a, b, c, d , $1 \le a, b, c, d \le 10^5$

count the number of x, y that $a \le x \le b$ and $c \le y \le d$ $\gcd(x, y) = 1$.

$\gcd(x, y) = d$ s.t. $d \mid x$ and $d \mid y$ and $c \mid x$, $c \mid y \Rightarrow c \le d$

$O(n^2 \log n)$ $O(1)$ space | $O(n^2)$ $O(n^2)$ space

for x from a to b
  for y from c to d
    check (x, y)
    cr++

$\gcd[x][y] = \gcd[x][y \% x]$.

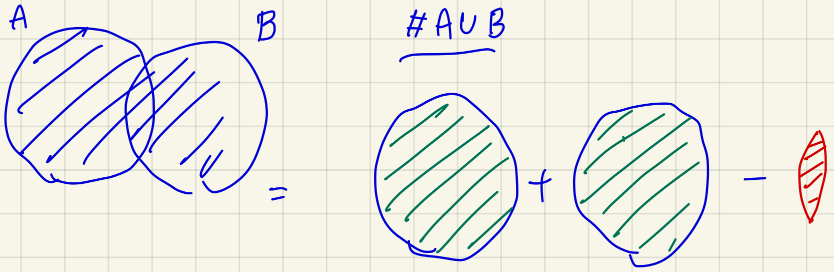
Inclusion - Exclusion $\Rightarrow$ $O(n)$

Mobius - inversion $\Rightarrow$ $O(n)$ $\Rightarrow$ $\cancel{O(n^{\frac{2}{3}})}$

Inclusion - Exclusion

A B

$\#A \cup B$



$= \#A + \#B - \#A \cap B$

Given two dices, what is the probability that at least one of them is 6.

A = the set of event that first dice is 6
B = " ——— second " is 6

$\frac{\#A \cup B}{36}$

$\underset{6}{\#A} + \underset{6}{\#B} - \underset{1}{\#A \cap B} = 11 \Rightarrow \boxed{\frac{11}{36}}$

$$\#(A_1 \cup A_2 \cup \dots \cup A_n) = \underbrace{\sum_{\text{orbit}} \#A}_{S_1} - \underbrace{\sum_{\text{orbit}} \#A_i \cap A_j}_{S_2} + \sum_{\text{orbit}} \#A_i \cap A_j \cap A_k + \dots + \underbrace{(-1)^{n+1} \#(A_1 \cap \dots \cap A_n)}_{S_n}.$$

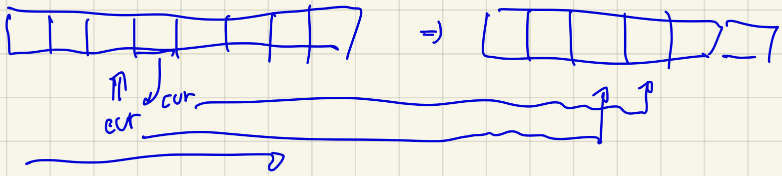
$$= \sum_{S \subseteq \{1, \dots, n\}} (-1)^{|S|+1} \left| \bigcap_{k \in S} A_k \right|$$

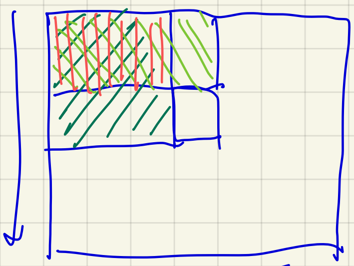
n dices what is the probability that $\sum a_i = k$. $1 \le n \le 10^5$, $1 \le k \le \min(10^5, 6 \cdot n)$

$1 \le a_i$ $O(n)$. $\Rightarrow$ $\frac{sm!}{sm_1 \, sm!}$ $\Rightarrow$ $O(n)$ precompute $O(1)$ query.

