CS 880: Quantum Algorithms 3/9/2021

Lecture 12: Fourier Transform

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In this lecture, we discuss the standard Fourier transform and its generalization from functions from $\mathbb{R}$ to $\mathbb{C}$ to functions with domains of more general groups G. We show that there exists a unique Fourier transform for all finite Abelian G. The lecture is mainly intended as background material about the notion of a Fourier transform. It can be skipped without much loss of continuity; a utilitarian definition of the Fourier transform over finite Abelian groups suffices for the rest of the course.

1 Fourier Sampling Exercise

We begin with the solution to the exercise from the last lecture, which posed the following question:

Exercise 1. *Suppose that we are given some one-to-one functions $f, g : \{0,1\}^n \to \{0,1\}^n$ with the property that $g(x) = f(x \oplus s)$ for some $s \in \{0,1\}^n$. Find s, with certainty, using $O(n)$ queries.*

Solution

We will use the Fourier sampling technique we discussed in the solution to Simon's problem. We begin with the initial superposition

$$\frac{1}{\sqrt{N}} \sum_{x \in \{0,1\}^n} |x\rangle |0^n\rangle$$

where $N = 2^n$ as usual.

In order to allow interference between the output of f and g, we introduce an additional control qubit which we use to select whether to apply U_f or U_g. To implement this, consider the function $h : \{0,1\}^{n+1} \to \{0,1\}^n$ where $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for $x \in \{0,1\}^n$. The controlled application of U_f and U_g is achieved by U_h.

We now have state

$$\frac{1}{\sqrt{N}} \sum_{x \in \{0,1\}^n} |x\rangle |0\rangle |0^n\rangle$$

and, after applying H to the control qubit, we have

$$\frac{1}{\sqrt{2N}} \sum_{x \in \{0,1\}^n} |x\rangle (|0\rangle + |1\rangle) |0^n\rangle .$$

We now perform the controlled application of U_f and U_g; we get

$$\frac{1}{\sqrt{2N}} \sum_{x \in \{0,1\}^n} |x\rangle (|0\rangle |f(x)\rangle + |1\rangle |g(x)\rangle)$$
$$= \frac{1}{\sqrt{2N}} \sum_{x \in \{0,1\}^n} |x\rangle (|0\rangle |f(x)\rangle + |1\rangle |f(x \oplus s)\rangle).$$

To enable interference, we first apply the n-fold Hadamard gate $H^{\otimes n}$ to the first register. Recall that

$$H^{\otimes n} |x\rangle = \frac{1}{\sqrt{N}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle .$$

Our state is now

$$\frac{1}{\sqrt{2}N} \sum_{x,y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle \left(|0\rangle |f(x)\rangle + |1\rangle |f(x \oplus s)\rangle\right).$$

Next, we apply H to the control qubit, resulting in a state of

$$\begin{aligned}
&\frac{1}{2N} \sum_{x,y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle \Big((|0\rangle + |1\rangle) |f(x)\rangle + (|0\rangle - |1\rangle) |f(x \oplus s)\rangle \Big) \\
=& \frac{1}{2N} \sum_{x,y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle |0\rangle \left(|f(x)\rangle + |f(x \oplus s)\rangle\right) + \\
&\frac{1}{2N} \sum_{x,y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle |1\rangle \left(|f(x)\rangle - |f(x \oplus s)\rangle\right) \\
=& \frac{1}{2N} \sum_{x,y \in \{0,1\}^n} \left((-1)^{x \cdot y} + (-1)^{(x \oplus s) \cdot y} \right) |y\rangle |0\rangle |f(x)\rangle + \\
&\frac{1}{2N} \sum_{x,y \in \{0,1\}^n} \left((-1)^{x \cdot y} - (-1)^{(x \oplus s) \cdot y} \right) |y\rangle |1\rangle |f(x)\rangle .
\end{aligned}$$

