Honors Thesis
A Bound on the Maximum Coefficient of Gaussian Polynomials

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1 Introduction

The Gaussian binomial coefficients $\binom{a+b}{a}_q$ are an extension of binomial coefficients that introduce a new parameter q. When $q = 1$, the behavior of the Gaussian binomial coefficients is identical to the standard binomial coefficients, but different values of q allow them to describe other phenomena. For example, the number of a-dimensional subspaces of a $(a + b)$-dimensional vector space in a finite field $\mathbb{F}_q$ is given by $\binom{a+b}{a}_q$.

When q varies and other parameters are kept fixed, the Gaussian binomial coefficient $\binom{a+b}{a}_q$ is in fact a polynomial in q of degree ab, which we will refer to as the Gaussian polynomial. Let the coefficient of q^k in $\binom{a+b}{a}_q$ be $G(a, b, k)$. These coefficients have many interesting combinatorial uses. Notably, the number of permutations of $\{1, \ldots, a + b\}$ with exactly k inversions between the first a elements and last b elements is given by $G(a, b, k)$. Among other things, this characterization implies that the coefficients $G(a, b, k)$ are nonnegative integers, and add up to $\binom{a+b}{a}$ for any fixed a and b. This means $G(a, b, k)/\binom{a+b}{a}$ defines a probability distribution on the integers from 0 to ab. Our goal is to establish a tight upper bound on the maximum probability of this distribution as a function of a and b.

The characterization involving inversions plays an important role in the Mann-Whitney U test in statistics [MW47]. The Mann-Whitney test compares two independent populations A and B, and has as null hypothesis that both have the same distribution. The test is nonparametric, meaning that the distributions of A and B are arbitrary and unknown. The test achieves this property by only considering the relative order of the samples from A and B. It takes a samples from A and b samples from B, arranges them in sorted order, and then counts the number of inversions, namely the pairs of elements of B that come before elements of A. Using the above characterization of the Gaussian polynomial coefficients, the probability that the number of inversions equals k is given by $G(a, b, k)/\binom{a+b}{a}$, if the null hypothesis holds. Mann and Whitney proved that the distribution converges to a normal distribution as a and b grow large.

Motivation Our goal is motivated by the combinatorial problem of minimizing inversions in trees as described in [HvMM22]. In this problem, an unordered tree T with root r has n leaf nodes, each with a distinct label in the set $\{1, \ldots, n\}$, and the question is finding the minimum number of possible inversions among the leaf nodes over all possible orderings of the tree. The goal is to determine the answer with as few comparisons between leaves as possible.

One can make an analogy to sorting, where $\log_2(n!)$ comparisons are needed, because there are a total of $n!$ possible permutations, and a sorting algorithm must make different decisions for each of them. However, in this scenario, the output is an integer between 0 and $\binom{n}{2}$, so this information theoretic framework only yields a lower bound of $\Omega(\log n)$ comparisons. This may seem reasonable at first – sorting requires $\Omega(n \log n)$ comparisons, but no comparisons are needed at all to find the minimum number of inversions (it is always zero).

However, for general binary trees, the problem of finding the minimum number of inversions is almost as hard as sorting up to a small gap. This relies on the sensitivity of the tree, which is the expectation over a random ranking of the number of pairs of adjacent-rank labels on leaf nodes that result in a change in the minimum number of inversions.

By considering a node v with children v_1, v_2, we can analyze when we swap the label of a leaf descendant of v_1 with a leaf descendant of v_2. The descendants of v_1 and the descendants of v_2 form two sets with a certain number of inversions between them. Swapping a descendant of v_1 with a descendant of v_2 will leave the minimum number of inversions unchanged unless the number of inversions between the descendants of v_1 and v_2 equals a specific value (based on the positions of the two elements being swapped). Thus, the insensitivity of T is upper bounded by the maximum value of $G(a, b, k)$ interpreted as a probability.

This analysis recursively applies to every non-leaf node in the tree, which means the sensitivity of T might be dominated by nodes close to the leaves or nodes close to the root, depending on the structure of T. This motivates a bound that applies without any assumptions on a and b.

Prior Work Because of the multiple uses of the coefficients of the Gaussian polynomials, there is a significant body of previous work on the asymptotic behavior of these coefficients. Mann and Whitney [MW47] argued that the underlying distribution converges to a normal distribution with mean $\mu = ab/2$ and variance $\sigma^2 = ab(a + b + 1)/12$ as a and b grow large, an important fact used in the Mann-Whitney U test mentioned previously.

As the normal distribution has a maximum density of $1/(\sqrt{2\pi}\sigma)$, their result suggests that the maximum of the underlying probability distribution is $O(1/\sigma) = O(1/\sqrt{ab(a + b + 1)})$. Takács [Tak86] managed to formally establish such a bound for all pairs (a, b) with $|a - b| = O(\sqrt{a + b})$, Stanley and Zanello [SZ16] for all pairs (a, b) with $\min(a, b)$ bounded, and Melczer, Panova, and Pak [MPP20] for all pairs (a, b) with $|a - b| \leq \alpha \cdot (a + b)$ for some constant $\alpha < 1$. However, these results do not cover all regimes, and leave open a single bound that applies to all pairs (a, b).

New Bound This work provides an unconditional bound on the maximum value of any coefficient in $\binom{a+b}{a}_q$. In particular, the maximum value of any coefficient in the Gaussian polynomial is at most $O(\binom{a+b}{a}/\sqrt{ab(a + b)})$ when $a, b \geq 1$. This gives an upper bound for the maximum coefficient of the Gaussian polynomial that matches the results of [MW47], [SZ16], and [MPP20] up to a constant factor, but with our result applying to all pairs a, b without assuming an asymptotic regime. Although it does not give more results on the precise distribution of the coefficients, the

upper bound is instrumental in the tight lower bound of $\log_2(n!) - O(\log n)$ for the number of comparisons in the tree inversion minimization in [HvMM22].

Organization Section 2 consists of background information on Gaussian binomial coefficients, starting with definitions in Section 2.1 and combinatorial characterizations in Section 2.2. Additional properties of the coefficients of the Gaussian polynomial are in Section 2.3, which can be skipped without loss of continuity. Section 2.4 discusses the characteristic function of the Gaussian polynomial coefficients. Our main result and the proof are presented in Section 3, with Sections 3.1 and 3.2 bounding an integral resulting from the characteristic function, and Section 3.3 proving an important combinatorial lemma. We compare our result with previous works in Section 4.

2 Background

We start by introducing the definition and basic properties of the Gaussian binomial coefficients, especially their interpretation as Gaussian polynomials. This section also presents the connections of the Gaussian polynomials to various combinatorial quantities. We also discuss the coefficients of the Gaussian polynomial, as well as their interpretation as a random variable and the resulting characteristic function.

2.1 Definitions and basic properties

Definition 1. The q-analog of a positive integer n is given by

$$[n]_q = q^{n-1} + \cdots + q + 1 = \begin{cases} \frac{1-q^n}{1-q} & q \neq 1 \\ n & q = 1 \end{cases}$$

We also formally define $[0]_q = 1$, even when $q = 1$. Therefore, $[n]_q$ is continuous as a function of q.

We can use this to define the q-factorial.

$$[n]_q! = [1]_q[2]_q \cdots [n]_q.$$

Definition 2. For nonnegative integers a, b, the *Gaussian binomial coefficient* $\binom{a+b}{a}_q$ is defined as the following:

$$\binom{a+b}{a}_q = \frac{[a+b]_q!}{[a]_q![b]_q!}.$$

When $q = 1$, the Gaussian binomial coefficient is identical to the standard binomial coefficient $\binom{a+b}{a}$, hence the name. When $q \neq 1$, we can write

$$\begin{aligned} \binom{a+b}{a}_q &= \frac{(1-q)(1-q^2)\cdots(1-q^{a+b})}{(1-q)\cdots(1-q^a)(1-q)\cdots(1-q^b)} \\ &= \frac{(1-q^{1+b})\cdots(1-q^{a+b})}{(1-q)\cdots(1-q^a)}. \end{aligned} \tag{1}$$

It is clear that from (1), the Gaussian binomial coefficient is a rational function in q. However, we can use the following recursive formula to show that $\binom{a+b}{a}_q$ is in fact a polynomial in q.

