Quantum Computing 2/10/2022

Lecture 6: Single-Qubit and Two-Party Systems

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Last lecture we introduced the formalism of density operators to describe mixed states. In the first part of this lecture we take a closer look at the formalism in the simplest possible setting, namely a system consisting of a single qubit. In the last part we consider systems that consist of parts that are under control of two distinct parties, where the formalism offers a way of capturing the views of the individual parties. In between we discuss the connection between intermediate measurements and classical randomness.

1 Single-Qubit Systems

In this section we discuss an elegant way to describe one-qubit systems and the effect of quantum operations, namely the Bloch sphere. We start by introducing the Pauli operators. Among other things, our coverage will explain the standard notation of X, Y, and Z.

Pauli matrices We have already introduced two of the Pauli matrices, namely

$$X \doteq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad Z \doteq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

They are both Hermitian and unitary. The matrix X corresponds to a bit flip, and the matrix Z to a phase flip. They anti-commute, and their product equals the remaining Pauli matrix

$$Y \doteq \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

up to a scalar of i:

$$ZX = iY = -XZ. \tag{1}$$

It follows that Y is unitary as well; the additional factor of i makes it Hermitian, just like X and Z. The matrix Y anti-commutes with X and Z, and the equation (1) continues to hold when applying the cyclic permutation (X, Y, Z) or its inverse. The relationships can be summarized as follows: For all vectors $\vec{a}, \vec{b} \in \mathbb{C}^3$:

$$(\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma}) = (\vec{a} \cdot \vec{b})I + i(\vec{a} \times \vec{b}) \cdot \vec{\sigma}, \tag{2}$$

where $\vec{\sigma}$ denotes the vector $(\sigma_1, \sigma_2, \sigma_3) \doteq (\sigma_x, \sigma_y, \sigma_z) \doteq (X, Y, Z)$, the inner product $\vec{u} \cdot \vec{v} \doteq u_x v_x + u_y v_y + u_z v_z$ for $\vec{u} = (u_x, u_y, u_z)$ and $\vec{v} = (v_x, v_y, v_z)$, and the cross product $\vec{a} \times \vec{b}$ equals the determinant expression

$$\vec{a} \times \vec{b} \doteq \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ e_x & e_y & e_z \end{vmatrix},$$

in which e_x, e_y, and e_z denote the unit vectors along the three positive axes x, y, and z.

As the Pauli matrices are Hermitian and unitary, their eigenvalues are real and have absolute value 1, i.e., they are 1 and/or -1. In fact, as each Pauli matrix has trace 0, they each have one eigenvalue 1 and the other one -1.

Pauli basis The Pauli matrices are linearly independent. Combined with the identity matrix I, they form a basis for all complex (2×2)-matrices: Every $A \in \mathbb{C}^{2\times 2}$ can be uniquely decomposed as

$$A = a_0 \cdot I + \vec{a} \cdot \vec{\sigma} = \begin{bmatrix} a_0 + a_z & a_x - ia_y \\ a_x + ia_y & a_0 - a_z \end{bmatrix}, \tag{3}$$

where $a_0, a_x, a_y, a_z \in \mathbb{C}$. Note that $\det(A) = a_0^2 - (a_x^2 + a_y^2 + a_z^2)$.

Hermitian matrices are characterized by the property that the coefficients a_0 and $\vec{a}$ are real. For unitary matrices we have the following.

Exercise 1. *Show that unitary matrices are characterized by the property that $|a_0|^2 + \|\vec{a}\|^2 = 1$ and that $a_0\vec{a}$ has vanishing real parts. Hint: Use* (2).

If further follows that unitary matrices U are also characterized by a decomposition of the form

$$U = e^{i\alpha} \left(\cos(\gamma) I + i \sin(\gamma)(\vec{u} \cdot \sigma)\right), \tag{4}$$

where all parameters α, γ, and $\vec{u}$ are real, and $\|\vec{u}\|_2 = 1$. With the restrictions that $\alpha \in [0, 2\pi)$ and $\gamma \in [0, \pi/2]$, the decomposition is unique.