Now, as we have shown that

$$(-1)^{(x \oplus s) \cdot y} = (-1)^{x \cdot y} (-1)^{s \cdot y},$$

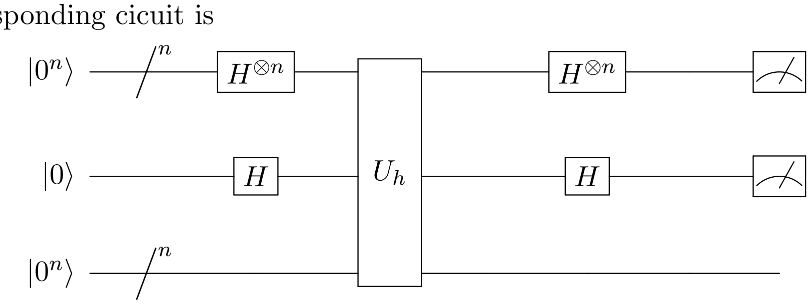
our state is

$$\begin{aligned}
&\frac{1}{2N} \sum_{x,y \in \{0,1\}^n} (-1)^{x \cdot y} \left(1 + (-1)^{s \cdot y}\right) |y\rangle |0\rangle |f(x)\rangle + \\
&\frac{1}{2N} \sum_{x,y \in \{0,1\}^n} (-1)^{x \cdot y} \left(1 - (-1)^{s \cdot y}\right) |y\rangle |1\rangle |f(x)\rangle .
\end{aligned}$$

Now, we observe the first register and control qubit. If $s \neq 0^n$, then note that $|s^\perp| = |\{0,1\}^n - s^\perp|$, so we observe $|y\rangle |0\rangle$ for some $y \in s^\perp$ or $|y\rangle |1\rangle$ for some $y \notin s^\perp$, both with probability $\frac{1}{2}$. If $s = 0^n$, we observe $|y\rangle 0$ for some $y \in \{0,1\}^n$. Note that in all cases y is distributed uniformly over its respective domain.

Hence, we observe $|y\rangle |y \cdot s \mod 2\rangle$ in all cases, where y is distributed uniformly over $\{0,1\}^n$.

The corresponding cicuit is



(1)

where U_h is the controlled application of U_f and U_g with h as defined above.

We repeat the process until we have obtained n linearly independent such y. By using amplitude amplification with error elimination in the same way as discussed in Simon's problem, we can ensure that each time we repeat the process, we obtain a y which is linearly independent from the previously obtained y using 3 applications of U_h. Once we have obtained n such y, which we can do with certainty in n iterations and $3n$ applications of U_h, we may solve the resulting system for s.

2 Standard Fourier Transform

Definition 1. *Let $f : \mathbb{R} \to \mathbb{C}$ such that $\int_x |f(x)|^2 \, dx < \infty$. Then its Fourier transform $\hat{f}$ is a function from $\mathbb{R}$ to $\mathbb{C}$ such that $\hat{f}(\omega) = \int_x f(x) e^{2\pi i \omega x} \, dx$ for all $\omega \in \mathbb{R}$. The inverse Fourier transform of $\hat{f}$ is $f(x) = \int_\omega \hat{f}(\omega) e^{-2\pi i \omega x} \, d\omega$.*

Properties of the Fourier Transform

Note that the Fourier transform is a linear transformation: $\widehat{af + bg} = a\hat{f} + b\hat{g}$. The Fourier tranform is also unitary; we can see this property using several equivalent definitions of the unitary property. A unitary transformation can be viewed as one which preserves norms or as one which transforms an orthonormal basis of its domain into an orthonormal basis of its range. Another important definition, which we have used heavily in this class, is that a transform is unitary when its inverse is equal to its adjoint.

First, consider the a unitary transformation as one which preserves inner products. If we consider the inner product space of functions from $\mathbb{R}$ to $\mathbb{C}$ with inner product $(f, g) = \int_x f(x)\overline{g(x)} \, dx$, then the Fourier transform preserves inner products, i.e., $(\hat{f}, \hat{g}) = (f, g)$, and is hence a unitary transformation.