A_i = event that $a_i > 6$

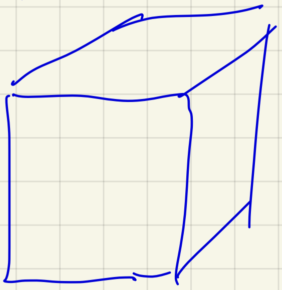
We want to compute $A_1^c \cap A_2^c \cap \dots \cap A_n^c = (A_1 \cup A_2 \cup \dots \cup A_n)^c = \#X - \#(A_1 \cup A_2 \cup \dots \cup A_n)$





$dp[i][j] = dp[i-1][j] + dp[i][j-1] - dp[i-1][j-1] + a[i][j]$

$a[i][j]$
$= dp[i][j]$
$- dp[i-1][j]$
$- dp[i][j-1]$
$+ dp[i-1][j-1]$



$dp[i, j, k] = \sum dp - - -$
$- \sum dp - - -$
$+ dp - - -$
$+ a[i][j][k]$

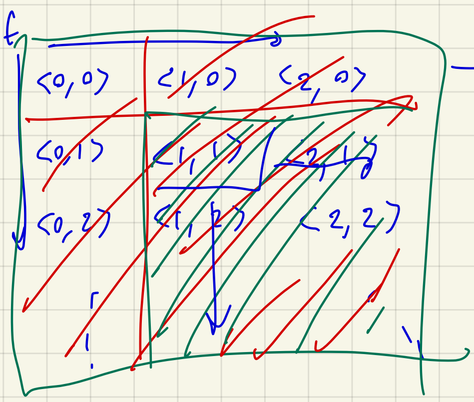
$[a, b] \ni x$
$[c, d] \ni y$

$\Rightarrow \begin{matrix} 1 \to b \\ 1 \to d \end{matrix} - \begin{matrix} 1 \to b \\ 1 \to c \end{matrix} - \begin{matrix} 1 \to a \\ 1 \to d \end{matrix} + \begin{matrix} 1 \to a \\ 1 \to b \end{matrix}$

$\boxed{\begin{matrix} 1 \to a \\ 1 \to b \end{matrix}}$

$2^n 3^m$

$120 = \langle \underset{2}{3}, \underset{3}{1}, \underset{5}{1}, \dots \rangle$



$f\langle 0,0 \rangle$ to be a number of pairs (x, y) such that $\gcd(x, y) = 1$ and $x \in [1, a]$, $y \in [1, b]$.

$g(n) = (x, y)$ that $n \mid \gcd(x, y) \Rightarrow g$ is a suffix sum.

$\# k \mid \gcd(x, y)$

$\Rightarrow \# k \mid x$ and $k \mid y$

$= \left\lfloor \frac{a}{k} \right\rfloor \times \left\lfloor \frac{b}{k} \right\rfloor$

$$\sum_{k=1}^{\min(a,b)} \boxed{\text{smth}} \times \left\lfloor \frac{a}{k} \right\rfloor \times \left\lfloor \frac{b}{k} \right\rfloor \longrightarrow \sum_{k=1}^{\min(a,b)} \mu(k) \times \left\lfloor \frac{a}{k} \right\rfloor \times \left\lfloor \frac{b}{k} \right\rfloor$$

$$\Downarrow \quad \left.\begin{array}{ll} \text{if } p^2 \mid k & \Rightarrow 0 \\ \text{else } k = p_1 \cdots p_n & \Rightarrow (-1)^n \end{array}\right\} \mu(k)$$

Given $\mu(k) \Rightarrow O(n)$

Computation of Mobiüs Inversion

if $n=1 \Rightarrow 1$

if $p^2 | n \Rightarrow 0$

$p_1 \cdots p_k = n \Rightarrow (-1)^k$

Do the seive!

Isprime(i)

v.push_back(i)

Mu[i] = -1;

Then $\Longrightarrow$ enumerate all prime & break

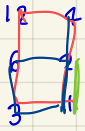
Isprime[i] = false

if (i % pr == 0) Mu = 0

else Mu = Mu[i] × (-1),

Property:

"Mobius Inversion Formula": If $f(x) = \sum_{d|n} g(d)$, then $g(n) = \sum_{d|n} \mu(\frac{n}{d}) f(d)$



Multiplicative Property: if x, y follows $\gcd(x,y)=1$, then $\mu(xy) = \mu(x)\mu(y)$.

Example: computation of $\varphi(x)$. $\Rightarrow$ #y that $\gcd(x,y)=1$.