Proposition 3 (Generalized Pascal)**.** For integers $a, b \geq 0$,

$$\binom{a+b}{a}_q = \begin{cases} 1 & \text{if } a = 0 \text{ or } b = 0 \\ \binom{a+b-1}{a-1}_q + q^a \cdot \binom{a+b-1}{a}_q & \text{if } a > 0 \text{ and } b > 0. \end{cases}$$

Proof. The case of $a = 0$ or $b = 0$ follows directly from definition. When $a > 0$ and $b > 0$, notice that when $q \neq 1$,

$$\binom{a+b-1}{a-1}_q = \frac{(1-q^{b+1})\cdots(1-q^{a+b-1})}{(1-q)\cdots(1-q^{a-1})} = \frac{(1-q^a)}{(1-q^{a+b})}\binom{a+b}{a}_q$$

and

$$\binom{a+b-1}{a}_q = \frac{(1-q^b)\cdots(1-q^{a+b-1})}{(1-q)\cdots(1-q^a)} = \frac{(1-q^b)}{(1-q^{a+b})}\binom{a+b}{a}_q.$$

Therefore

$$\begin{aligned} \binom{a+b-1}{a-1}_q + q^a\binom{a+b-1}{b}_q &= \frac{(1-q^a)+q^a(1-q^b)}{1-q^{a+b}}\binom{a+b}{a}_q \\ &= \binom{a+b}{a}_q. \end{aligned}$$

The recursion must also hold for $q = 1$ because the q-binomial coefficient is continuous. (The case $q = 1$, in particular, is identical to Pascal's identity for binomial coefficients.) □

A variation of the recurrence in Proposition 3 is given below, which may be proven in a similar manner.

$$\binom{a+b}{a}_q = \binom{a+b-1}{a}_q + q^b\binom{a+b-1}{a-1}_q \tag{2}$$

This allows us to prove the following generalization of the binomial theorem – thus, the name "Gaussian binomial coefficients."

Proposition 4 (Generalized Binomial Theorem)**.** For integers $n \geq 0$,

$$\prod_{k=1}^{n}(1+q^{k-1}x) = \sum_{m=0}^{n}\binom{n}{m}_q \cdot q^{m(m-1)/2}x^m.$$

Proof. We use induction on n. When $n = 0$, both sides are 1, with the left hand side being the empty product. We can write

$$\prod_{k=1}^{n+1}(1+q^{k-1}x) = (1+q^n x)\prod_{k=1}^{n}(1+q^{k-1}x).$$

Using the inductive hypothesis, we can say for $1 \leq m \leq n$, the coefficient of x^m is

$$\begin{aligned} &\binom{n}{m}_q \cdot q^{m(m-1)/2} + \binom{n}{m-1}_q \cdot q^{n+(m-1)(m-2)/2} \\ &= q^{m(m-1)/2}\left(\binom{n}{m}_q + q^{n-m+1}\binom{n}{m-1}_q\right) \\ &= q^{m(m-1)/2} \cdot \binom{n+1}{m}_q, \end{aligned}$$

which follows from Eq. (2). One can also check that the coefficients of x^0 and x^{n+1} match. $\square$

From Proposition 3, we can use induction to deduce that $\binom{a+b}{a}_q$ is a polynomial in q, which we will refer to as the *Gaussian polynomial*. By inspecting the numerator and denominator in (1), we can see that the degree of $\binom{a+b}{a}_q$ is ab.

Definition 5. Let $G(a, b, k)$ be the coefficient of q^k in the Gaussian polynomial $\binom{a+b}{a}_q$. In other words, let

$$\binom{a+b}{a}_q = \sum_{k=0}^{ab} G(a, b, k) q^k.$$

When $k < 0$ or $k > ab$, we define $G(a, b, k) = 0$.

From Proposition 3, we can get a recursive definition for $G(a, b, k)$.

Proposition 6. For nonnegative integers a, b and any integer k,

$$G(a, b, k) = G(a - 1, b, k) + G(a, b - 1, k - a).$$

By induction from the cases where $a = 0$ or $b = 0$, Proposition 6 implies that $G(a, b, k)$ is a positive integer when $0 \leq k \leq ab$, and zero otherwise. Furthermore, by plugging in $q = 1$ into Definition 5, we get that

$$\sum_{k=0}^{ab} G(a, b, k) = \binom{a+b}{a}.$$

From these two observations, $G(a, b, k)/\binom{a+b}{a}$ represents a probability distribution of a random variable in the range $k \in \{0, \ldots, ab\}$.

2.2 Characterizations of the Gaussian polynomial coefficients

There are several combinatorial quantities that are described by the coefficients of the Gaussian polynomial as a result of this recurrence. Notably, these coefficients describe the number of cross inversions between two sets. The notion of cross inversions is useful in the merge step of mergesort.

For any array $A = [x_1, \ldots, x_a]$ of distinct elements from an ordered set, say $\mathbb{Z}$ or $\mathbb{R}$, an *inversion* is a pair of indices $i < j$ such that $x_i > x_j$, and let $\mathrm{Inv}(A)$ be the number of inversions in A. For two disjoint arrays $A = [x_1, \ldots, x_a]$ and $B = [y_1, \ldots, y_b]$, we can define the number of *cross inversions* $\mathrm{Inv}(A : B)$ as the number of pairs of indices $i \in \{1, \ldots, a\}$ and $j \in \{1, \ldots, b\}$ such that $x_i > y_j$.

Characterization 7. Consider a set $X = \{x_1, \ldots, x_{a+b}\}$ permuted into an array, with the first a elements forming a subarray A and the last b elements forming a subarray B. Then the number of permutations of X such that $\mathrm{Inv}(A : B) = k$ is $G(a, b, k)$.

Proof. We use strong induction. Suppose the smallest element of X is x_1. We consider two scenarios: x_1 is either contained in A or B.

If x_1 is contained in A, then removing x_1 from A will not change the number of inversions, as x_1 cannot form any inversions with elements of B. From this, we can construct a bijection with each array X with $x_1 \in A$ and k cross inversions, to an array X' with $|A| = a - 1$, $|B| = b$, and k cross inversions. There are $G(a - 1, b, k)$ of these arrays.

If x_1 is contained in B, then removing x_1 from B will reduce the number of inversions by a, as x_1 forms an inversion with every element of A. Constructing a similar bijection as before shows that there are $G(a, b-1, k-a)$ of these arrays (in the edge case where $k < a$, $G(a, b-1, k-a) = 0$ is valid).

Using Proposition 6, we can show that there are $G(a-1, b, k) + G(a, b-1, k-a) = G(a, b, k)$ arrangements of X with k cross inversions. □

Here are some additional characterizations of the Gaussian polynomial coefficients.

Characterization 8. Consider all strings composed of a 0s and b 1s. An *inversion* is a pair of indices $1 \leq i < j \leq a+b$ such that the character at position i is 1, while the character at position j is 0. The number of such strings with exactly k inversions is $G(a, b, k)$.

Characterization 9. Consider a rectangular grid of width a and height b. A path starts from the bottom left corner and ends at the top right corner, taking a unit steps rightwards and b unit steps upwards in the process. Then $G(a, b, k)$ counts the number of such paths such that the area under the path is k.

Characterization 10. The number of ways to place k indistinguishable balls into a indistinguishable bins such that each bin contains at most b balls is $G(a, b, k)$.

The connection to the coefficients of the Gaussian polynomial is similar to Characterization 7: we are showing that these examples follow the recurrence in Proposition 6. In Characterization 8, we can categorize the strings into strings starting with 0 and strings starting with 1. In Characterization 9, we can categorize the paths into two types, ones that start with a rightward step and ones that start with an upward step. In Characterization 10, we can consider configurations where every bin has at least one ball, and configurations where at least one bin is empty.

2.3 More properties of the Gaussian polynomial

In this section, we prove a few elementary properties of the coefficients of the Gaussian polynomial.

