural numbers $n \geq 1$.

Alternately, we can give a direct proof using so-called telescoping sums.

$$\begin{aligned}
\sum_{i=1}^{n} \frac{1}{i(i+1)} &= \sum_{i=1}^{n} \frac{(i+1)-i}{i(i+1)} = \sum_{i=1}^{n} (\frac{1}{i} - \frac{1}{i+1}) \\
&= \sum_{i=1}^{n} \frac{1}{i} - \sum_{i=1}^{n} \frac{1}{i+1} = 1 + \sum_{i=2}^{n} \frac{1}{i} - \sum_{i=1}^{n-1} \frac{1}{i+1} - \frac{1}{n+1} = 1 - \frac{1}{n+1} \\
&= \frac{n}{n+1}.
\end{aligned}$$

Problem 4

First note that, if the statement is correct, then we can compute the number of breaks by following a simple pattern. For example, we can first break the $m \times n$ bar into m bars of size $1 \times n$ by making $m-1$ horizontal breaks. We can break up those bars by by making $n-1$ vertical break for each one. This gives us a total of $(m-1) + m(n-1) = mn - 1$ breaks.

We now give a proof by strong induction on mn that every strategy requires $mn - 1$ breaks.

For the base case, consider a bar consisting of just one square of chocolate. We don't need to break this chocolate bar any further to get individual 1×1 squares, and note that $1 \cdot 1 - 1 = 0$, so the base case holds.

For the induction step, suppose we have a bar with m rows and n columns of squares of chocolate. Thus, this bar consists of a total of mn squares. As our induction hypothesis, we assume that for every bar with $k \times l$ squares of chocolate, where $kl \leq mn - 1$, it takes the same number of breaks, $kl - 1$, to break it into individual 1×1 squares.

Our first break of an $m \times n$ chocolate bar can be either a horizontal break or a vertical break. First consider a horizontal break. This separates the chocolate bar into two bars, one with k rows and n columns of squares, and one with $m - k$ rows and n columns of squares. Since moves are not allowed to break apart two chocolate bars at once, we need to break these two chocolate bars into 1×1 pieces of chocolate separately. Moreover, both of these bars contain strictly fewer than mn squares, so our strong induction hypothesis implies that it takes $kn - 1$ breaks to split the first one into 1×1 squares, and $(m-k)n - 1$ breaks to split the second one. The initial horizontal break of the $m \times n$ bar into two smaller bars gives us one additional break for a total of $kn - 1 + (m-k)n - 1 + 1 = (k + m - k)n - 2 + 1 = mn - 1$ breaks to break up our $m \times n$ chocolate bar. That proves the induction step in the case the first break is a horizontal break.

Now suppose the first move is a vertical break. We can treat this break, instead, as a horizontal break of an $n \times m$ chocolate bar, and the result follows by the previous paragraph since a $n \times m$ chocolate bar also consists of mn squares, and the first break is going to break it into two bars of at most $mn - 1$ squares each.

This finishes the induction step and the proof by strong induction.

Extra Credit

We will state that the prize is in the first box. This will be proven using a proof by cases.

We know that each box must be packed by either Alice or Bob. This gives four total combinations of packing, or cases, to examine. We will label the box reading "The prize is not here" as B1, and the other box as B2.

Case One: Alice packs both boxes.

In this case, B2's statement "Exactly one box was packed by Bob" is false; however, since Alice packed B2, its statement should be true. So this case is contradictory, and thus not possible.

Case Two: Alice packs B1, Bob packs B2.

In this case, B2's statement is true; however, since Bob packed B2, its statement should be false. So this case is contradictory, and thus not possible.

Case Three: Bob packs B1, Alice packs B2.

In this case, B2's statement is true, as it should be. Since Bob packed B1, its statement is false, so "It is not the case that the prize is not here [in box one]," or "The prize is in box one."

Case Four: Bob packs both boxes.

In this case, B2's statement is false, as it should be. Since Bob packed B1, its statement is false, so "It is not the case that the prize is not here [in box one]," or "The prize is in box one."

So, for all valid cases, the prize is in box one; thus the prize is in box one. This completes our proof by cases.