Lecture 16 : Relations and Functions

Instructor: Dieter van Melkebeek   Scribe: Dalibor Zelený

DRAFT

In Lecture 3, we described a correspondence between predicates on one variable and sets. A predicate defines a set, namely the set of all elements of the domain that satisfy the predicate. Conversely, any set S defines a predicate $P(x)$: $x \in S$. Today we discuss an extension of this correspondence to predicates on two variables.

16.1   Relations

Consider a predicate $P(x, y)$ where $x \in D_1$ and $y \in D_2$ (recall that D_1 and D_2 are called domains). This predicate defines a (binary) relation. We can think of it as a set of pairs of elements, one from D_1 and one from D_2, that satisfy the predicate.

16.1.1   Examples

Example 16.1: We have already seen some predicates on two variables in this course.

- $P(x, y)$: x is a parent of y

- $C(x, y)$: x is a child of y

- $M(x, y)$: x is the mother of y

- $S(x, y)$: x is the spouse of y

$\boxtimes$

Example 16.2: Some more mathematically flavored predicates we have seen are

- $x \in y$: x is an element of y

- $x \subseteq y$: x is a subset of y

- $x < y$: x is less than y

- $x \leq y$: x is less than or equal to y

- $x \mid y$: x divides y

- $x \iff y$: x is logically equivalent to y (in other words, x and y have the same truth table)

- $x \equiv_3 y$: x and y have the same remainder after division by 3 (in other words, x and y are *congruent modulo* 3).

$\boxtimes$

Observe that sometimes both variables of a predicate come from the same domain, and sometimes they come from different domains. For example, both variables in $\subseteq$ are sets. On the other hand, in the relation $\in$, the domain for x is some set S, and the domain for y is the set of all subsets of S. We could make x and y be from the same domain if we chose the set of all sets as the domain.

16.1.2   Formal Definition of a Relation

In Lecture 4 we saw operations that make new sets out of old sets. These operations included taking intersections, unions, or power sets. In order to define relations formally, we introduce another operation on sets.

Definition 16.1. *The* Cartesian product *of sets A and B, denoted $A \times B$, is the set of all pairs of elements $a \in A$ and $b \in B$, that is,*

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

The parentheses in the notation (a, b) indicate that order matters. Thus, when we describe an element of $A \times B$, we always list the element of A first and the element of B second. Two elements of $A \times B$ are the same if and only if both of their components are the same. Formally, $(a_1, b_1) = (a_2, b_2)$ if and only if $a_1 = a_2$ and $b_1 = b_2$.

We now define a relation as a subset of the Cartesian product.

Definition 16.2. *A relation R between A and B (sometimes from A to B) is a subset of $A \times B$. We call A the* domain *of R, and B the* codomain *of R. If $A = B$, we say R is a* relation on A.

The notation $(x, y) \in R$ means that x is related to y by R, and we often denote this by xRy instead of using the former notation. For example, we say $x \subseteq y$ and $x < y$, and don't say $(x, y) \in \subseteq$ or $(x, y) \in <$.

16.1.3   Specifying Relations

The examples in this section illustrate multiple ways of describing relations.

We can define a relation by listing all pairs.

Example 16.3: Let $A = B = \{1, 2, 3, 4, 5, 6, 7\}$. We define the divisibility relation $\mid$ between A and B by listing all its elements. Observe that since $|A| = |B| = 7$, $|A \times B| = 49$, so the relation $\mid$ consists of at most 49 pairs. Therefore, there is some hope that we can list all the elements.

For each possibility for the first component, we list all its multiples in the second component. This gives us a representation of the relation $\mid$ as the set $\{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7),$ $(2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6), (7, 7)\}$.   ⊠

We can also represent a relation using a bipartite graph (we will define what a bipartite graph is later in this course). We list all elements of A one one side and elements of B on the other side. If aRb, we connect the nodes corresponding to elements $a \in A$ and $b \in B$ with an edge.

Example 16.4: Figure 16.1a is a graphical representation of the divisibility relation from Example 16.3. For example, 1 divides every integer, so the node labeled 1 on the left side is connected to all nodes on the right side. In general, a node labeled x on the left side is connected to nodes representing multiples of x on the right side.   ⊠

