CS/Math 240: Introduction to Discrete Mathematics	4/7/2011

Lecture 19 : Graphs

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DRAFT

Last time we finished the discussion of digraphs and started talking about graphs. A graph is like a digraph, except edges don't have direction. Today we focus on graphs that can be drawn in a plane without any edges crossing, and then talk about what it means for two graphs to be the same.

19.1 Planar Graphs

We have been drawing graphs on a piece of paper or the blackboard. One can ask whether we can draw a particular graph so that no two lines representing two different edges intersect.

Definition 19.1. *A graph is called* planar *if it can be drawn in a plane (such as a piece of paper) without any two edges intersecting.*

Intuitively, if a graph has a lot of edges, we should not be able to draw it in a plane. Thus, we conjecture that some graphs are not planar. We will make this more precise in a moment. First let's see a few examples.

Example 19.1: The complete graph K_4 consisting of 4 vertices and with an edge between every pair of vertices is planar. Figure 19.1a shows a representation of K_4 in a plane that does not prove K_4 is planar, and 19.1b shows that K_4 is planar. The graphs K_5 and $K_{3,3}$ are nonplanar graphs. No matter how we draw them in a plane, some pair of edges will intersect. We show K_5 (the complete graph on five vertices) if Figure 19.1c, and we show $K_{3,3}r$ (the complete bipartite graph with 3 vertices on each side and all possible edges between vertices on opposite sides) in Figure 19.1d.

⊠

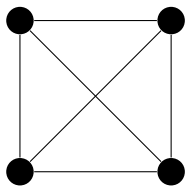
It turns out that in some sense (which we will specify soon), every nonplanar graph contains either K_5 or $K_{3,3}$. In other words, K_5 and $K_{3,3}$ are the two smallest possible nonplanar graphs. We start with a definition.

Definition 19.2. *Consider a drawing of a connected planar graph in a plane. A* face *of a planar graph is a part of the plane delimited by a cycle such that no edges are drawn inside of that cycle in the drawing.*

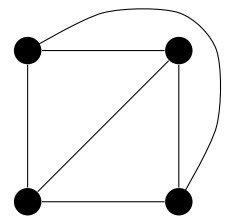
In other words, take any point P in the plane. The set of all points to which it is possible to draw a curve from P without crossing an edge is a face. For example the planar graph K_4 from Figure 19.1b has 4 faces.

Example 19.2: Consider the graph in Figure 19.2. It has four faces. The three faces inside are shaded. There is a fourth face on the outside of the graph, and is not shaded. The inside faces are delimited by the cycles (i) $1, 2, 6, 5, 1$, (ii) $2, 3, 7, 6, 2$, and (iii) $3, 4, 8, 7, 3$. The outside face has the cycle $1, 2, 3, 4, 8, 7, 6, 5, 1$ as its boundary.

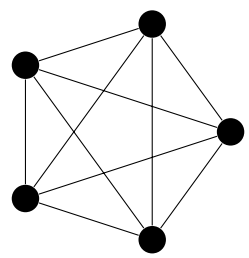
Pick any point P that is in one of the shaded regions (or on the outside) in Figure 19.2. We can draw a curve from P to any point inside the region with the same shading without crossing



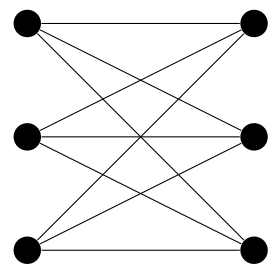
(a) The planar graph K_4 drawn with two edges intersecting.



(b) The planar graph K_4 drawn without any two edges intersecting.



(c) The nonplanar graph K_5.



(d) The nonplanar graph $K_{3,3}$

Figure 19.1: Some examples of planar and nonplanar graphs.

edges, but it is impossible to draw a curve from P to a point in a region with a different shading than P's region. We draw a curve in Figure 19.2 that shows points P and Q are in the same face. Here we also point out that the curves we draw can be arbitrarily complicated.

Not every cycle delimits a face, but every cycle has a face inside of it. For example the cycle $1, 2, 3, 7, 6, 5, 1$ does not delimit any face, but has two faces inside of it. ⊠

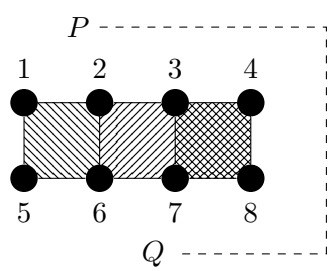


Figure 19.2: A graph with four faces. The curve from point P to point Q does not cross any edges, so it shows that P and Q belong to the same face.