Now, recall that the inverse Fourier transform is

$$f(x) = \int_\omega \hat{f}(\omega) e^{-2\pi i \omega x} \, d\omega.$$

As $e^{-2\pi i \omega x}$ is the conjugate of $e^{2\pi i \omega x}$, we can see that the inverse Fourier transform is the conjugate transpose, or the adjoint, of the standard Fourier transform, which is thus unitary.

We can also see that the Fourier transform is unitary as it tranforms the standard orthonormal basis (consisting of the Dirac delta functions) into an orthonormal basis, which is referred to as the Fourier basis or as the harmonics,

$$e^{2\pi i \omega x} = \cos(2\pi \omega x) + i \sin(2\pi \omega x).$$

Functions in the standard basis is referred to as being in the time domain, and functions in the Fourier basis are referred to as being in the frequency domain.

Definition 2. *The convolution of $f : \mathbb{R} \to \mathbb{C}$ with $g : \mathbb{R} \to \mathbb{C}$ is $f * g : \mathbb{R} \to \mathbb{C}$ where*

$$(f * g)(x) = \int_y f(x) g(x - y) \, dy.$$

One particularly important property of the Fourier transform is that convolution in the time domain is equivalent to multiplication in the frequency domain, i.e., that $\widehat{f * g}(\omega) = \hat{f}(\omega)\hat{g}(\omega)$ for all $\omega \in \mathbb{R}$.

We will next discuss the more general form of the Fourier transform, which will be of particular interest in developing quantum algorithms. This form is also the one most commonly used in the practical applications which rely on the $O(N \log N)$ complexity of the fast Fourier tranform, an efficient algorithm to compute the discrete Fourier transform on the group $\mathbb{Z}_N$. Due to the convolution property of the Fourier transform, the fast Fourier transform makes it possible to perform convolutions in time $O(N \log N)$ which would take $O(N^2)$ if done in the time domain instead. For this reason, the Fourier transform is heavily used in fields such as digital signal processing, computer vision, and statistics.

3 General Fourier Transform

In order to apply the Fourier Transform to quantum algorithms, we need to generalize it to a transformation of functions whose domain is a more general group; these exist for many important groups (for example, $\mathbb{R}$ under addition as above), though note that Fourier transforms do not exist for all such groups. We will show in this lecture that it is guaranteed to exist for an important class of groups, finite Abelian groups; the Fourier transform is also unique (up to permutations of the basis elements) for this class of groups.

Definition 3. *Let G be a group. A Fourier transform on G is a transformation on the space of functions $\{f : G \to \mathbb{C}\}$, mapping f to $\hat{f}$, that is:*

- *linear*
- *unitary*
- *turns convolutions into point-wise products: $\widehat{f * g}(x) = \hat{f}(x)\hat{g}(x)$ for $f, g : G \to \mathbb{C}$ and $x \in G$.*

The convolution of f and g on a finite group G is defined as $(f * g)(x) = \sum_y f(y)g(x - y)$. The group operation is used in the subtraction $x - y$; the other operations are in $\mathbb{C}$.

3.1 Characters of a Group

In constructing the Fourier transform for finite Abelian G, *characters* take the place of harmonics.

Definition 4. *A character of a group G is a homomorphism from G to the multiplicative group $\mathbb{C}$, or equivalently, a mapping $\chi : G \to \mathbb{C}$ such that $\chi(x + y) = \chi(x) \cdot \chi(y)$.*

The properties of characters χ, χ' of a finite group G include the following (proofs follow):