The Gaussian binomial coefficients are symmetric. From the q-factorial definition in Definition 2, we can see that $\binom{a+b}{a}_q = \binom{a+b}{b}_q$, which implies the following.

Proposition 11. For nonnegative integers a, b and any integer k,

$$G(a, b, k) = G(b, a, k).$$

Another property is that the coefficients of $\binom{a+b}{a}_q$ are identical when you write them in reverse.

Proposition 12. For nonnegative integers a, b and any integer k,

$$G(a, b, k) = G(a, b, ab-k).$$

Proof. Given a polynomial $P(q)$ with degree d, writing the coefficients of P in reverse results in the polynomial $q^d P(1/q)$. Applying this fact to the Gaussian polynomials, we need to check that $\binom{a+b}{a}_q = q^{ab}\binom{a+b}{a}_{1/q}$.

Expanding Definition 2, we can see that

$$\begin{aligned} q^{ab}\binom{a+b}{a}_{1/q} &= q^{ab} \cdot \frac{(1-q^{-(1+b)})\cdots(1-q^{-(a+b)})}{(1-q^{-1})\cdots(1-q^{-a})} \\ &= q^{ab} \cdot \frac{q^{-(1+b)}(q^{1+b}-1)\cdots q^{-(a+b)}(q^{a+b}-1)}{q^{-1}(q-1)\cdots q^{-a}(q^a-1)} \\ &= q^{ab} \cdot \frac{q^{-ab-1-\ldots-a}}{q^{-1-\ldots-a}} \cdot \frac{(1-q^{1+b})\cdots(1-q^{a+b})}{(1-q)\cdots(1-q^a)} \\ &= \frac{(1-q^{1+b})\cdots(1-q^{a+b})}{(1-q)\cdots(1-q^a)} \\ &= \binom{a+b}{a}_q, \end{aligned}$$

as desired. □

Notice that these properties also follow from the combinatorial interpretations in Characterization 7 and Characterization 9. Proposition 12 follows from Characterization 9, by measuring the area above the path rather than below. Reflecting the grid about a diagonal means we are measuring the area above the curve in a $b \times a$ grid, which implies $G(a,b,k) = G(b,a,ab-k)$ and therefore Proposition 11.

A deep property is that the coefficients of the Gaussian polynomial are *unimodal*, more precisely that as k goes from 0 to $\lfloor ab/2 \rfloor$, $G(a,b,k)$ increases, and then decreases when $k > \lfloor ab/2 \rfloor$. This difficult fact was first proven by [Syl78] using algebraic methods, with no combinatorial proof found for over a century until [O'H90].

Proposition 13 (Unimodality)**.** If $k_1 < k_2 \leq \lfloor ab/2 \rfloor$, then $G(a,b,k_1) \leq G(a,b,k_2)$. If $\lceil ab/2 \rceil \leq k_1 < k_2$, then $G(a,b,k_1) \geq G(a,b,k_2)$. In particular, $G(a,b,k)$ is maximized when $k = \lfloor ab/2 \rfloor$ for fixed a, b.

When $a, b \geq 8$, these inequalities are in fact strict for nonzero values of $G(a,b,k)$, a fact first proven by [PP13] and later proven combinatorially by [Dha14]. More recently, [PP17] established a concrete lower bound on the differences between consecutive values of $G(a,b,k)$, which were further improved by [MPP20] in the asymptotic regime where $|a-b| \leq C \cdot (a+b)$ for some constant $C < 1$.

2.4 Characteristic function of the Gaussian polynomial coefficients

From Definition 5 and Proposition 6, we can define an integer valued random variable $\mathcal{G}_{a,b}$ that achieves the value k with probability $G(a,b,k)/\binom{a+b}{a}$. Our main result will be based on the characteristic function of $\mathcal{G}_{a,b}$.

Every random variable X has an associated characteristic function $\varphi_X(t) : \mathbb{R} \to \mathbb{C}$ that encodes important information about X. This characteristic function is given by $\varphi_X(t) = \mathbb{E}[e^{itX}]$, and it exists for all random variables X. These functions satisfy an important property:

Fact 14. For independent random variables X, Y,

$$\varphi_{X+Y}(t) = \varphi_X(t)\,\varphi_Y(t).$$

Proof.

$$\varphi_{X+Y}(t) = \mathbb{E}[e^{it(X+Y)}] = \mathbb{E}[e^{itX}e^{itY}] = \mathbb{E}[e^{itX}]\mathbb{E}[e^{itY}] = \varphi_X(t)\,\varphi_Y(t).$$

□

Let $\varphi_{a,b}(t)$ be the characteristic function of $\mathcal{G}_{a,b}$. In other words,

$$\varphi_{a,b}(t) = \mathbb{E}[\exp(it\,\mathcal{G}_{a,b})] = \sum_{k=0}^{ab} \frac{G(a,b,k)}{\binom{a+b}{a}} e^{itk}.$$

The characteristic function of $\mathcal{G}_{a,b}$ has the following explicit form:

Proposition 15. For integers $n \geq 0$, let $s_n(t) = \prod_{k=1}^{n} \frac{\sin(kt)}{k}$. Then

$$\varphi_{a,b}(t) = e^{itab/2} \frac{s_{a+b}(t/2)}{s_a(t/2)s_b(t/2)}.$$

We present two proofs of this fact: one using induction based on the recurrence, and a direct one based on the connections to inversions in an array.

2.4.1 Proof using recurrence

Using Proposition 6 and the definition of $\varphi_{a,b}$, we can get a recurrence for $\varphi_{a,b}$:

$$\varphi_{a,b}(t) = \frac{a}{a+b}\,\varphi_{a-1,b}(t) + \frac{b}{a+b} e^{ita}\,\varphi_{a,b-1}(t).$$

We can now prove Proposition 15 by induction. When $a = 0$ or $b = 0$, $\varphi_{a,b}(t) = 1$. Otherwise, we can use the recurrence to get

$$\begin{aligned}
\varphi_{a,b}(t) &= \frac{a}{a+b}\,\varphi_{a-1,b}(t) + \frac{b}{a+b} e^{ita}\,\varphi_{a,b-1}(t) \\
&= e^{itab/2} \frac{s_{a+b}(t/2)}{s_a(t/2)s_b(t/2)} \left(e^{-itb/2} \frac{\sin(at/2)}{\sin((a+b)t/2)} + e^{ita/2} \frac{\sin(bt/2)}{\sin((a+b)t/2)} \right) \\
&= e^{itab/2} \frac{s_{a+b}(t/2)}{s_a(t/2)s_b(t/2)} \left(\frac{e^{-itb/2}(e^{ita/2} - e^{-ita/2}) + e^{ita/2}(e^{itb/2} - e^{-itb/2})}{2\sin((a+b)t/2)} \right) \\
&= e^{itab/2} \frac{s_{a+b}(t/2)}{s_a(t/2)s_b(t/2)} \left(\frac{e^{it(a+b)/2} - e^{-it(a+b)/2}}{2\sin((a+b)t/2)} \right) \\
&= e^{itab/2} \frac{s_{a+b}(t/2)}{s_a(t/2)s_b(t/2)}.
\end{aligned}$$

2.4.2 Proof using inversions

This proof of Proposition 15 is based on a connection between the cross inversions as described in Characterization 7 and inversions in a single array, reminiscent of divide and conquer.

Claim 16. Let A and B be arrays, and let AB be the concatenation of A with B. Then

$$\mathrm{Inv}(AB) = \mathrm{Inv}(A) + \mathrm{Inv}(B) + \mathrm{Inv}(A : B).$$

Proof. Any inversion in AB is either between two elements of A, two elements of B, or one element of A and one element of B. In each case, the inversion is counted in $\mathrm{Inv}(A)$, $\mathrm{Inv}(B)$, or $\mathrm{Inv}(A : B)$, respectively. □

Let $\mathcal{I}_a$ be the random variable equal to $\mathrm{Inv}(A)$ over a uniformly random permutation of a set A with a elements, and let $\varphi_a(t)$ be the characteristic function of $\mathcal{I}_a$.