If a relation, such as the one from Example 16.3, is defined on a set A, we can represent it using a graph that has just one vertex set. This introduces an ambiguity because if we connect vertices a and b with an edge, this edge does not tell us whether aRb or bRa. Thus, to disambiguate, we use directed edges, where the arrow points at b if aRb, and there will be two edges, one going from a to b, and one going from b to a if both aRb and bRa. Such a graph is called a directed graph, or simply a digraph. A digraph representing the divisibility relation from Example 16.3 is in Figure 16.1b.
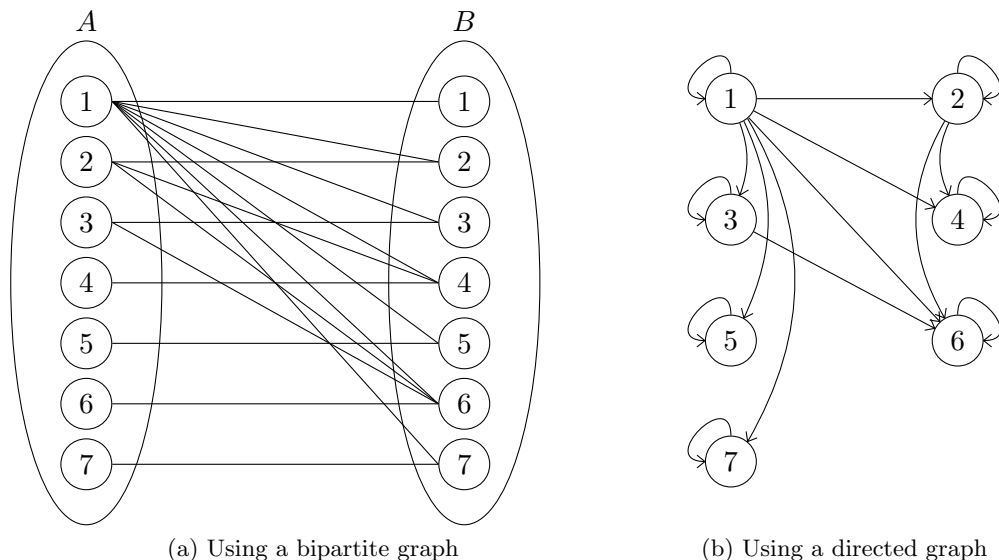
(a) Using a bipartite graph   (b) Using a directed graph

Figure 16.1: Representing the divisibility relation of Example 16.3.

Example 16.5: Congruence modulo 3 ($\equiv_3$) is a relation on $A = \{1, 2, 3, 4, 5, 6, 7\}$. The set representing this relation is $\{(1, 1), (1, 4), (1, 7), (2, 2), (2, 5), (3, 3), (3, 6), (4, 1), (4, 4), (4, 7), (5, 2), (5, 5), (6, 3), (6, 6), (7, 1), (7, 4), (7, 7)\}$.

Note that since pairs are ordered, we need to list both $(1, 7)$ and $(7, 1)$ in the enumeration of all elements of $\equiv_3$. We also remark here that congruence modulo 3 is an *equivalence relation*. We'll say more about equivalence relations in the next lecture.

Figure 16.2 shows a directed graph representing the relation $\equiv_3$.   ⊠
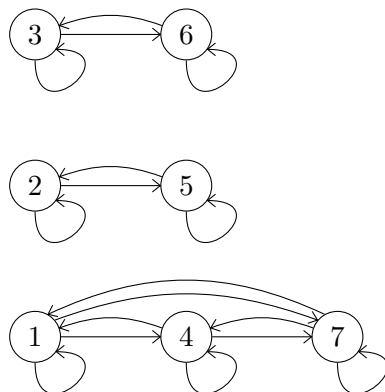
Figure 16.2: A directed graph representing the relation $\equiv_3$.

16.2   Types of Relations

There are many special types of relations that deserve closer attention. These include

- Functions

3

- Equivalence relations

- Order relations

We will discuss them in more detail today and in the next lecture.

16.2.1   Functions

You are probably familiar with the concept of a function. Functions turn out to be special cases of relations.

Definition 16.3. *A function f from A to B is a relation R from A to B where for every $a \in A$ there is at most one $b \in B$ such that aRb. We say a function is* total *if for every $a \in A$ there is one $b \in B$ such that aRb.*

We use the notation $f : A \to B$ to denote that f is a function from A to B, and write $f(a)$ to denote the the unique $b \in B$ (if any) such that aRb. We also say b is the *image of a under f*.

We also mention the notion of an inverse of a relation. Among other things, it is used to describe some properties of relations. You may be familiar with the notion of an inverse function. Unlike inverse functions, inverse relations always exist.

Definition 16.4. *The* inverse relation *of R from A to B, denoted R^{-1}, is a relation from B to A such that for all $a \in A$ and $b \in B$, $aRb \iff bR^{-1}a$.*

Example 16.6: Let's consider the relations from Example 16.1 and see if they are functions.

$P(x, y)$ is not a function. For a fixed x, there is not necessarily a unique y such that $P(x, y)$ holds. Person x can be a parent of multiple children.

$C(x, y)$ is also not a function. A child has two parents.