Bloch sphere representation Recall that there is a one-to-one and onto correspondence between one-qubit states and (2×2)-density matrices ρ, which are exactly (2×2)-matrices over $\mathbb{C}$ that (a) are Hermitian, (b) have trace 1, and (c) are positive semi-definite. Each of those three properties can be expressed in term of the Pauli decomposition (3) of $A \doteq \rho$:

(a) The Hermitian property is equivalent to the coefficients a_0 and $\vec{a}$ in the Pauli decomposition being real.

(b) As the Pauli matrices all have trace 0 and I has trace 2, having tracing 1 is equivalent to $a_0 = 1/2$.

(c) The product of the eigenvalues equals $\det(A) = a_0^2 - a_z^2 - |a_x + ia_y|^2$, which equals $a_0^2 - \|\vec{a}\|_2^2$ in case of real coefficients. Thus, a necessary condition for positive semi-definiteness is that $\|\vec{a}\|_2 \leq |a_0| = 1/2$. The condition is also sufficient because the trace is the sum of the eigenvalues, and equals 1, which implies that at least one of the eigenvalues is positive.

Writing $\vec{a} \doteq 2\vec{r}$, we conclude that one-qubit states ρ are in one-to-one and onto correspondence with the vectors $\vec{r} \in \mathbb{R}^3$ with $\|\vec{r}\|_2 \leq 1$, where the correspondence is given by

$$\rho = \frac{I + \vec{r} \cdot \vec{\sigma}}{2}. \tag{5}$$

The vector $\vec{r}$ is called the Bloch vector of the state ρ, and can be pictured as a point in the sphere of unit radius centered at the origin. See Figure 1.

The pure states are represented by the points on the outer shell of the Bloch sphere in Figure 1, i.e., by Bloch vectors $\vec{r}$ with $\|\vec{r}\|_2 = 1$. One way to see this is that pure states correspond to density matrices where one of the eigenvalues is 1, or equivalently, one of the eigenvalues is 0. This happens exactly when the determinant is zero, i.e., $\|\vec{a}\|_2 = 1/2$, or equivalently, $\|\vec{r}\|_2 = 1$. Another way to see this is that by (5) the Bloch vector of a convex combination equals the convex combination of the Bloch vectors, combined with the facts that (i) the pure states are the only mixed states that

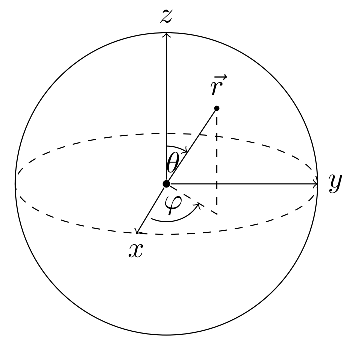


Figure 1: Bloch sphere

cannot be written as a nontrivial convex combination, and (ii) the points on the outer shell are the only points of the sphere that cannot be written as a nontrivial convex combination.

Here are some examples of pure states and their Bloch vectors.

pure state $\vert\psi\rangle$	density matrix ρ	Bloch vector $\vec{r}$
$\vert 0\rangle$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{I+Z}{2}$	$(0,0,1)$
$\vert 1\rangle$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{I-Z}{2}$	$(0,0,-1)$
$\vert +\rangle$	$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{I+X}{2}$	$(1,0,0)$
$\vert -\rangle$	$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{I-X}{2}$	$(-1,0,0)$

Exercise 2. *Consider two pure states $|\psi_j\rangle$, $j \in \{1,2\}$, with corresponding density matrices $\rho_j = |\psi_j\rangle\langle\psi_j|$ and Bloch vectors $\vec{r}_j$.*

(a) Show that

$$|\langle\psi_1|\psi_2\rangle|^2 = \frac{1+\vec{r}_1\cdot\vec{r}_2}{2}. \tag{6}$$

Hint: Show that both sides equal $\mathrm{Tr}(\rho_1\rho_2)$.