Claim 17. We have

$$\varphi_a(t) = \prod_{k=1}^{a} \left(\frac{e^{it(k-1)}}{k} \cdot \frac{\sin(kt/2)}{\sin(t/2)} \right).$$

Proof. Consider the process of placing the elements $1, \ldots, a$ one by one, each time placing each new element between two elements or on some end of the array, to form an array A.

For $k = 1, \ldots, a$, let X_k be a random variable representing the number of new inversions formed with k when k is placed. First of all, when k is placed, the number of new inversions is equal to the number of elements to the left of k at the time of placement (only the elements $1, \ldots, k-1$ have been placed at this point). So, X_k has a uniform distribution over $\{0, \ldots, k-1\}$. Furthermore, this situation applies regardless of the placement of the other elements, so X_k is independent from all other X_i, $i \neq k$.

Therefore, $\mathcal{I}_a$ can be written as the following sum of independent variables:

$$\mathcal{I}_a = X_1 + \cdots + X_a.$$

We can use Fact 14 to calculate the characteristic function:

$$\varphi_a(t) = \prod_{k=1}^{a} \mathbb{E}[e^{itX_k}] = \prod_{k=1}^{a} \left(\frac{1}{k} \sum_{m=0}^{k-1} e^{itm} \right) = \prod_{k=1}^{a} \left(\frac{e^{it(k-1)}}{k} \cdot \frac{\sin(kt/2)}{\sin(t/2)} \right).$$

The last step follows from the geometric series formula and the identity $e^{it} - e^{-it} = 2\sin(t)$. □

Let X be a uniformly random permutation of $\{1, \ldots, a+b\}$, letting A be the subarray of the first a elements and B be the subarray of the last b elements. Here, the values of $\mathrm{Inv}(A)$, $\mathrm{Inv}(B)$, and $\mathrm{Inv}(A : B)$ are independent, and by Claim 16, their sum is $\mathrm{Inv}(AB) = \mathrm{Inv}(X)$. Recall that the distribution of cross inversions between two arrays of lengths a and b is described by $\mathcal{G}_{a,b}$. We can recast this in terms of random variables using Characterization 7 and Claim 17:

$$\mathcal{I}_{a+b} = \mathcal{I}_a + \mathcal{I}_b + \mathcal{G}_{a,b}\,.$$

We can use Fact 14 to see that

$$\varphi_{a+b}(t) = \varphi_a(t)\,\varphi_b(t)\,\varphi_{a,b}(t),$$

or

$$\varphi_{a,b}(t) = \frac{\varphi_{a+b}(t)}{\varphi_a(t)\,\varphi_b(t)}$$

which by Claim 17 proves Proposition 15.

3 Main Result

Our main result is a bound on the maximum value of any coefficient of a Gaussian polynomial that does not depend on any asymptotic regime for a and b. By Proposition 13, this is a bound on $G(a, b, \lfloor ab/2 \rfloor)$, although this fact is not necessary for the proof.

Theorem 18. *There exists a constant C such that for all integers $a, b \geq 1$ and $0 \leq k \leq ab$,*

$$G(a, b, k) \leq \binom{a+b}{a} \cdot \frac{C}{\sqrt{ab(a+b)}}.$$

Recall that the characteristic function $\varphi_{a,b}(t)$ of $G(a, b, k)/\binom{a+b}{a}$ is given by

$$\varphi_{a,b}(t) = \sum_{k=0}^{ab} \frac{G(a, b, k)}{\binom{a+b}{a}} e^{itk}.$$

From this, the original coefficients $G(a, b, k)$ can be extracted from the characteristic function by taking the inverse Fourier transform:

$$\frac{G(a, b, k)}{\binom{a+b}{a}} = \int_{-\pi}^{\pi} \varphi_{a,b}(t) e^{-itk}\, dt. \tag{3}$$

This is because the characteristic function of a random variable is the Fourier transform of its density function. In the case of a finite random variable, the density function can be extended to a periodic function, which can be retrieved from its Fourier transform by applying the general formula (3) for periodic functions.

A bound on $G(a, b, k)$ therefore follows from a bound on the characteristic function, using the form in Proposition 15.

$$\frac{G(a, b, k)}{\binom{a+b}{a}} = \int_{-\pi}^{\pi} e^{it(ab/2-k)} \frac{s_{a+b}(t/2)}{s_a(t/2) s_b(t/2)}\, dt \leq 2 \int_{-\pi/2}^{\pi/2} \left| \frac{s_{a+b}(t)}{s_a(t) s_b(t)} \right| dt.$$

When $a, b \geq 2$, the following Lemma 19 is enough to prove Theorem 18. In the case where $a = 1$ or $b = 1$, it is enough to notice that $G(a, b, k) = 1$ for $k = 0, \ldots, ab$.

Lemma 19. *Let $b \geq a \geq 2$, let*

$$\psi_{a,b}(t) = \frac{s_{a+b}(t)}{s_a(t) s_b(t)} \tag{4}$$

$$= \prod_{k=1}^{a} \left(\frac{k}{b+k} \cdot \frac{\sin((b+k)t)}{\sin(kt)} \right). \tag{5}$$

Then there exists a constant C such that

$$\int_{-\pi/2}^{\pi/2} |\, \psi_{a,b}(t)|\, dt \leq \frac{C}{b\sqrt{a}}. \tag{6}$$

Since $b \geq a$, $\sqrt{ab(a+b)}$ is within a constant factor of $b\sqrt{a}$.

Because $|\psi_{a,b}(t)|$ is an even function, it suffices to take the integral (6) over the domain $[0, \pi/2]$. We can divide the domain of integration into two regions: one close to zero, and the other far away from zero. The function is well-behaved in the center near zero, with it being approximated accurately by a normal curve. It is harder to analyze the behavior of the function away from zero. In this region, a pole reduction lemma (captured by Lemma 25) that hinges on a combinatorial matching result (Lemma 27) plays a crucial role in eliminating most of the messy behavior of the function and still providing an effective bound.

3.1 Center bound

For the first piece, we will integrate $|\psi_{a,b}(t)|$ over the interval $[0, \pi/(a+b)]$. Because every term in the product in $\psi_{a,b}$ is nonnegative on this interval, we can omit the absolute value signs.

Lemma 20 (Center)**.** *For integers $b \geq a \geq 2$,*

$$\int_0^{\frac{\pi}{a+b}} \psi_{a,b}(t)\, dt = O\left(\frac{1}{b\sqrt{a}}\right).$$

We can write

$$\psi_{a,b}(t) = \prod_{k=1}^{a} \frac{k}{b+k} \cdot \frac{\sin((b+k)t)}{\sin(kt)},$$

so we start with proving the following claim.

Claim 21. For positive integers $k \leq b$ and $x \in [0, \pi/(b+k)]$,

$$\frac{k}{b+k} \cdot \frac{\sin((b+k)x)}{\sin(kx)} \leq 1 - \frac{b^2}{2\pi^2}x^2.$$

To prove this claim, we first prove two trigonometric bounds. Refer to Fig. 1 for a plot of the functions and bounds.

Claim 22. For positive integers k, and $x \in [0, \pi/k]$,

$$\sin(kx) \leq kx - \frac{(kx)^3}{\pi^2}.$$

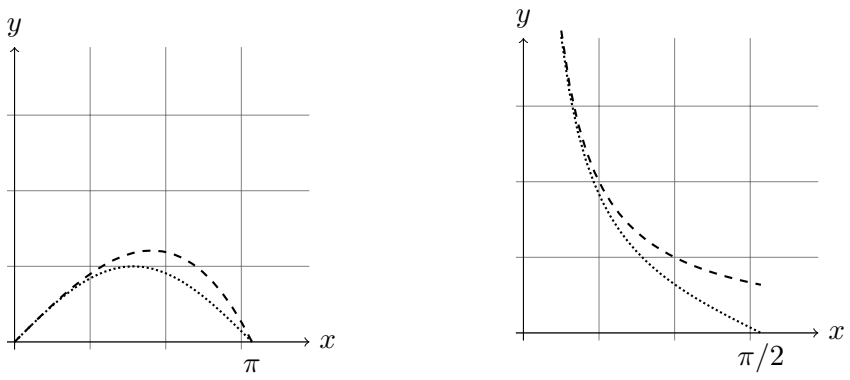
Proof. Let $y = kx$. It is enough to argue that $\sin(y) \geq y - \frac{y^3}{\pi^2}$ in the range $y \in [0, \pi]$.