We now relate the number of vertices, edges, and faces using one formula. The following theorem is known as *Euler's formula*.

Theorem 19.3. *Let $G = (V, E)$ be a connected simple planar graph, and let F be the set of faces of G. Then*

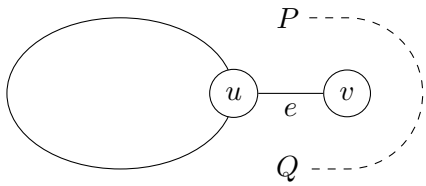
$$|V| - |E| + |F| = 2. \tag{19.1}$$

We prove Theorem 19.3 by induction. Here we remark that we have two options what to induct on: the number of vertices and the number of edges. We have been using the former for all inductive proofs about graphs so far. This is the first time we induct on the number of edges.

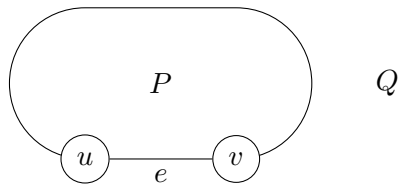
Proof of Theorem 19.3. For the base case, consider a graph G with no edges. Since there are no edges, any drawing of that G in the plane has just one face, namely the entire plane. Furthermore,

the graph has only one vertex because otherwise it would not be connected. Therefore, we have $|V| - |E| + |F| = 1 - 0 + 1 = 2$, and the base case is proved.

Now consider a graph $G = (V, E)$ and assume that (19.1) holds for any graph with $|E| - 1$ edges. We consider two cases.



(a) Case 1: The graph contains a vertex v of degree 1. Then both sides of the only edge incident on v belong to the same face as indicated by the dashed line from P on one side of the edge to Q on the other.



(b) Case 2: All vertices have degree at least 2. Then there is some cycle in G, and an edge e on that cycle is on the boundary of two faces. The remaining edges of e's cycle are represented by the unlabeled curve from u to v. The point P is inside of the cycle, and Q is outside of that cycle.

Figure 19.3: A visual aid for the proof of Euler's formula.

Case 1: G has a vertex of degree 1. Let v be a vertex of degree 1 in G, and let e be the only edge incident on v. The same face appears on both sides of the edge. To see that, pick any point P on one side of the edge. To draw a curve from that point to the point Q that is on the other side of that edge, first follow the edge towards v, then make a semicircle around v and follow the edge in the other direction on the other side until you get to Q. We can go around v because it has only one edge incident on it, so there isn't going to be another edge in our way. We show this situation in Figure 19.3a.

So consider the subgraph $G' = (V', E')$ of G consisting of all vertices of G except for v and ell edges of G except for e. This subgraph is a simple planar because it is a subgraph of a simple planar graph. It has $|V'| = |V| - 1$ vertices and $|E'| = |E| - 1$ edges. It also has $|F'| = |F|$ faces because removing e did not remove any faces. We could have gotten from one side of e to the other side of e without crossing any edges, so removing e does not add any more points to the face corresponding to point x.

Finally, G' is connected. To see that, consider any two vertices $x, y \in V'$. Also let $u \in G'$ be the other endpoint of e. Since G is connected, there is a path in G from x to y. If this path has e in it, the next edge on the path is also e because the only edge we can follow from v is e. Thus, this path goes to u, then to v, and then back to u. So we can remove the two occurrences of e from that path and still have a path from x to y. After removing all occurrences of e from the path in this fashion, we get a path from x to y that uses only edges in E'.

Since G' is a simple connected planar graph with $|E|-1$ edges, the induction hypothesis applies, and implies that $2 = |V'| - |E'| + |F'| = |V| - 1 - (|E| - 1) + |F| = |V| - |E| + |F|$.

Case 2: G has no vertices of degree 1. Then there is at least one edge that is part of a cycle. (In fact, every edge is part of some cycle, but we don't need that for this proof.) We argue similarly as in the proof that every directed graph has a topological ordering. In particular, we exhibit a cycle in G.

Start at an arbitrary edge e connecting vertices v_1 and v_2.