(b) Show the following geometric interpretation of (6)*: the angle between $\vec{r}_1$ and $\vec{r}_2$ equals twice the acute angle between $|\psi_1\rangle$ and $|\psi_2\rangle$.*

In particular, orthogonal pure states correspond to antipodal points in the Bloch sphere representation. In particular, $|0\rangle$ is located at the north pole, and $|1\rangle$ at the south pole in Figure 1. Similarly, $|+\rangle$ corresponds to the unit vector along the positive x-axis, and $|-\rangle$ to the one along the negative x-axis.

An explicit expression for the Bloch vector $\vec{r}$ corresponding to the pure state $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ can be obtained as follows. Since global phases do not matter, we can assume that α_0 is a nonnegative real. As $|\alpha_0|^2 + |\alpha_1|^2 = 1$, this allows us to write $\alpha_0 = \cos(\gamma)$ for some $\gamma \in [0, \pi/2]$, and $\alpha_1 = e^{i\varphi}\sin(\gamma)$ for some $\varphi \in [0, 2\pi)$.

Exercise 3. *Consider the Bloch vector $\vec{r}$ of the density matrix $\rho = |\psi\rangle\langle\psi|$ of the pure state $|\psi\rangle = \cos(\gamma)|0\rangle + e^{i\varphi}\sin(\gamma)|1\rangle$.*

(a) Show that $r_z = \cos(\theta)$ and $r_z + ir_y = e^{i\varphi}\sin(\theta)$ where $\theta = 2\gamma$.

(b) Show that the above angles θ and φ correspond to the ones indicated in Figure 1.

Inner points of the Bloch sphere correspond to mixed non-pure states. For example, the origin corresponds to the totally mixed state $\rho = I/2$, which can be obtained as an equal mixture of $|0\rangle$ and $|1\rangle$, or as an equal mixture of $|+\rangle$ and $|-\rangle$.

Bloch sphere action Consider applying a unitary operation U to the single qubit in state ρ. We showed that the new state ρ' is given by $\rho' = U\rho U^*$. What is the corresponding action on the Bloch sphere, i.e, how do we obtain the new Bloch vector $\vec{r'}$ out of the old one $\vec{r}$?

A critical property is that the action preserves inner products. This follows from (6) combined with the fact that unitary operators preserve the inner product. Morover, $\rho' = U\rho U^*$ is equivalent to

$$\vec{r'} \cdot \vec{\sigma} = U(\vec{r} \cdot \vec{\sigma})U^*, \tag{7}$$

which shows that the mapping from $\vec{r}$ to $\vec{r'}$ is linear. The only linear transformations in $\mathbb{R}^3$ that preserve the inner product are rotations around the origin. What remains it to determine the axis of rotation, and the angle. To determine those, consider the Pauli decomposition of U, which has the form (4), where we can assume that $\alpha = 0$ as global phases do not matter:

$$U = \cos(\gamma)I + i\sin(\gamma)(\vec{u} \cdot \vec{\sigma}) \tag{8}$$

for some $\gamma \in [0, \pi/2]$ and $\vec{u} \in \mathbb{R}^3$ with $\|\vec{u}\|_2 = 1$.

- Note that $\vec{u} \cdot \vec{\sigma}$ commutes with U. It follows that

  $$\vec{u'} \cdot \vec{\sigma} = U(\vec{u} \cdot \vec{\sigma})U^* = (\vec{u} \cdot \vec{\sigma})UU^* = (\vec{u} \cdot \vec{\sigma}),$$

  which shows that $\vec{u'} = \vec{u}$. Since the only invariant vectors are those along the axis of rotation, the line connecting $\vec{u}$ and the origin is the axis of rotation.