Let $f(y) = \sin(y) - y + \frac{y^3}{\pi^2}$. Notice that $f(0) = f(\pi) = 0$ and $f'(\pi) > 0$. We will argue that there is a unique point $y^* \in (0, \pi)$ such that $f'(y^*) = 0$, which will ensure that $f(y) \leq 0$ for all $y \in [0, \pi]$.

We can calculate that

$$\begin{aligned} f'(y) &= \cos(y) - 1 + \frac{3y^2}{\pi^2} \\ &= \frac{3y^2}{\pi^2} - 2\sin^2\left(\frac{y}{2}\right). \end{aligned}$$

So $f'(y) = 0$ if and only if $\sin(y/2) = \pm(\sqrt{6}/\pi) \cdot (y/2)$, which is satisfied by one unique point $y^* \in (0, \pi)$. □



(a) Claim 22 (b) Claim 23

Figure 1: Plots of Trigonometric Bounds
Trigonometric functions are dotted, upper bounds are dashed

Claim 23. For positive integers k, and $x \in (0, \pi/2k]$,

$$\cot(kx) \leq \frac{1}{kx}.$$

Proof. Let $y = kx$. We will prove that $\cot(y) \leq \frac{1}{y}$ for all $y \in (0, \pi/2]$. It is enough to prove that $y\cos(y) \leq \sin(y)$ for all $y \in [0, \pi/2]$.

The latter inequality follows because both sides are zero when $y = 0$, and the derivative of the left hand side is bounded above by the derivative of the right hand side when $y \in [0, \pi/2]$:

$$\cos(y) - y\sin(y) \leq \cos(y).$$

□

Now we finish the proof of Claim 21.

Proof. Notice that

$$\begin{aligned}\frac{\sin((b+k)x)}{\sin(kx)} &= \frac{\sin(kx)\cos(bx) + \sin(bx)\cos(kx)}{\sin(kx)} \\ &\leq \cos(bx) + \cot(kx)\sin(bx).\end{aligned}$$

Of course, $\cos(bx) \leq 1$. Furthermore, from Claim 22 and Claim 23, we can see that in this domain of x, $\sin(bx) \leq bx - \frac{b^3x^3}{\pi^2}$ and $\cot(kx) \leq \frac{1}{kx}$. Additionally, $\sin(bx) \geq 0$. Therefore, using the fact that $k \leq b$,

$$\begin{aligned}\frac{k}{b+k} \cdot \frac{\sin((b+k)x)}{\sin(kx)} &\leq \frac{k}{b+k}\left(1 + \frac{1}{kx}\left(bx - \frac{b^3x^3}{\pi^2}\right)\right) \\ &\leq 1 - \frac{b^3}{(b+k)\pi^2}x^2 \leq 1 - \frac{b^2}{2\pi^2}x^2.\end{aligned}$$

□

From this, we can now prove Lemma 20.

Proof. Recall

$$\psi_{a,b}(t) = \prod_{k=1}^{a} \frac{k}{b+k} \cdot \frac{\sin((b+k)t)}{\sin(kt)}.$$

We can drop the absolute value signs as every term is positive over this interval. Claim 21 applies on all t in the domain because $\pi/(b+a) \le \pi/(b+k)$ for all k.

Therefore,

$$\int_0^{\frac{\pi}{a+b}} \psi_{a,b}(t)\,dt \le \int_0^{\frac{\pi}{a+b}} \left(1 - \frac{b^2t^2}{2\pi^2}\right)^a dt \le \int_0^{\frac{\pi}{a+b}} \exp\left(-\frac{ab^2t^2}{2\pi^2}\right) dt = O\left(\frac{1}{b\sqrt{a}}\right).$$

Here, we use the fact that $1 - x \le \exp(-x)$ for all x, and the Gaussian integral: the integral of $\exp(-t^2)$ over $\mathbb{R}$ is constant, and by scaling the argument, the integral of $\exp(-ct^2)$ over $\mathbb{R}$ is a constant factor of $c^{-1/2}$ for any parameter c. □

3.2 Peripheral bound

We will now bound $|\psi_{a,b}(t)|$ in the region away from 0, namely, the interval $[\pi/(a+b), \pi/2]$.

Lemma 24 (Peripheral)**.** *For integers $b \ge a \ge 2$,*

$$\int_{\frac{\pi}{a+b}}^{\frac{\pi}{2}} |\psi_{a,b}(t)|\,dt = O\left(\frac{1}{b\sqrt{a}}\right).$$

In this region, the main problem is that the denominator of $\psi_{a,b}(t)$ often goes to zero, which could potentially blow up the integrand. However, the terms in the numerator always cancel out these blowups. The following lemma will be our main tool for bounding $|\psi_{a,b}(t)|$ in this region, which will allow terms in the numerator to cancel out bad terms in the denominator.

Lemma 25 (Pole Reduction)**.** *For every $t \in \mathbb{R}$, there exists a bijection $\beta_t : \{1, \ldots, a\} \to \{b+1, \ldots, b+a\}$ (depending on t) such that for every $k = 1, \ldots, a$,*

$$\left|\frac{1}{k}\sin(kt)\right| \ge \left|\frac{1}{\beta_t(k)}\sin(\beta_t(k)t)\right|.$$

A basic bound can be found by applying Lemma 25 on $\psi_{a,b}(t)$ for $k = 2, \ldots, a$, resulting in an upper bound of $1/b\sin(t)$. This bound is usable, but is weak on points closer to 0. To remedy this, we can divide the domain of integration into multiple parts, where the points closer to 0 can safely include more terms in the denominator.

Let $2 \le n \le a$ be an integer. We split the domain of integration into three intervals: $[\frac{\pi}{a+b}, \frac{\pi}{2n}]$, $[\frac{\pi}{2n}, \frac{\pi}{2} - \frac{\pi}{2n}]$, and $[\frac{\pi}{2} - \frac{\pi}{2n}, \frac{\pi}{2}]$. By selecting a good value of n, we can get a reasonable upper bound on this region.

Here, we will make use of Lemma 25 and the following linear approximation to sine:

Fact 26. For a positive integer k and $t \in [0, \pi/2k]$,

$$\sin(kt) \ge \frac{2kt}{\pi}.$$

Proof. Notice that $\sin(kt) = \frac{2kt}{\pi}$ when $t = 0$ and $t = \frac{\pi}{2k}$. This fact then follows since sine is concave on this interval. □

3.2.1 Region I

The first region of integration is $[\pi/(a+b), \pi/(2n)]$.

$$\begin{aligned}
\int_{\frac{\pi}{a+b}}^{\frac{\pi}{2n}} |\,\psi_{a,b}(t)|\,dt &\le \int_{\frac{\pi}{a+b}}^{\frac{\pi}{2n}} \frac{n!}{\beta_t(1)\cdots\beta_t(n)} \left|\frac{\sin(\beta_t(1)t)\cdots\sin(\beta_t(n)t)}{\sin(t)\cdots\sin(nt)}\right| dt \\
&\le \frac{n!}{b^n} \int_{\frac{\pi}{a+b}}^{\frac{\pi}{2n}} \left|\frac{1}{\sin(t)\cdots\sin(nt)}\right| dt \\
&\le \frac{1}{b^n} \int_{\frac{\pi}{a+b}}^{\frac{\pi}{2n}} \left(\frac{\pi}{2}\right)^n \frac{1}{t^n}\,dt \\
&\le \frac{1}{b^n} \cdot \left(\frac{\pi}{2}\right)^n \cdot \frac{1}{n-1} \cdot \left(\frac{a+b}{\pi}\right)^{n-1} \\
&\le \frac{1}{b^n} \cdot \left(\frac{\pi}{2}\right)^n \cdot \frac{1}{n-1} \cdot \left(\frac{2b}{\pi}\right)^{n-1} \\
&\le \frac{\pi}{2b(n-1)}.
\end{aligned}$$

The first two steps involve applying Lemma 25 on all $k = n+1, \ldots, a$, and then using the fact that $|\sin(x)| \le 1$ and $\beta_t(k) \ge b$. The third step uses Fact 26 for $k = 1, \ldots, n$, which applies on the interval $[\pi/(a+b), \pi/(2n)]$ for these values of k. From there, we bound with the left limit of integration.