1. **Roots of Unity** All members of the range of a character of G are roots of unity and, in particular, $|G|$-th roots of unity, i.e., $\chi(x)^{|G|} = 1$ for all $x \in G$.
2. **Orthogonality** Distinct charaters of G are orthogonal to each other: if $\chi \neq \chi'$, then $(\chi, \chi') = 0$ where the inner product of $f, g : G \to \mathbb{C}$ is defined as

$$(f, g) = \sum_{x \in G} f(x)\overline{g(x)}.$$

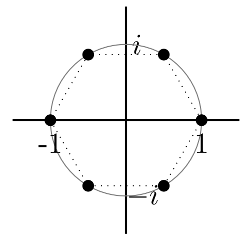


Figure 1: The sixth roots of unity

Notation: if x is a member of a group with operation $+$ and n is a positive integer, we write $n \cdot x$ to represent $x + x + \cdots + x$ where the additions are performed n times.

3.1.1 Roots of Unity Property

In the reals, the only roots of unity (i.e., of 1) are 1 and -1 (for even powers). However, given some positive integer n, there are n roots of unity in $\mathbb{C}$: specifically $e^{2k\pi i/n}$ for $0 \leq k < n$. Visualizing them in the complex plane, these values will form the vertices of a regular n-gon inscribed in the unit circle: see Figure 1 for an example with $n = 6$.

We wish to show that $\chi(x)$ is a $|G|$-th root of unity for every character χ of a finite group G.

Proof. First, we will show that $\chi(0) = 1$, which follows from the homomorphism property of χ. We have that $\chi(0) = \chi(0 + 0) = \chi(0)^2$. Since $\chi(0)$, an element of the multaplicative group $\mathbb{C}$, is invertible, we must have $\chi(0) = 1$.

Suppose that $x \in G$. We wish to show that $\chi(x)^{|G|} = 1$, i.e., that x is a $|G|$-th root of unity. Let $\langle x \rangle = \{x, x + x, x + x + x, \dots\} = \{1 \cdot x, 2 \cdot x, 3 \cdot x, \dots\}$ be the subgroup of G generated by x.

As $|G|$ is finite, $|\langle x \rangle| \leq |G|$ is as well, so we have some finite k such that $k \cdot x = 0$. Take the smallest such k, which we call the order of x (and which equals $|\langle x \rangle|$), and consider $\chi(x)^k$.

As χ is a homomorphism, $\chi(x)^k = \chi(k \cdot x) = \chi(0) = 1$. Now, from group theory we have that the order of a subgroup of a finite group divides the size of the group, so k divides $|G|$. Hence, $\chi(x)^{|G|} = 1$. ☐

3.1.2 Orthogonality Property

We now wish to show that distinct characters of a group G are orthogonal.

Proof. Suppose that $a \in G$. As a is invertible, we have that $x = y$ if and only if $a + x = a + y$. Now, as G is closed under addition, we have that

$$\begin{aligned}\sum_{x \in G} \chi(x) &= \sum_{a+x \in G} \chi(a + x) \\ &= \sum_{x \in G} \chi(a + x) \\ &= \sum_{x \in G} \chi(a)\chi(x) \\ &= \chi(a) \sum_{x \in G} \chi(x)\end{aligned}$$

as χ is a homomorphism.

Hence, we have either that $\sum_{x \in G} \chi(x) = 0$ or $\chi(a) = 1$ for all $a \in G$.

Noting that the conjugate of a root of unity is its inverse, we have, by the property shown above, that $\overline{\chi} = \chi^{-1}$ for all characters χ of G.

Suppose that χ_1, χ_2 are distinct characters of G. Now, let $\chi = \chi_1 \cdot \overline{\chi_2}$. As the conjugate of a character and the product of two characters both satisfy the homomorphism properties, they are also characters of G, and consequently, χ is a character of G. If χ is identically equal to 1, then we must have that $\overline{\chi_2} = \chi_1^{-1}$, and, by the above, that $\chi_1 = \chi_2$, a contradiction.