- The angle of rotation β is determined by the inner product of any unit vector $\vec{r} \in \mathbb{R}^3$ that is orthogonal to the axis of rotation $\vec{u}$, with the image $\vec{r'}$ of $\vec{r}$. We have that $\vec{r} \cdot \vec{r'} = \cos(\beta)$. This inner product equals the minimum value of the inner product $\vec{r} \cdot \vec{r'}$ over all unit vectors $\vec{r} \in \mathbb{R}^3$, which itself is related by (6) to the minimum of $|\langle\psi|U|\psi\rangle|^2$ over all pure states $|\psi\rangle$. We now compute the latter minimum using the spectral decomposition of U.

  We know that U has a full orthonormal basis of eigenvectors, and that all its eigenvalues have absolute value 1. Since $\det(U) = (\cos\gamma)^2 - (i\sin\gamma)^2\|\vec{u}\|_2^2 = (\cos\gamma)2 + (\sin\gamma)^2\|\vec{u}\|_2^2 = 1$, and $\mathrm{Tr}(U) = 2\cos\gamma$, it follows that the eigenvalues of U are $\lambda_\pm = \exp(\pm i\gamma)$. Any pure state $|\psi\rangle$ can be decomposed as $|\psi\rangle = \alpha_+|\psi_+\rangle + \alpha_-|\psi_-\rangle$, where $|\psi_\pm\rangle$ is an eigenstate corresponding to the eigenvalue $\lambda_\pm$, and $|\alpha_+|^2 + |\alpha_-|^2 = 1$. It follows that

  $$\begin{aligned} \langle\psi|U|\psi\rangle &= |\alpha_+|^2\exp(i\gamma) + |\alpha_-|^2\exp(-i\gamma) \\ |\langle\psi|U|\psi\rangle|^2 &= \left[(|\alpha_+|^2 + |\alpha_-|^2)\cos\gamma\right]^2 + \left[(|\alpha_+|^2 - |\alpha_-|^2)\sin\gamma\right]^2 \\ &= (\cos\gamma)^2 + \left[(|\alpha_+|^2 - |\alpha_-|^2)\sin\gamma\right]^2. \end{aligned} \tag{9}$$

The right-hand side of (9) is minimized for $|\alpha_+| = |\alpha_-|$, which yields the value $(\cos\gamma)^2$. By Exercise 2, this means that the minimum of $\vec{r} \cdot \vec{r'}$ equals $2\cos(\gamma)^2 - 1 = \cos(2\gamma)$. As the minimum of $\vec{r} \cdot \vec{r'}$ equals $\cos(\beta)$, we conclude that $\beta = 2\gamma$.

To summarize:

Fact 1. *The action of the unitary operation* (8) *on the Bloch sphere is a rotation with axis $\vec{u}$ over 2γ.*

In particular, for $U = X$, we have that $\gamma = \pi/2$ and $\vec{u} = (1, 0, 0)$, so the effect of X on the Bloch sphere is a rotation about the x-axis over π. Similarly, Y corresponds to a rotation about the y-axis over π, and Z to a rotation about the z-axis over π. This is what the notation X, Y, and Z refers to.

What about the effect of a measurement? Recall that $\rho' = P_0 \rho P_0 + P_1 \rho P_1$, where P_b represents the projector onto the basis state $|b\rangle$. In terms of matrices, the effect is to leave the diagonal elements untouched and cancel the off-diagonal elements:

$$\rho = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \rho' = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}. \tag{10}$$

In terms of the Pauli decomposition, the diagonal elements determine the coefficients of I and Z, whereas the off-diagonal elements determine the coefficients of X and Y. Thus, the z-component of the Bloch vector remains unaffected, whereas the x- and y-components are canceled. We conclude:

Fact 2. *The action of a measurement in the standard basis on the Bloch sphere is a projection onto the z-axis.*

Conformal mappings As a final side note, we point out another way to arrive at the Bloch sphere representation and expressions for $\vec{r}$ given in Exercise 3.

- First represent $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$ as the point α_1/α_0 in the complex plane (extended with the point ∞ at infinity to handle the case $\alpha_0 = 0$). The action of a unitary U in this representation is through a fractional linear transformation (rational map with numerator and denominator of degree 1). Such maps are conformal, i.e., they locally preserve angles (but not necessarily inner products and distances).
- Next apply a reverse stereographic projection.