We continue constructing the cycle in stages. At the beginning of stage i with $2 \le i \le n$, the following invariant holds: we have distinct vertices $v_1, \ldots, v_i$ such that there is a path from v_1 to v_i

using edges $\{v_1, v_2\}, \ldots, \{v_{i-1}, v_i\}$. We have already shown it holds in the base case with $i = 2$. So now assume the invariant holds at the beginning of stage i. Since G is simple and v_i has degree at least 2, there is a vertex $u \neq v_{i-1}$ such that there is an edge $\{v_i, u\} \in E$. If $u = v_j$ for $j \in \{1, \ldots, i\}$, this edge completes the cycle $\{v_j, v_{j+1}\}, \ldots, \{v_{i-1}, v_i\}, \{v_i, v_j\}$ and we are done.

Otherwise u is different from all of $v_1, \ldots, v_i$. In that case define $v_{i+1} = u$ and move to stage $i + 1$. Notice that we have $i + 1$ distinct vertices and a path that goes from v_1 to v_{i+1}, namely the path $\{v_1, v_2\}, \ldots, \{v_{i-1}, v_i\}, \{v_i, v_{i+1}\}$. Thus, the invariant holds at the beginning of stage $i + 1$ as well.

Then, at the beginning of stage n, we have a path that goes through all vertices, namely the path $\{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}$, and since v_n has degree at least 2, there is another vertex, say $v_j \neq v_{n-1}$, such that $\{v_j, v_n\}$ is an edge. This edge completes the cycle $\{v_j, v_{j+1}\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_j\}$. Thus, we have shown that there is an edge in G that is part of a cycle.

Pick any edge $e \in E$ that is part of a cycle, and say its endpoints are u and v. Observe that this cycle splits the plane into two parts: the inside of the cycle and the outside of the cycle, and that this edge is on the boundary of both of them. The face on the inside of the cycle whose border e is part of is different from the face on the other side of e.

Now consider the subgraph $G' = (V', E')$ of G consisting of all vertices of G and all edges of G except for e. Thus, we have $|V'| = |V|$ and $|E'| = |E| - 1$. Moreover, the removal of e joins the two faces of G whose border e was, and merges them into one face. Thus, the number of faces in G', $|F'|$, is $|F| - 1$. As before, observe that G' is planar because it's a subgraph of a planar graph. The graph G' is also connected. To see that, consider a path from x to y in G. If this path does not use e, it is still present in G'. If it uses e, say by going from u to v, then, instead of traversing e, we can go from u to v by following the rest of the cycle e is part of.

Thus, G' is a simple planar graph. Furthermore, it has $|E|-1$ edges, so the induction hypothesis applies, and implies that $2 = |V'| - |E'| + |F'| = |V| - (|E| - 1) + |F| - 1 = |V| - |E| + 1$. This completes the proof. □

Now we use Theorem 19.3 to prove that neither K_5 nor $K_{3,3}$ is planar.

Theorem 19.4. *The complete graph on 5 vertices, K_5, is not planar.*

Proof. We argue by contradiction. Assume that K_5 is planar. Note that K_5 has 5 vertices and 10 edges. Let F be the set of faces in the planar representation of K_5. By Theorem 19.3, we have $|F| = 2 - |V| + |E| = 2 - 5 + 10 = 7$.

Now consider the following set of edge-face pairs: $B = \{(e, f) \in E \times F \mid e \text{ is on the border of } f\}$. We bound its cardinality in two different ways.

First, every edge is the border of at most two faces, so we have

$$|B| = \sum_{e \in E} (\text{number of faces with } e \text{ on their border}) \leq \sum_{e \in E} 2 = 2|E| = 20.$$

Second, the graph K_5 is simple, so every face has at least three edges on its border. The border cannot consist of only one edge because then that edge would be a self-loop, and it cannot consist of only two edges because those two edges would have to be connecting the same two vertices, thus contradicting the fact that K_5 is simple. Now

$$|B| = \sum_{f \in F} (\text{number of edges on } f\text{'s border}) \geq \sum_{f \in F} 3 = 3|F| = 21.$$

So we see that $|B| \leq 20$, and also $|B| \geq 21$. This cannot happen, so the assumption that K_5 is planar is wrong, and we have that K_5 is not planar. □

We use the same high-level proof structure to prove $K_{3,3}$ is not planar. We start with Euler's formula and derive a contradiction, but this time we consider vertex-face pairs instead of edge-face pairs.

Theorem 19.5. *The complete bipartite graph with three vertices on each side, $K_{3,3}$, is not planar.*

Proof. Assume that $K_{3,3}$ is planar. $K_{3,3}$ has 6 vertices and 9 edges. Let F be the set of faces in the planar representation of $K_{3,3}$. By Theorem 19.3, we have $|F| = 2 - |V| + |E| = 2 - 6 + 9 = 5$.