Thus, we must instead have

$$\begin{aligned} 0 &= \sum_{x \in G} \chi(x) \\ &= \sum_{x \in G} \chi_1(x)\overline{\chi_2(x)} \\ &= (\chi_1, \chi_2) \end{aligned}$$

and we are done. □

From the fact that $\chi(x)\overline{\chi(x)} = 1$ for all $x \in G$ and characters χ of G we immediately derive the following corollary.

Corollary 1. *The normalized characters* $\frac{1}{\sqrt{|G|}}\chi$ *are orthonormal.*

3.2 Properties of the General Fourier Transform

If $f : G \to \mathbb{C}$ can be written as

$$f = \frac{1}{\sqrt{|G|}} \sum_{\chi} \hat{f}(\chi)\overline{\chi} \tag{2}$$

for some $\hat{f}$, then $\hat{f}$ is our candidate Fourier transform of f.

3.2.1 Unitary Property

The linearity property of the Fourier transform is clearly satisfied; consider the unitary property. We will now show that the unitary property is satisfied. Suppose that f and g can be written in the form (2).

Then, by the orthogonality property of the characters χ, and the fact that $(\chi, \chi) = |G|$ for all characters χ of G we must have that

$$\begin{aligned}(f,g) &= \frac{1}{|G|}\sum_{x\in G}\sum_{\chi_1,\chi_2}\hat{f}(\chi_1)\chi_1(x)\overline{\hat{g}(\chi_2)\chi_2(x)} \\ &= \frac{1}{|G|}\sum_{\chi_1,\chi_2}\sum_{x\in G}\hat{f}(\chi_1)\overline{\hat{g}(\chi_2)}\chi_1(x)\overline{\chi_2(x)} \\ &= \frac{1}{|G|}\sum_{\chi_1,\chi_2}\hat{f}(\chi_1)\overline{\hat{g}(\chi_2)}\sum_{x\in G}\chi_1(x)\overline{\chi_2(x)} \\ &= \frac{1}{|G|}\sum_{\chi_1,\chi_2}\hat{f}(\chi_1)\overline{\hat{g}(\chi_2)}(\chi_1,\chi_2) \\ &= \sum_{\chi}\hat{f}(\chi)\overline{\hat{g}(\chi)} \\ &= (\hat{f},\hat{g})\end{aligned}$$

and so our candidate Fourier transform preserves norms and is unitary.

We will also show that our candidate Fourier transform is unitary by showing that its inverse is equal to its adjoint. By the orthagonality of the characters χ, we must also have that our candidate Fourier transform satisfies

$$\hat{f}(\chi) = (f,\overline{\chi}) = \frac{1}{\sqrt{|G|}}\sum_{x\in G} f(x)\chi(x)$$

we can see that its inverse, expressed by equation (2) for f of that form, is its conjugate transpose, or adjoint, showing again that our candidate Fourier transform is unitary.

3.2.2 Exercise: Convolutional Property

Exercise 2. *Show that our candidate Fourier tranform satisfies the convolution property of the Fourier transform, i.e., that it transforms convolutions into point-wise products.*

(a) Show that if f and g can be written in the form (2), then so can $f * g : x \to \mathbb{C}$, defined by $(f*g)(x) = \sum_{y\in G} f(y)g(x-y)$.

(b) Show that $\widehat{f*g}(\chi) = c(G)\cdot\hat{f}(\chi)\cdot\hat{g}(\chi)$.

(c) Determine $c(G)$.

We have now shown that our candidate Fourier transform satisfies the basic properties of a Fourier transform. It remains to be shown that all $f : G \to C$ can be written in form (2), i.e., that the characters of G form a basis for $\{f : G \to \mathbb{C}\}$.

We will show that this holds for finite Abelian G, and thus that the Fourier transform exists for those groups.

3.2.3 Uniqueness of Fourier Basis

Theorem 2. *If the characters of a group G span the space of all functions $f : G \to \mathbb{C}$ then the normalized characters form the unique Fourier basis up to a permutation of the basis elements and global phase.*

Proof. We have already shown above that, if the characters span the space of all functions $f : G \to \mathbb{C}$ that they form a Fourier basis; it remains to show uniqueness.