The combined effect is the same as that of the Bloch representation. Since the combined effect is conformal (and, in fact, preserves inner products), as is the first step, the second step is conformal as well. This is one way to see why stereographic projection is conformal.

2 Intermediate Measurements vs Classical Randomness

We have already seen that a classical random bit can be obtained by starting a qubit in the basis state $|0\rangle$, applying a unitary (namely the Hadamard gate H), and measuring the qubit. In the other direction, we can simulate a measurement on a qubit using a classical random bit and unitary operations.

Let us first consider a single-qubit system. Recall from (10) that the effect of a measurement on the density matrix

$$\rho = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

of a single-qubit system is to cancel out the off-diagonal elements. The same affect can be achieved as follows:

1. Pick $b \in \{0, 1\}$ uniformly at random.
2. If $b = 1$ then apply the phase flip operator Z.

Applying a phase flip to the state ρ yields the state

$$Z\rho Z^* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}.$$

Thus, the effect of the above two steps is:

$$\rho = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \frac{1}{2} \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \frac{1}{2} \begin{bmatrix} a & -b \\ -c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix},$$

which is the same as (10). This means that a measurement in a single-qubit system can be replaced by picking a classical random bit and applying unitary operations. The same holds for a single-qubit measurement in a multi-qubit system as the above analysis applies to the subspace determined by fixing the other qubits to some basis state.

We conclude that, up to unitary operations, measurements are equivalent to classical randomness. This connection was exploited in the recent result on time-space efficient eliminations of intermediate measurements [GR22]. The authors first replace the intermediate measurements by classical randomness, then apply a classical pseudorandom generator (that can be computed efficiently reversibly) to reduce the number of random bits from t to $O(\log t)$, revert the remaining classical bits back to intermediate measurements, and then postpone those $O(\log t)$ intermediate measurements using the folklore method of deferring measurements that we presented. The resulting simulation only needs $O(\log t)$ extra ancillas, and only incurs a polynomial overhead in running time.

3 Solution to exercise

Consider the EPR pair (Einstein-Podolsky-Rosen): a two-component system in state

$$\tfrac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right), \tag{11}$$

where Alice holds first component, and separately Bob holds the second.

1. Describe the density operator for the entire system
   (a) at the start,
   (b) after Bob's component is measured,and
   (c) after the outcome is announced.
2. Describe Alice's view at each of the above points. *Hint:* Deferred measurement / purification exercise.

3.1 Solution to part 1

We describe the system using density operators. In part (a) the density operator is that of the pure state, which is the outer product of that state with itself:

$$\rho_a = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

After measuring the second qubit, we have a state that is an equal mixture of $|00\rangle$ and $|11\rangle$, so the density operator is an equal mixture of those of $|00\rangle$ and $|11\rangle$:

$$\rho_b = \tfrac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} + \tfrac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Another way of obtaining the result is by applying the rule for obtaining the density operator after a measurement: $\rho_b = P_{|*0\rangle} \rho_a P_{|*0\rangle} + P_{|*1\rangle} \rho_a P_{|*1\rangle}$. The first term keeps the top left 2×2 submatrix of ρ_a and cancels the rest. The second term similarly keep the bottom right 2×2 submatrix of ρ_a and cancels the rest.

For outcome $b \in \{0, 1\}$, the density operator after b is announced is given by $P_{|*b\rangle} \rho_a P_{|*b\rangle}$ (up to normalization). Thus, ρ_c can be either of two options:

$$\rho_c \in \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \right\}$$

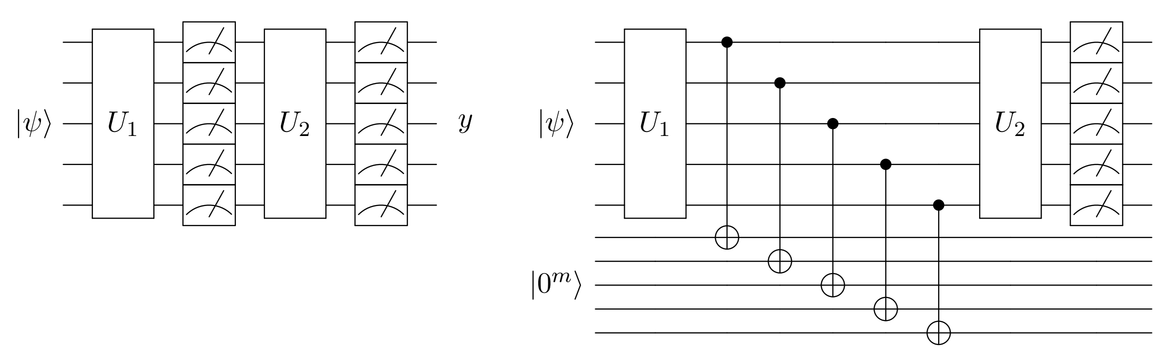
3.2 Solution to part 2

Before embarking on the solution, we gain some intuition for what Alice's part of the system looks like. We peruse the exercise on the folklore way of deferring measurements and purification, where

Figure 2: Deferring intermediate measurements

(a) Intermediate measurement

(b) Deferred measurement



we showed that for every possible pure m-qubit state $|\psi\rangle$, the distribution of the output y of the circuit in Figure 2a is the same as the distribution of the output z of the circuit in Figure 2b.

We apply the result for $m = 1$, $|\psi\rangle = |+\rangle$, and $U_1 = I$. Note that the state of the entire system right before U_2 in Figure 2b is the start state (11). The output z then is Alice's view after applying an arbitrary unitary U_2 to her part of the system, and measuring her part of the system. The output y is obtained by applying the same quantum circuit (U_2 followed by a full measurement) to Alice's qubit starting from the result of applying a measurement to $|+\rangle$, i.e., starting from the completely mixed state on one qubit. Thus, when the combined system is in state (11), for Alice it is as if her qubit is in the complete mixed state, which has density operator $\rho_a^{(\text{Alice})} = \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$.

After Bob measuring his qubit, the system is in an equal mixture of $|00\rangle$ and $|11\rangle$. This means Alice's part is in an equal mixture of $|0\rangle$ and $|1\rangle$, and thus her state still is the completely mixed state: $\rho_b^{(\text{Alice})}\frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \rho_a^{(\text{Alice})}$ She doesn't perceive any change when Bob makes his measurement.

After Bob announces the outcome of his measurement, Alice's qubit can be in either $|0\rangle$ or $|1\rangle$, so her density correspondingly can be either of the following two options: $\rho_c^{(\text{Alice})} \in \left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right\}$.

3.3 Spooky action at a distance

The collapse of an entire system (two parties') after a local observation by only one of two parties for the EPR pair and similar systems has been called "spooky action at a distance" by some. Einstein, Podolsky, and Rosen described a similar experiments with change occurring faster than light speed. That is, if Alice and Bob are separated by some distance, then the instantaneous change in density operator from Bob's measurement changed the whole system. However, because Alice cannot perceive any change in her density operator until she hears what state Bob measured, there's no actual information traveling faster than light here.

Probabilistic spooky action The same phenomenon occurs in classical probabilistic computation. If we set up the same system with probabilistic measurement and separated the two bits far apart, then we achieve the same result.

Hidden variables It should be noted that in the quantum setting, there is still *more* going on than in the classical setting. In the classical setting, such things can be described by hidden variables such as secret coin flips which neither party can see, but determine all the information. This is *not* the case for quantum systems. A well known collection of result codifying the fact that quantum systems are not described by probabilistic hidden variables can be found by reading about Bell's theorem or Bell's inequality.