Consider the set $B = \{(v, f) \in V \times F \mid v \text{ is on the border of } f\}$. We bound its size in two different ways.

Every face has a cycle as its border. Note that the shortest this cycle can be is 4. It cannot be 1 or 2 for the same reason as in the previous proof. Moreover, it cannot be 3 because $K_{3,3}$ is bipartite. Every cycle begins and ends at the same vertex, and edges go between vertices in two different halves of the graph. Thus, after 3 steps, a path cannot end in the same half of the vertex set as the half where it started. But after four steps, it can, and in that case the path visits three more vertices in addition to the starting/ending one. Hence, we have

$$|B| = \sum_{f \in F} (\text{number of vertices on } f\text{'s border}) \geq \sum_{f \in F} 4 = 4|F| = 20.$$

Second, each vertex of $K_{3,3}$ has degree at 3. Each face that has v on its border must have at least two edges incident on v as its border. There are three ways to form a pair out of three edges, so v can be on the boundary of at most 3 faces. It follows that

$$|B| = \sum_{v \in V} (\text{number of faces with } v \text{ on their border}) \leq \sum_{v \in V} 3 = 3|V| = 18.$$

So we see that $|B| \geq 20$, and also $|B| \leq 18$. This cannot happen, so the assumption that $K_{3,3}$ is planar is wrong, and we have that $K_{3,3}$ is not planar. □

Now that we know K_5 and $K_{3,3}$ are not planar, let's discuss what we meant when we said K_5 and $K_{3,3}$ were the smallest nonplanar graphs. For that we need the following definition.

Definition 19.6. *A* minor *of a graph G is a graph obtained from G by applying a sequence of the following operations.*

(i) Removing an edge.

(ii) Removing a vertex and all edges it's incident on.

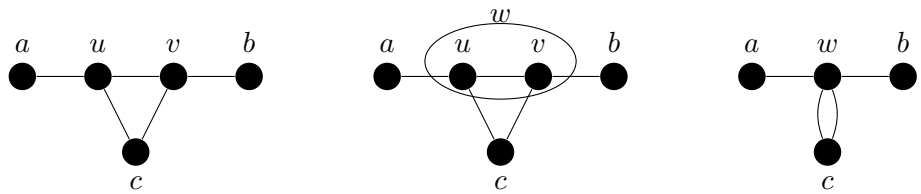
(iii) Contracting an edge. This means we combine two vertices $u, v \in V$ connected by an edge into one vertex w. We remove that edge. If an endpoint of any other edge was in u or v, we move that endpoint to w.

We call a minor obtained using only operations (i) and (ii) a subgraph *of G.*

Example 19.3: Removing edges and vertices are straightforward operations. Edge contraction may be a little less intuitive, so let's look at an example in Figure 19.4. In Figure 19.4a we see a graph G. We contract the edge e and obtain the graph in Figure 19.4c. ⊠

Theorem 19.7. *Every nonplanar graph contains at least one of K_5 and $K_{3,3}$ as a minor.*

We do not prove this theorem in this course.



(a) Before contracting e.

(b) The vertices u and v and the edge e connecting them become one vertex.

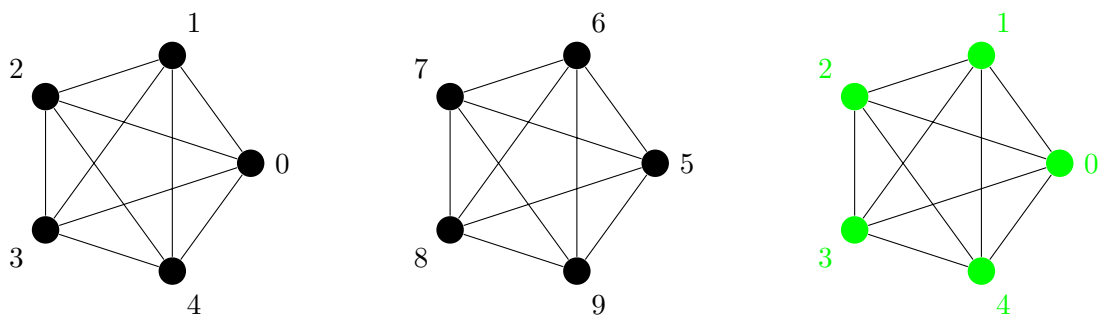
(c) After contracting e.

Figure 19.4: Contracting an edge.