3.2.2 Region II

We now bound $|\,\psi_{a,b}(t)|$ on the interval $[\frac{\pi}{2n}, \frac{\pi}{2} - \frac{\pi}{2n}]$. We can use Lemma 25 to eliminate all terms except $k = 1, 2$ this time.

$$\begin{aligned}
\int_{\frac{\pi}{2n}}^{\frac{\pi}{2}-\frac{\pi}{2n}} |\,\psi_{a,b}(t)|\,dt &\le \int_{\frac{\pi}{2n}}^{\frac{\pi}{2}-\frac{\pi}{2n}} \frac{2}{\beta_t(1)\,\beta_t(2)} \left|\frac{\sin(\beta_t(1)t)\sin(\beta_t(2)t)}{\sin(t)\sin(2t)}\right| dt \\
&\le \frac{2}{b^2} \int_{\frac{\pi}{2n}}^{\frac{\pi}{2}-\frac{\pi}{2n}} \left|\frac{1}{\sin(t)\sin(2t)}\right| dt.
\end{aligned}$$

Because $|\sin(t)|$ is increasing here and $|\sin(2t)|$ is symmetric about $\frac{\pi}{4}$, the value of the integral on the interval $[\frac{\pi}{2n}, \frac{\pi}{4}]$ exceeds the value on the interval $[\frac{\pi}{4}, \frac{\pi}{2} - \frac{\pi}{2n}]$. Using Fact 26,

$$\frac{2}{b^2} \int_{\frac{\pi}{2n}}^{\frac{\pi}{4}} \left|\frac{1}{\sin(t)\sin(2t)}\right| dt \le \frac{1}{b^2} \int_{\frac{\pi}{2n}}^{\frac{\pi}{4}} \left(\frac{\pi}{2}\right)^2 \frac{1}{t^2}\,dt \le \frac{\pi^2}{4b^2} \cdot \frac{2n}{\pi} = \frac{\pi n}{2b^2}.$$

We have now established that

$$\int_{\frac{\pi}{2n}}^{\frac{\pi}{2}-\frac{\pi}{2n}} |\,\psi_{a,b}(t)|\,dt \le \frac{\pi n}{b^2}.$$

3.2.3 Region III

Notice that $|\sin(t)|$ is increasing on the interval $[\frac{\pi}{2} - \frac{\pi}{2n}, \frac{\pi}{2}]$, so we can bound $|\sin(t)|$ by $|\sin(\frac{\pi}{2} - \frac{\pi}{2n})|$. Similar to before, the first step follows from Lemma 25, this time applied to $k = 2, \ldots, a$.

$$\begin{aligned}
\int_{\frac{\pi}{2}-\frac{\pi}{2n}}^{\frac{\pi}{2}} |\,\psi_{a,b}(t)|\,dt &\leq \int_{\frac{\pi}{2}-\frac{\pi}{2n}}^{\frac{\pi}{2}} \frac{1}{\beta_t(1)} \left|\frac{\sin(\beta_1(t))}{\sin(t)}\right| dt \\
&\leq \int_{\frac{\pi}{2}-\frac{\pi}{2n}}^{\frac{\pi}{2}} \frac{1}{|b\sin(t)|}\,dt \\
&\leq \frac{\pi}{2n} \cdot \frac{1}{b\sin(\frac{\pi}{2} - \frac{\pi}{2n})} \\
&\leq \frac{\pi}{2n} \frac{1}{b(1-\frac{1}{n})} \\
&= \frac{\pi}{2b(n-1)}.
\end{aligned}$$

3.2.4 Overall bound

Summing the above bounds, we can deduce that

$$\int_{\frac{\pi}{a+b}}^{\frac{\pi}{2}} |\,\psi_{a,b}(t)|\,dt \leq \frac{\pi}{b(n-1)} + \frac{\pi n}{b^2}.$$

By choosing $n = \lceil\sqrt{a}\rceil$ (keeping in mind that $b \geq a$), we can deduce Lemma 24.

3.3 Proof of pole reduction lemma

Finally, we will prove the pole reduction lemma (Lemma 25). The main essence of the result is based on an interval matching lemma, Lemma 27.

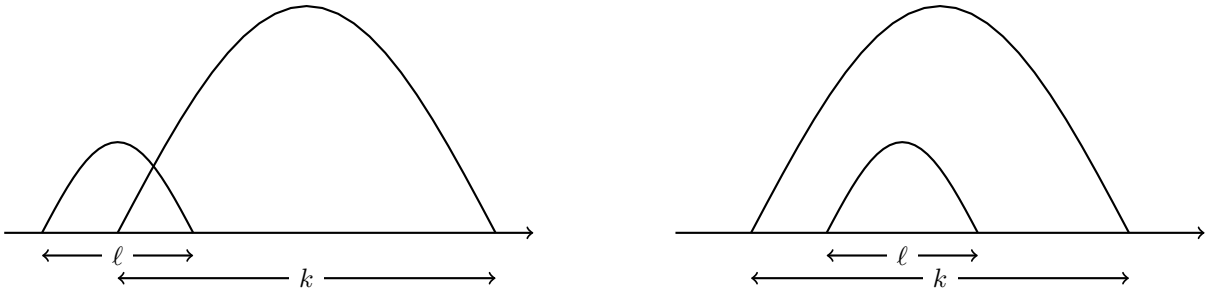
For every real number t and positive integer k, there is a unique integer n such that t is contained in the half-open interval $[n/k, (n+1)/k)$. We call this interval the *k-interval* of t. For positive integers k, ℓ, we can say that the k-interval of t *encloses* the ℓ-interval of t if the ℓ-interval of t is a subset of the k-interval of t. We use the shorthand that k encloses ℓ at t.

For any integer k, the function $|\frac{\sin(kt)}{k}|$ is periodic and concave on k-intervals. If the k-interval contains the ℓ-interval for a particular t, then $|\frac{\sin(kt)}{k}| \geq |\frac{\sin(\ell t)}{t}|$, as in Fig. 2 and formally proven in Claim 34. To establish Lemma 25, we can ensure that for each t, every $k \in \{1, \ldots, a\}$ can be matched with some $\ell \in \{b+1, \ldots, b+a\}$ with k containing ℓ. Without this containment, there might be nonzero terms in the numerator of $|\,\psi_{a,b}(t)|$ with zero terms in the denominator, jeopardizing a possible bound.

This bijection can be different depending on t. In fact, this is necessary as otherwise ℓ would need to be a multiple of k, which is not possible with a bijection between the sets $\{1, \ldots, a\}$ and $\{b+1, \ldots, b+a\}$.

Lemma 27 (Interval Matching)**.** *Let a, b be positive integers. For any real t, there exists a bijection β_t between $\{1, \ldots, a\}$ and $\{b+1, \ldots, b+a\}$ such that for all $k = 1, \ldots, a$, k encloses $\ell = \beta_t(k)$ at t.*

We can interpret Lemma 27 as a matching on a bipartite graph by using Hall's marriage lemma.



(a) Bad case: no interval enclosure

(b) Good case: interval enclosure

Figure 2: Enclosing intervals are needed for Lemma 25.

Lemma 28 (Hall)**.** *Let G be a bipartite graph with partitions L, R. For any $A \subseteq L$, let $N(A)$ be the set of all vertices in R with at least one neighbor in A. The graph G admits a perfect matching if and only if for all such A, $|N(A)| \geq |A|$.*

For $A \subseteq \{1, \ldots, a\}$, let $N_t(A)$ be the set of all $\ell \in \{b+1, \ldots, b+a\}$ such that there exists $k \in A$ where k encloses ℓ at t. To produce the desired bijection β_t in Lemma 27, it is sufficient to prove that $|N_t(A)| \geq |A|$ for all t and A.