Suppose that χ_1 and χ_2 are characters of G. From the convolution property,

$$\widehat{\chi_1 * \chi_2} = c(G) \cdot \hat{\chi}_1 \cdot \hat{\chi}_2.$$

By the definition of convolutions of $f : G \to \mathbb{C}$ and the homomorphism properties of χ_2

$$\begin{aligned}(\chi_1 * \chi_2)(x) &= \sum_{y \in G} \chi_1(y)\chi_2(x-y) \\ &= \sum_{y \in G} \chi_1(y)\chi_2(x)\overline{\chi_2(y)} \\ &= (\chi_1, \chi_2) \cdot \chi_2(x)\end{aligned}$$

as $\chi_2(-y) = \chi_2(y)^{-1} = \overline{\chi_2(y)}$.

Hence, if $\chi_1 \neq \chi_2$, then we have

$$c(G) \cdot \hat{\chi}_1 \cdot \hat{\chi}_2 = \widehat{\chi_1 * \chi_2} = (\chi_1, \chi_2) \cdot \hat{\chi}_2 = 0$$

and so $\operatorname{supp}(\hat{\chi}_1) \cap \operatorname{supp}(\hat{\chi}_2) = \emptyset$.

As the vector space of functions $f : G \to \mathbb{C}$ is $|G|$-dimensional, and as the characters span the set of all such functions, we must have at least $|G|$ characters. Furthermore, because the characters are orthogonal by the above, we can have no more than $|G|$ characters and thus there exist exactly $|G|$ distinct characters of G.

Since $\hat{\chi}(\chi) = (\chi, \chi) = |G| \neq 0$ for all characters of G, we have $\operatorname{supp}(\chi) \geq 1$ for all χ and hence

$$|G| \leq \sum_{\chi} |\sup(\hat{\chi})|.$$

But as, by the above, the supports of distict χ_1 and χ_2 are disjoint and there are exactly $|G|$ characters of G, we must also have that

$$\sum_{\chi} |\sup \hat{\chi}| \leq |G|.$$

Hence, $\sum_{\chi} |\sup \hat{\chi}| = |G|$ and we must have $|\sup(\hat{\chi})| = 1$ for all χ.

Therefore, in order for χ to be in the span of the Fourier basis, some scalar multiple of it must be in the Fourier basis. As the Fourier basis is orthonormal, $\frac{\chi}{\sqrt{|G|}}$ must, in particular, be a member of the basis up to global phase. Consequently, the Fourier basis consisting of the normalized characters is unique up to a permutation of the basis elements and global phase. □

3.3 Characters of Finite Abelian Groups

We will use the following result from group theory:

Theorem 3 (Structure Theorem). *Every finite Abelian group is isomorphic to*

$$\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3} \times \cdots \times \mathbb{Z}_{N_k}$$

under component-wise addition for some $N_1, N_2, \ldots, N_k \in \mathbb{N}$.

By our previous result it suffices to find $|G|$ distinct characters. We will first find N characters for $\mathbb{Z}_N$ for $N \in \mathbb{N}$ and then find $|G_1| \cdot |G_2|$ characters of $G_1 \times G_2$ where G_1 and G_2 have $|G_1|$ and $|G_2|$ characters, respectively.

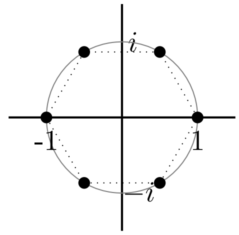


Figure 2: The sixth roots of unity

3.3.1 Characters of Modular Addition

We simply construct the following N distinct characters. Recall that the range of the characters of G is the set of N-th roots of unity (see Figure 2 for an example for $N = 6$). For each element n of $\mathbb{Z}_N$ we construct a mapping from $\mathbb{Z}_N$ onto a unique cyclic subgroup of the N-th roots of unity by mapping 0 to 1 and rotating successive elements of $\mathbb{Z}_N$ by an angle of $\frac{2\pi n}{N}$ in the complex plane.