19.2 Graph Isomorphism

We've been describing K_5 as *the* complete graph on five vertices. The article "the" requires an explanation. There are certainly many graphs on 5 vertices such that every pair of vertices is connected with an edge. We show a few drawings of K_5 in Figure 19.5. The first one just has its vertices labeled 0 through 4. Another one has them labeled 5 through 9. The last one has all its vertices colored and labeled green.

The concept of isomorphism ignores such differences, and only focuses on structural differences such as the number of connected components or the number of vertices. For example, all three graphs in Figure 19.5 are isomorphic.



(a) K_5 from Figure 19.1c.

(b) K_5 with different vertex labels.

(c) K_5 with green vertices.

Figure 19.5: Different ways of drawing K_5. The three graphs above are isomorphic.

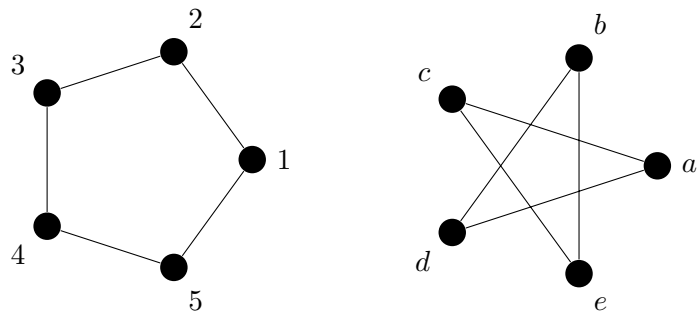
Definition 19.8. *Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are* isomorphic *if there is a total bijection $f : V_1 \to V_2$ such that $(\forall u, v \in V_1) \quad (u, v) \in E_1 \iff (f(u), f(v)) \in E_2$. We write $G_1 \cong G_2$ to denote that G_1 and G_2 are isomorphic.*

In other words, the map from Definition 19.8 renames the vertices of G_1 using labels from G_2 in a way that preserves edges.

Example 19.4: In Figure 19.6 we show two isomorphic graphs and describe the total bijection f. Note that since the two graphs have the same number of vertices, there is some hope we can get a total bijection.

We set $f(1) = a$. Now f needs to preserve edges, so for $f(2)$ we need to pick one of the vertices that are connected to vertex a by an edge because vertices 1 and 2 are connected by an edge. So we choose $f(2) = c$. The other vertex connected to 2 with an edge is 3, and f must map it to the

other vertex that is connected to $f(2) = c$ because setting $f(3) = a$ would violate the requirement that f be a bijection. So we set $f(3) = e$. We finish by setting $f(4) = b$ and $f(5) = c$. $\boxtimes$



(a) G_1 (b) G_2

v	1	2	3	4	5
$f(v)$	a	c	e	b	d

(c) Isomorphism between the two graphs.

Figure 19.6: Two isomorphic graphs and the isomorphism between them.

Graph isomorphism is an equivalence relation. This fact should remind you of the fact that the last relation of problem 2 on homework 7 is an equivalence relation. In fact, the proof is almost the same.

Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, and $G_3 = (V_3, E_3)$.

Every graph is isomorphic to itself. Just define $f(v) = v$ for all $v \in V_1$. Thus, graph isomorphism is reflexive.

Now suppose G_1 is isomorphic to G_2. Then there is a total bijection $f : V_1 \to V_2$ such that $(u, v) \in E_1 \iff (f(u), f(v)) \in E_2$. Since f is a bijection, we can "reverse" it and get the map $f^{-1} : V_2 \to V_1$ which satisfies $(w, x) \in E_2 \iff (f^{-1}(w), f^{-1}(x)) \in E_1$. It follows that graph isomorphism is symmetric.

Finally, if G_1 is isomorphic to G_2 and G_2 is isomorphic to G_3, there are total bijective functions $f : V_1 \to V_2$ and $g : V_2 \to V_3$ that satisfy the equivalences $(u, v) \in E_1 \iff (f(u), f(v)) \in E_2$ and $(w, x) \in E_2 \iff (g(w), g(x)) \in E_3$. Then the map $h : V_1 \to V_3$ defined by $h(v) = g(f(v))$ is also a total bijection. Moreover, note $(u, v) \in E_1 \iff (f(u), f(v)) \in E_2 \iff (g(f(u)), g(f(v)) \in E_3$, so h also preserves edges. Thus, graph isomorphism is transitive.

The fact that graph isomorphism is an equivalence relation explains why we can talk about K_5 as *the* complete graph on five vertices. Every other complete graph on five vertices is isomorphic to K_5.