$\varphi = \mu * ID. \leftrightarrow n = 1 * \varphi.$

Proof: $n = \sum_{d|n} \varphi(d)$ $\Rightarrow$ Example: $n=12$ $\frac{1}{12}, \underset{\frac{1}{6}}{\cancel{\frac{2}{12}}}, \underset{\frac{1}{4}}{\cancel{\frac{3}{12}}}, \underset{\frac{1}{3}}{\cancel{\frac{4}{12}}}, \frac{5}{12}, \underset{\frac{1}{2}}{\cancel{\frac{6}{12}}}, \frac{7}{12}, \underset{\frac{2}{3}}{\cancel{\frac{8}{12}}}, \underset{\frac{3}{4}}{\cancel{\frac{9}{12}}}, \underset{\frac{5}{6}}{\cancel{\frac{10}{12}}}, \frac{11}{12}, \underset{1}{\cancel{\frac{12}{12}}}$

$\sum_{d|n} \mu(d) = \mathbb{1}_{n=1}.$ $\Leftrightarrow$ $I = 1 * \mu \overset{\text{Mobius}}{\Longleftrightarrow} 1 = \sum_{d|n} I = 1.$ ✓

Back to original problem, we wish to compute

$$\sum_{i=1}^{a}\sum_{j=1}^{b} \mathbb{1}_{\gcd(i,j)=1} = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{t|\gcd(i,j)} \mu(t)$$

$$= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{t=1}^{\infty} \mathbb{1}_{t|\gcd(i,j)}\mu(t)$$

$$= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{t=1}^{\infty} \mathbb{1}_{t|i \wedge t|j}\,\mu(t) \Rightarrow \mathbb{1}_{t|i} * \mathbb{1}_{t|j}\,\mu(t).$$

$$= \sum_{t=1}^{\infty} \mu(t)\left(\sum_{i=1}^{a} \mathbb{1}_{t|i}\right)\left(\sum_{j=1}^{b} \mathbb{1}_{t|j}\right)$$

$$= \sum_{t=1}^{\infty} \mu(t) \left\lfloor \frac{a}{t} \right\rfloor \left\lfloor \frac{b}{t} \right\rfloor \quad \blacksquare$$

Exercise 2: Find sum of $\gcd(i,j)$ from $(1,a), (1,b)$ $\quad 1 \le a, b \le 10^5$

$$\sum_{i=1}^{a}\sum_{j=1}^{b} \gcd(i,j) = \sum_{k=1}^{\infty} k \sum_{i=1}^{a}\sum_{j=1}^{b} \mathbb{1}_{\gcd(i,j)=k}$$

$$= \sum_{k=1}^{\infty} k \sum_{i=1}^{\lfloor \frac{a}{k} \rfloor}\sum_{j=1}^{\lfloor \frac{b}{k} \rfloor} \mathbb{1}_{\gcd(i,j)=1}$$

$$= \sum_{k=1}^{\infty} k \sum\sum\sum_{t|\gcd} \mu(t)$$

$$= \sum_{k=1}^{\infty} k \sum_{i}\sum_{j}\sum_{t} \mu(t)\, \mathbb{1}_{t|i}\, \mathbb{1}_{t|j}.$$

$$= \sum_{k=1}^{\infty} k \sum_{t=1}^{\infty} \mu(t)\left(\sum_{i} \mathbb{1}_{t|i}\right)\left(\sum_{j} \mathbb{1}_{t|j}\right) = \sum_{k=1}^{\infty} k \sum_{t=1}^{\infty} \mu(t) \left\lfloor \frac{a}{kt} \right\rfloor \left\lfloor \frac{b}{kt} \right\rfloor. \quad \Rightarrow O(n \log n).$$

Question: Why can we swap the sum?