Explicity, for $y \in \mathbb{Z}_N$, let $\chi_y : \mathbb{Z}_N \to \mathbb{C}$ such that $\chi_y(1) = (e^{2\pi i/N})^y = e^{2\pi i y/N}$ and $\chi_y(x) = \chi_y(1)^x = e^{2\pi i x y/N}$ for $x \in \mathbb{Z}_N$. As χ_y is a homomorphism and distinct for each $y \in \mathbb{Z}_N$, we are done.

3.3.2 Characters of Direct Product

As before, we construct the following $|G_1| \cdot |G_2|$ characters. For $y_1 \in G_1$ and $y_2 \in G_2$ let

$$\chi_{y_1,y_2}(x_1, x_2) = \chi_{y_1}^{(G_1)}(x_1) \cdot \chi_{y_2}^{(G_2)}(x_2).$$

As we have given a distinct $\chi_{y_1}^{(G_1)}$ for each $y_1 \in G_1$ and similarly for G_2, we have $|G_1|\cdot|G_2| = |G_1 \times G_2|$ of these, which are distinct because the $\chi_{y_1}^{(G_1)}$ and $\chi_{y_2}^{(G_2)}$ are.

To show this, note that, where 0_1 and 0_2 are the identities of G_1 and G_2, respectively, we have that $\chi_{y_1,y_2}(0_1, x_2) = \chi_{y_2}^{(G_2)}(x_2)$ and $\chi_{y_1,y_2}(x_1, 0_2) = \chi_{y_1}^{(G_1)}(x_1)$ since homomorphisms map identities to identities (in this case, to 1). If $(y_1, y_2) \neq (y_1', y_2')$ we can clearly see that χ_{y_1,y_2} and $\chi_{y_1',y_2'}$ will disagree on some point. It remains to show that they are characters, i.e., that they are homomorphisms.

Proof. By the definition of χ_{y_1,y_2} and as $\chi_{y_1}^{(G_1)}$ and $\chi_{y_2}^{(G_2)}$ are homomorphisms,

$$\begin{aligned}
\chi_{y_1,y_2}(x_1 + z_1, x_2 + z_2) &= \chi_{y_1}^{(G_1)}(x_1 + z_1) \cdot \chi_{y_2}^{(G_2)}(x_2 + z_2) \\
&= (\chi_{y_1}^{(G_1)}(x_1)\chi_{y_1}^{(G_1)}(z_1)) \cdot (\chi_{y_2}^{(G_2)}(x_2)\chi_{y_2}^{(G_2)}(z_2)) \\
&= (\chi_{y_1}^{(G_1)}(x_1)\chi_{y_2}^{(G_2)}(x_2)) \cdot (\chi_{y_1}^{(G_1)}(z_1)\chi_{y_2}^{(G_2)}(z_2)) \\
&= \chi_{y_1,y_2}(x_1, x_2) \cdot \chi_{y_1,y_2}(z_1, z_2).
\end{aligned}$$

□

As we have constructed N distinct characters for each $\mathbb{Z}_N$ for all $N \in \mathbb{N}$, the result which we have just shown that the direct product of groups G_1 and G_2 with $|G_1|$ and $|G_2|$ distinct characters has $|G_1| \cdot |G_2|$ characters allows us to show by induction that all groups of the form

$$\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3} \times \cdots \times \mathbb{Z}_{N_k}$$

for $N_1, N_2, \ldots, N_k \in \mathbb{N}$ have

$$N_1 \cdot N_2 \cdot \cdots \cdot N_k = |\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3} \times \cdots \times \mathbb{Z}_{N_k}|$$

distinct characters.

By the Structure Theorem, all finite Abelian groups are isomorphic to such a group, and hence have a unique Fourier transform, up to a permutation of the basis elements and global phase.