4 Two-party systems

In a two-party system, two parties, Alice and Bob, have access to two different parts A and B of a quantum register. Alice applies unitary transformations and observations to her part of the register without affecting Bob's part, and vice versa. The general form of the state is $\sum_{s,t} \alpha_{s,t} |s\rangle |t\rangle$ where the first component (the state $|s\rangle$) belongs to Alice and the second component belongs to Bob. To

Alice, the state of the system looks like a mixed state over all possible states that Bob's part of the quantum register could be in. Thus, Alice's state is

$$\left\{\left(\Pr[t], \frac{\sum_s \alpha_{s,t} |s\rangle}{\sqrt{\Pr[t]}}\right)\right\}_t \quad \text{where} \quad \Pr[t] = \sum_s |\alpha_{s,t}|^2,$$

and there is a symmetric expression for Bob's state.

Suppose ρ is the density operator for the whole system. Then we wish to reduce ρ to $\rho^{(\text{Alice})}$, the density operator for Alice. We call this the *reduced density operator*.

$$\begin{aligned}
\rho^{(\text{Alice})} &= \sum_t \Pr[t] \cdot \frac{\sum_s \alpha_{s,t} |s\rangle}{\sqrt{\Pr[t]}} \cdot \frac{\sum_{s'} \overline{\alpha_{s',t}} \langle s'|}{\sqrt{\Pr[t]}} \\
&= \sum_{s,s'} \left(\sum_t \alpha_{s,t}\overline{\alpha_{s',t}}\right) |s\rangle \langle s'| \\
&= \sum_{s,s'} \left(\sum_t \rho_{\underbrace{(s,t)}_{\substack{\text{row}\\ \text{index}}},\underbrace{(s',t)}_{\substack{\text{column}\\ \text{index}}}}\right) |s\rangle \langle s'| \,. \doteq \operatorname{Tr}_B(\rho) \qquad (12)
\end{aligned}$$

This expression can be extended by linearity to any mixed combined state. The final expression in (12) $\operatorname{Tr}_B(\rho)$ is read as *trace with respect to B of ρ* and is known as a partial trace. This may look a little confusing, so let's look at an example.

Example 3. *Suppose Alice and Bob operate on a two-qubit system, where the first qubit A belongs to Alice and the second qubit B belongs to Bob. The density operator is*

$$\rho = \begin{pmatrix} \rho_{00,00} & \rho_{00,01} & \rho_{00,10} & \rho_{00,11} \\ \rho_{01,00} & \rho_{01,01} & \rho_{01,10} & \rho_{01,11} \\ \rho_{10,00} & \rho_{10,01} & \rho_{10,10} & \rho_{10,11} \\ \rho_{11,00} & \rho_{11,01} & \rho_{11,10} & \rho_{11,11} \end{pmatrix}.$$

Then the trace with respect to B is the matrix

$$\operatorname{Tr}_B(\rho) = \begin{pmatrix} \rho_{00,00} + \rho_{01,01} & \rho_{00,10} + \rho_{01,11} \\ \rho_{10,00} + \rho_{11,01} & \rho_{10,10} + \rho_{11,11} \end{pmatrix}.$$

We see that the top left entry of $\operatorname{Tr}_B(\rho)$ is the trace of a submatrix of ρ where Alice's part of the first index (i.e., the first bit of the first index in our case) is fixed to s and Alice's part of the second index is fixed to s'. Using this observation, we see that the trace with respect to A is the matrix

$$\operatorname{Tr}_A(\rho) = \begin{pmatrix} \rho_{00,00} + \rho_{10,10} & \rho_{00,01} + \rho_{10,11} \\ \rho_{01,00} + \rho_{11,10} & \rho_{01,01} + \rho_{11,11} \end{pmatrix}.$$

Example 4. *Consider the density operator for the EPR pair $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$:*

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Then we have $\operatorname{Tr}_A(\rho) = \operatorname{Tr}_B(\rho) = \frac{1}{2}I$.

References

[GR22] Uma Girish and Ran Raz. Eliminating intermediate measurements using pseudorandom generators. In Mark Braverman, editor, *13th Innovations in Theoretical Computer Science Conference, ITCS 2022, January 31 - February 3, 2022, Berkeley, CA, USA*, volume 215 of *LIPIcs*, pages 76:1–76:18. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022.