Exercise 3: Find sum of LCM $\quad n \le 10^5$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{lcm}(a,b) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{ij}{\gcd(i,j)} = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{n} \frac{ij}{t}\, \mathbb{1}_{\gcd(i,j)=t}$$

$$= \sum_{t=1}^{n} t \sum_{i=1}^{n/t}\sum_{j=1}^{n/t} ij\, \mathbb{1}_{\gcd(i,j)=1}$$

$$= \sum_{t=1}^{n} t \sum_{l=1}^{n} \mu(l)\left(\sum_{i=1}^{n/tl} il\right)^2$$

$$= \sum_{t=1}^{n} t \sum_{l=1}^{n/t} \mu(l)\cdot l^2\left(\frac{\frac{n}{tl}\left(1+\frac{n}{tl}\right)}{2}\right)^2 \Rightarrow O(n\log n)$$

Exercise 4: $\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} d(i\cdot j\cdot k)$ — ① (CF 235 E). $\quad 1 \le a,b,c \le 2000$

Lemma: $d(a*b) = \sum_{i|a}\sum_{j|b} \mathbb{1}_{\gcd(i,j)=1}$

Proof: Write each number as vector $a = \langle x_1, x_2, \dots \rangle$ then $d(a) = (x_1+1)(x_2+1)(x_3+1)\dots$

$\gcd(i,j)=1 \Leftrightarrow$ cannot have positive number in same column $\Rightarrow$ x_i+1 ways for each column.

$$\therefore ① = \sum_{x|i}\sum_{y|j\cdot k} \mathbb{1}_{\gcd(x,y)=1} \overset{\text{why?}}{=} \sum_{x|i}\sum_{y|j}\sum_{z|k} \mathbb{1}_{\gcd(x,yz)=1}\, \mathbb{1}_{\gcd(y,z)=1}$$

$$= \sum_{x|i}\sum_{y|j}\sum_{z|k} \mathbb{1}_{\gcd(x,y)=1}\, \mathbb{1}_{\gcd(y,z)=1}\, \mathbb{1}_{\gcd(x,z)=1}.$$

$$\therefore \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} d(i\cdot j\cdot k) = \underbrace{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}}\ \underbrace{\sum_{x|i}\sum_{y|j}\sum_{z|k}}\ \underbrace{\mathbb{1}_{\gcd(x,y)=1}\, \mathbb{1}_{\gcd(y,z)=1}\, \mathbb{1}_{\gcd(x,z)=1}}_{A}$$

$$= \sum_{x=1}^{a}\sum_{y=1}^{b}\sum_{z=1}^{c} A \left\lfloor \frac{a}{x}\right\rfloor \left\lfloor \frac{b}{y}\right\rfloor \left\lfloor \frac{c}{z}\right\rfloor$$

$$= \sum \left\lfloor \frac{a}{x}\right\rfloor \left\lfloor \frac{b}{y}\right\rfloor \left\lfloor \frac{c}{z}\right\rfloor \mathbb{1}_{\gcd(x,y)=1}\, \mathbb{1}_{\gcd(y,z)=1} \sum_{t=1}^{\infty} \mu(t)\cdot \mathbb{1}_{t|x}\cdot \mathbb{1}_{t|z}$$

$$= \sum_{y=1}^{b} \left\lfloor \frac{b}{y}\right\rfloor \sum_{t=1}^{\infty} \mu(t) \left(\sum_{x=1}^{a} \left\lfloor \frac{a}{x}\right\rfloor \mathbb{1}_{\gcd(x,y)=1}\, \mathbb{1}_{t|x}\right)\left(\sum_{z=1}^{c} \left\lfloor \frac{c}{z}\right\rfloor \mathbb{1}_{\gcd(y,z)}\, \mathbb{1}_{t|y}\right).$$

$$= \underbrace{\sum_{y=1}^{b} \left\lfloor \frac{b}{y}\right\rfloor}_{n} \underbrace{\sum_{t=1}^{\infty} \mu(t)}_{n} \underbrace{\left(\sum_{x=1}^{\lfloor a/t\rfloor} \left\lfloor \frac{a}{xt}\right\rfloor \mathbb{1}_{\gcd(y,xt)=1}\right)}_{\log n} \underbrace{\left(\sum_{z=1}^{\lfloor c/t\rfloor} \left\lfloor \frac{c}{zt}\right\rfloor \mathbb{1}_{\gcd(y,zt)=1}\right)}_{\log n}$$

$\therefore O(n^2 \log n)$

If time permits, NAC 2020 A.