$M(x, y)$ is not a function because a person can be the mother of multiple children; however, the inverse relation is a function. The predicate $M^{-1}(y, x)$ says "y has x as a mother", and every person has exactly one mother. Thus, M^{-1} is a total function.

$S(x, y)$ is a function because a person has only one spouse. The function isn't total because some people are not married.   ⊠

There are some properties of relations that most commonly appear when discussing functions, so we mention them now.

Definition 16.5. *A relation R is* one-to-one *if its inverse relation R^{-1} is a function. If R is a function, we also say R is* injective.

Observe that f is injective if and only if $(\forall a_1, a_2 \in A)\ (f(a_1) = f(a_2)) \Rightarrow (a_1 = a_2)$.

Definition 16.6. *A relation R is* onto *if for every $b \in B$, there is some $a \in A$ such that aRb. If R is a function, we also say R is* surjective.

Definition 16.7. *A function that is injective and surjective is called* bijective.

Suppose f is an injective total function. Then every element of A maps to a different element of B, and for this to be possible, there must be at least one different element of B for each element of A. Hence, $|A| \le |B|$. We don't get equality because there could be some elements $b \in B$ that are not related to any $a \in A$. Also note that we can have $|A| > |B|$ if f is injective but not total.

If f is a surjective function, we have $|A| \ge |B|$. This does not require f to be total.

If f is a bijective total function, then $|A| = |B|$.

We can use the observations about injective, surjective, and bijective functions to compare cardinalities of sets. For example, if we can find a total function from A to B and prove that it is injective, we also get a proof that $|A| \le |B|$.

16.2.2   Relations on a Set A

We have seen many examples of relations where the domain and the codomain are equal. We now introduce some properties such relations can have. We use some of the relations from Example 16.2 to give examples of relations with those properties. A summary of relations and their properties is in Table 16.1 at the end of this section.

Definition 16.8. *A relation R on set A is* reflexive *if $(\forall a \in A)$ aRa. It is* antireflexive *if $(\forall a \in A)$ $\neg aRa$.*

Example 16.7: Any set is a subset of itself, so $\subseteq$ is reflexive.

The relation $<$ is not reflexive. No number can be less than itself, so $<$ is actually antireflexive. On the other hand, $\leq$ is reflexive because every number is equal to itself, so it is less than or equal to itself.

Every positive number divides itself, so we list $|$ as being reflexive in Table 16.1. Note, however, that if the domain were all integers, then $|$ would not be reflexive because zero does not divide any integer, and, in particular, does not divide itself.

A propositional formula has the same truth table as itself, and an integer has the same remainder after dividing by 3 as itself, so both $\iff$ and $\equiv_3$ are reflexive.   ⊠

Definition 16.9. *A relation R on set A is* symmetric *if $(\forall a, b \in A)$ aRb $\iff$ bRa. It is* antisymmetric *if $(\forall a, b \in A)$ $(aRb \wedge bRa) \Rightarrow (a = b)$.*

Example 16.8: The relation $\subseteq$ is not symmetric. For example, $\mathbb{N} \subseteq \mathbb{Z}$, but the other containment doesn't hold. If fact, whenever $A \subseteq B$ and $A \neq B$, we have $A \not\subseteq B$. Also note that if $A \subseteq B$ and $B \subseteq A$, then $A = B$, so $\subseteq$ is actually antisymmetric. The relations $\leq$ and $|$ are antisymmetric by a similar argument.

The relation $<$ is also not symmetric. In fact, it is as non-symmetric as it can be. Observe that $a < b$ implies b is not less than a, so it does not happen for any pair $a \neq b$ that both $a < b$ and $b < a$. Thus, $<$ is antisymmetric.

The relations $\iff$ and $\equiv_3$ are symmetric. Every relation that can be characterized by "a and b have the same . . . " is symmetric.   ⊠

Definition 16.10. *A relation R on set A is* transitive *if $(\forall a, b, c \in A)$ $(aRb \wedge bRc) \Rightarrow aRc$.*

Example 16.9: All the relations $\subseteq, <, \leq, |, \iff$ and $\equiv_3$ are easily seen to be transitive. The only one that is a little less obvious is divisibility, and for that just observe that if $a \mid b$ and $b \mid c$, there exist k and l such that $b = ka$ and $c = lb$, so $c = lb = lka$, which means a divides c.   ⊠

Finally, we summarize all our findings in Table 16.1.

relation	reflexive	symmetric	transitive
$\subseteq$	yes	anti	yes
$<$	anti	anti	yes
$\leq$	yes	anti	yes
$|$	yes	anti	yes
$\iff$	yes	yes	yes
$\equiv_3$	yes	yes	yes

Table 16.1: Properties of some relations from Example 16.2.