There are some values of ℓ that are always contained in $N_t(A)$ regardless of the value of t. Let $N(A)$ be the set of all $\ell \in \mathbb{N}$ such that for all t, there exists $k \in A$ where k encloses ℓ (this k can vary depending on t). As $N(A) \cap \{b+1, \ldots, b+a\} \subseteq N_t(A)$, it is sufficient to prove that $|N(A) \cap \{b+1, \ldots, b+a\}| \geq |A|$ for all A and apply Lemma 28 to prove Lemma 27.

Example 29. $5 \in N(\{2, 3\})$.

Proof. We only consider $t \in [0, 1)$ for clarity. When $t \in [0, 0.4)$, the 2-interval for t is $[0, 0.5)$, while the 5-interval for t is either $[0, 0.2)$ or $[0.2, 0.4)$, which means 2 encloses 5. When $t \in [0.4, 0.6)$, the 3-interval for t is $[1/3, 2/3)$, which encloses the 5-interval $[0.4, 0.6)$. When $t \in [0.6, 1)$, the 2-interval is $[0.5, 1)$ while the 5-interval is either $[0.6, 0.8)$ or $[0.8, 1)$, so 2 encloses 5. These enclosures are shown in Fig. 3, where the colored 5-intervals are contained within the respective colored 2 or 3-intervals.

For all t, either 2 encloses 5, or 3 encloses 5. In other words, $5 \in N(\{2, 3\})$. □

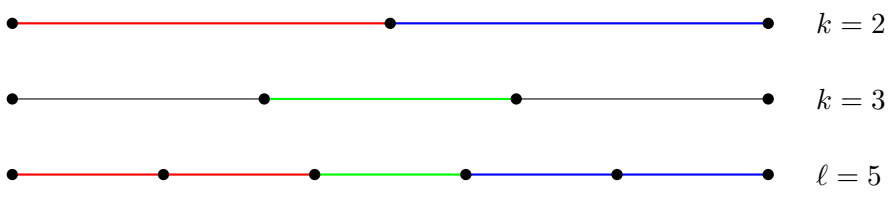


Figure 3: Example 29

We first consider a different characterization of $N(A)$.

Claim 30. Let ℓ be a positive integer, and let I be any open interval of $\mathbb{R}$ that contains a fraction of denominator k for every $k \in A$ (these fractions do not have to be distinct or reduced). Then $\ell \in N(A)$ if and only if every such I must also contain a fraction with denominator ℓ.

The idea is that if no $k \in A$ encloses ℓ for some t, then the endpoints of each k-interval form a fraction of denominator k contained strictly within the ℓ-interval of t. As a result, we have a contiguous interval I containing a fraction of denominator k, and I is contained strictly within the ℓ-interval of t. As such, I cannot contain a fraction of denominator ℓ. This is illustrated in Fig. 4. Of course, the formal proof would also need to consider the edge cases involving the endpoints of the intervals.

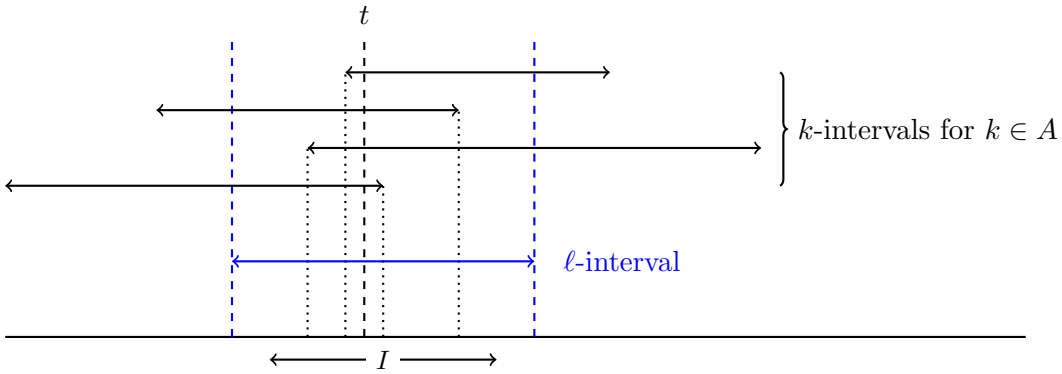


Figure 4: I contains a fraction of denominator k for all k, but no fraction of denominator ℓ.

For Claim 30, we prove the following proposition.

Proposition 31. The following statements are equivalent for any half-open interval $[c, d)$ and positive integer k:

(1) $[c, d)$ is contained within a k-interval.

(2) (c, d) is contained within a k-interval.

(3) (c, d) contains no fraction of denominator k.

Proof. (1) $\implies$ (2). This follows as (c, d) is a subset of $[c, d)$.

(2) $\implies$ (3). Suppose (c, d) is contained in the k-interval $[n/k, (n + 1)/k)$. The interval (c, d) cannot contain a fraction of denominator k, otherwise it would be strictly between n/k and $(n + 1)/k$.

(3) $\implies$ (1). Let n/k be the largest fraction of denominator k less than or equal to c. It must be true that $(n + 1)/k \geq d$, since $(n + 1)/k$ cannot be contained in (c, d). Therefore, $[c, d)$ is contained in the k-interval $[n/k, (n + 1)/k)$. □

From this, we can prove Claim 30.

Proof. By definition, the condition that $\ell \in N(A)$ is that any ℓ-interval J is contained in some k-interval for some $k \in A$. By condition (2) in Proposition 31, this is equivalent to saying $\text{Int}(J)$ is contained in some k-interval, where $\text{Int}(J)$ is the interior of J. This is true if and only if any open subinterval I of $\text{Int}(J)$ is contained in some k-interval. Equivalently, if I is an open interval that is not contained in any k-interval, then I is not contained in any ℓ-interval. Using condition (3) in Proposition 31, this is finally equivalent to the condition that if I contains a fraction of denominator k for every $k \in A$, then I contains a fraction of denominator ℓ. □

The characterization in Claim 30 allows us to prove the following key claim about $N(A)$.

Claim 32. If $\ell_1, \ell_2 \in N(A)$, then $\ell_1 + \ell_2 \in N(A)$.

Proof. By Claim 30, any interval I that contains a fraction of denominator k for every $k \in A$ must also contain two fractions x/ℓ_1 and y/ℓ_2. For positive integers a, b, c, d, define the *mediant* of a/b and c/d to be $(a+c)/(b+d)$. Then the mediant is always between the two fractions. In other words, if $a/b \leq c/d$,

$$\frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d}.$$

This fact can be proven with elementary algebra, as both sides are equivalent to $a/b \leq c/d$.

From this, we see that the mediant $(x+y)/(\ell_1+\ell_2)$ is contained in I, as it is between two elements of I. As this applies to every such I, we can use Claim 30 to conclude that $\ell_1+\ell_2 \in N(A)$.

Note that x/ℓ_1 and y/ℓ_2 do not have to be distinct. ℓ_1 and ℓ_2 might be equal, or $x/\ell_1 = y/\ell_2$. □

As k encloses k for every t, we have that $A \subseteq N(A)$. Let $\text{Sums}(A)$ be the set of positive integers that can be written as the sum of not necessarily distinct elements of A. Claim 32 implies that $\text{Sums}(A) \subseteq N(A)$.

Claim 33. For nonnegative integers n,

$$|\,\text{Sums}(A) \cap \{n+1, \ldots, n+a\}| \geq |A \cap \{n+1, \ldots, n+a\}|.$$

Proof. Let $m = \max(A)$. Because $A \subseteq \text{Sums}(A)$, $\text{Sums}(A)$ contains every positive integer congruent to an element of A modulo m, since we can repeatedly add m to any element of A. As $a \geq m$, the set $\{n+1, \ldots, n+a\}$ contains at least one element for every residue mod m. Therefore, $\text{Sums}(A) \cap \{n+1, \ldots, n+a\}$ contains at least $|A|$ elements of $\{n+1, \ldots, n+a\}$. □

Putting it all together, we have that

$$|A| \leq |\,\text{Sums}(A) \cap \{b+1, \ldots, b+a\}| \leq |N(A) \cap \{b+1, \ldots, b+a\}| \leq |N_t(A)|,$$

which proves Lemma 27 by our previous reasoning. To finish, we need to show that this implies Lemma 25.

Claim 34. Lemma 27 implies Lemma 25.

Proof. We prove the following more general claim. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that has period 1, and additionally, f is concave on $[0,1]$ and $f(0) = f(1) = 0$. For $t \in [0,1]$, let β_t be a bijection that satisfies the conditions of Lemma 27. Then for any $k \in \{1, \ldots, a\}$, let $\ell = \beta_t(k)$. We seek to prove that

$$\frac{f(kt)}{k} \geq \frac{f(\ell t)}{\ell}$$

for all $k = 1, \ldots, a$. Notice that $f(t) = |\sin(t)|$ satisfies all the conditions of the claim, albeit after scaling the period from π to 1.

To prove the general claim, let $\kappa = kt - \lfloor kt \rfloor$ and $\lambda = \ell t - \lfloor \ell t \rfloor$, noting that $\kappa, \lambda \in [0,1]$. By the enclosing property, it is true that $\kappa/k \geq \lambda/\ell$, since they represent the distance from t to the left endpoints of the k and ℓ-intervals, respectively, and k encloses ℓ at t. This can be seen in Fig. 5.

Suppose $\kappa \leq \lambda$. We have

$$f(kt) = f(\kappa) \geq \frac{\kappa}{\lambda} f(\lambda) \geq \frac{k}{\ell} f(\lambda) = \frac{k}{\ell} f(\ell t).$$

The first step follows from the periodicity of f, the second from the concavity of f on $[0, 1]$ and the fact that $f(0) = 0$ (the point $(\kappa, f(\kappa))$ is above the line segment connecting $(\lambda, f(\lambda))$ and $(0, 0)$), the third from the aforementioned inequality $\kappa/k \geq \lambda/\ell$, and the last from the periodicity of f again.

If $\lambda < \kappa$, we can instead consider the function $\widetilde{f}(t) = f(1 - t)$. Here, $\widetilde{\kappa} = 1 - \kappa$ and $\widetilde{\lambda} = 1 - \lambda$, and we can use our previous reasoning. This proves the general claim. □

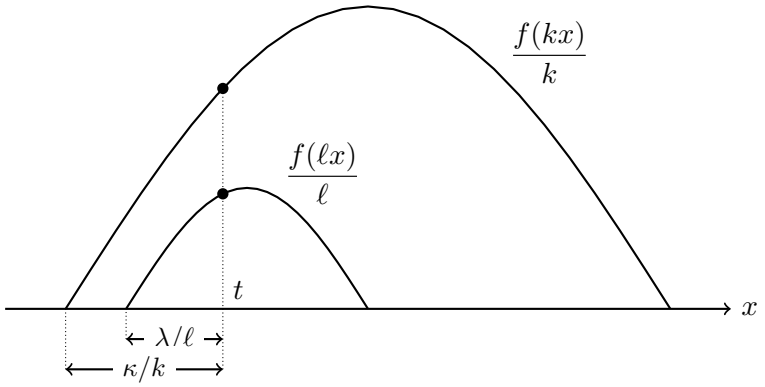


Figure 5: Proof of Claim 34: $\lambda/\ell \leq \kappa/k$.

4 Comparisons to Previous Work

In this section, we will discuss previous bounds on the coefficients of the Gaussian polynomial and how they corroborate our result that $G(a, b, \lfloor ab/2 \rfloor) = O(1/\sqrt{ab(a+b)})$. Our results apply on all possible asymptotic regimes of a and b, but they are in fact optimal in the settings described in previous work. We start with the results of [MW47] that apply in the asymptotic regime where $a \to \infty$ and $b \to \infty$, and then discuss the results of [Tak86] and [SZ16].

Once again, we use the language of probability to discuss the distribution induced by $G(a, b, k)$. The result of [MW47] shows that as both $a \to \infty$ and $b \to \infty$, $\mathcal{G}_{a,b}$ approaches a normal distribution. A proof of this fact is also given in [Tak86].

Theorem 35 ([MW47], [Tak86, Theorem 4])**.** *Let $\mu_{a,b} = ab/2$ and $\sigma_{a,b} = \sqrt{\frac{ab(a+b+1)}{12}}$. We have*

$$\lim_{a,b\to\infty} \Pr\left(\frac{\mathcal{G}_{a,b} - \mu_{a,b}}{\sigma_{a,b}} \leq x\right) = \Phi(x),$$

where Φ is the cdf of the normal random variable $\mathcal{N}(0, 1)$.

This result suggests that the highest probability outcomes of $\mathcal{G}_{a,b}$ have probability $1/\sigma_{a,b} = O(1/\sqrt{ab(a+b)})$ in the regime $a \to \infty$, $b \to \infty$, which matches with our bound in Theorem 18.

Takács also proved a local limit theorem [Tak86, Theorem 5] that gives an estimate for the coefficient of the Gaussian polynomial that is effective for the largest central coefficients.

Theorem 36 ([Tak86])**.** *For positive integers a, b,*

$$\frac{G(a,b,\lfloor ab/2\rfloor)}{2^{a+b}} = \phi\left(\frac{a-b}{\sqrt{a+b}}\right)\frac{4\sqrt{6}}{\sqrt{\pi}\cdot(a+b)\sqrt{(a+b+1)(a+b-1)}} + O((a+b)^{-3}),$$

where ϕ is the normal density function.

When $|a-b| < C\sqrt{a+b}$, we can use the following approximation:

$$\binom{a+b}{a}\frac{1}{2^{a+b}} \sim \phi\left(\frac{a-b}{\sqrt{a+b}}\right)\frac{2}{\sqrt{a+b}}.$$

This gives an estimate for the value of the constant in our bound for $G(a,b,\lfloor ab/2\rfloor)$:

$$G(a,b,\lfloor ab/2\rfloor) \sim \binom{a+b}{a}\sqrt{\frac{6}{\pi ab(a+b+1)}}.$$

[MPP20, Theorem 1] gives a more accurate estimate for $G(a,b,k)$ for all k in the asymptotic domain where $|a-b| \le C\cdot(a+b)$ for some constant $C<1$. This estimate applied to the central coefficient $G(a,b,\lfloor ab/2\rfloor)$ gives:

$$G(a,b,\lfloor ab/2\rfloor) \sim \frac{\sqrt{3}}{\pi ab}\left(\frac{(a+b)^{a+b}}{a^a b^b}\right).$$

This estimation is equivalent to the one by Takács, using Stirling's approximation $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, which we can rewrite as $n^n \sim \frac{n!e^n}{\sqrt{2\pi n}}$. However, it applies to the more general domain $|a-b| \le C(a+b)$, $C<1$.

Now we consider the results of [SZ16], which apply to the domain where a is fixed and $b\to\infty$.

Theorem 37 ([SZ16, Theorem 2.4])**.** *Fix $r \ge 0$ ($r\in\mathbb{R}$), and $a\in\mathbb{N}$. Then*

$$G(a,b,\lfloor rb\rfloor) = C(r,a)b^{a-1} + O(b^{a-2})$$

for $b\to\infty$, where $C(r,a)$ depends only on r and a.

Since we are only concerned about the maximum coefficient, we can fix $r=a/2$, in which case Theorem 37 implies that $G(a,b,\lfloor ab/2\rfloor) = O(b^{a-1})$ Because $\binom{a+b}{a} = O(b^a)$ when a is fixed, our result in the context where a is fixed gives

$$G(a,b,\lfloor ab/2\rfloor) = \frac{C(a)}{b}\binom{a+b}{a} = O(b^{a-1}),$$

which is consistent with Theorem 37.

Both of these results show that Theorem 18 is tight in different asymptotic regimes, which suggests that our upper bound is optimal up to a constant factor.

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