Syntax of First-Order Predicate Calculus (FOPC): 1. Alphabet

- Countable set of predicate symbols, each with specified arity ≥ 0.
- Countable set of function symbols, each with specified arity ≥ 0. Function symbols with arity 0 are also called constants or individual symbols.
- Countable set of variables.

1. Alphabet (Continued)

- (Consistent with Prolog, we will begin variables with an upper-case letter and predicate/function symbols with a lower-case letter.)
- Logical symbols: $\forall, \exists, \wedge, \vee, \leftarrow, \rightarrow, \leftrightarrow$

2. Terms

- A variable is a term.
- If *f* is a function symbol of arity n and $t_1,\ldots,t_n$ are terms then $f(t_1,\ldots,t_n)$ is a term.

Examples of Terms

- *0*
- *s(s(s(0)))*
- *nil*
- *cons(1,nil)*
- *cons(1,cons(2,nil))*
- *cons(1,cons(2,cons(3,nil)))*

3. Formulas

- If p is a predicate symbol of arity n and $t_1,\ldots,t_n$ are terms, then $p(t_1,\ldots,t_n)$ is an atomic formula.
- If α and β are formulas then so are $\neg\alpha$, $\alpha\vee\beta$, $\alpha\wedge\beta$, $\alpha\rightarrow\beta$, $\alpha\leftarrow\beta$, $\alpha\leftrightarrow\beta$.
- If X is a variable and α is a formula then $\forall X\alpha$ and $\exists X\alpha$ are formulas. We say that X is *quantified* in the formulas $\forall X\alpha$ and $\exists X\alpha$.

Some Notes

- Predicates of arity 0 are also called *propositions*, the only atomic formulas allowed in *propositional logic*.
- An *expression* is a term or formula. A formula with no *free* (unquantified) variables is a *sentence*.

An *interpretation* is a pair $(\mathcal{D}, \mathcal{A})$ where:

- $\mathcal{D}$ is a set of individuals. $\mathcal{D}$ is called the *domain*.
- $\mathcal{A}$ (the *assignment function*) is a function that maps
  - * each constant to an individual
  - * each n-ary function symbol to an n-ary function over $\mathcal{D}$
  - * each n-ary predicate symbol to an n-ary relation

A *valuation* is a function from variables to $\mathcal{D}$. If φ is a valuation then $\varphi[X \mapsto i]$ is the same valuation with the (possible) exception that φ maps X to i.

Let $\mathfrak{A} = (\mathcal{O}, \mathcal{A})$ be an interpretation and φ be a valuation. If α is a symbol of the alphabet (non-logical) or an expression then $[\![\alpha]\!]^{\mathfrak{A},\varphi}$ is the semantic value of α relative to $\mathfrak{A}$ and φ. $[\![\alpha]\!]^{\mathfrak{A},\varphi}$ is defined compositionally as follows.

- If α is a predicate symbol or function symbol then $[\![\alpha]\!]^{\mathfrak{A},\varphi} = \mathcal{A}(\alpha)$.
- If α is a variable, then $[\![\alpha]\!]^{\mathfrak{A},\varphi} = \varphi(\alpha)$.
- If α is a term $f(t_1, \ldots, t_n)$ then $[\![\alpha]\!]^{\mathfrak{A},\varphi}$ is: $[\![f]\!]^{\mathfrak{A},\varphi}\left([\![t_1]\!]^{\mathfrak{A},\varphi}, \ldots, [\![t_n]\!]^{\mathfrak{A},\varphi}\right)$.

- If α is an atomic formula $p(t_1, \ldots, t_n)$ then $[\![\alpha]\!]^{\mathcal{A},\varphi} = \text{TRUE}$ if $\left([\![t_1]\!]^{\mathcal{A},\varphi}, \ldots, [\![t_n]\!]^{\mathcal{A},\varphi}\right) \in [\![p]\!]^{\mathcal{A},\varphi}$, and $[\![\alpha]\!]^{\mathcal{A},\varphi} = \text{FALSE}$ otherwise.

- If α is $\neg\beta$ then $[\![\alpha]\!]^{\mathcal{A},\varphi} = \text{TRUE}$ if $[\![\beta]\!]^{\mathcal{A},\varphi} = \text{FALSE}$, and $[\![\alpha]\!]^{\mathcal{A},\varphi} = \text{FALSE}$ otherwise.

- If α is $\beta \vee \gamma$ then $[\![\alpha]\!]^{\mathcal{A},\varphi} = \text{TRUE}$ if $[\![\beta]\!]^{\mathcal{A},\varphi}$ or $[\![\gamma]\!]^{\mathcal{A},\varphi}$ is TRUE, and $[\![\alpha]\!]^{\mathcal{A},\varphi} = \text{FALSE}$ otherwise.

- If α is $\beta \wedge \gamma$ then $[\![\alpha]\!]^{\mathcal{A},\varphi} = \text{TRUE}$ if $[\![\beta]\!]^{\mathcal{A},\varphi}$ and $[\![\gamma]\!]^{\mathcal{A},\varphi}$ are TRUE, and $[\![\alpha]\!]^{\mathcal{A},\varphi} = \text{FALSE}$ otherwise.

- If α is $\beta \rightarrow \gamma$ then $[\![\alpha]\!]^{\mathcal{A},\varphi} = \text{TRUE}$ if $[\![\beta]\!]^{\mathcal{A},\varphi}$ is FALSE or $[\![\gamma]\!]^{\mathcal{A},\varphi}$ is TRUE, and $[\![\alpha]\!]^{\mathcal{A},\varphi}$ is FALSE otherwise.

- If α is $\forall X \beta$ then $[\![\alpha]\!]^{\mathfrak{A},\varphi}$ = TRUE if there is no $i \in \mathcal{D}$ such that $[\![\beta]\!]^{\mathfrak{A},\varphi[X \mapsto i]}$ = FALSE, and $[\![\alpha]\!]^{\mathfrak{A},\varphi}$ = FALSE otherwise.

- If α is $\exists X \beta$ then $[\![\alpha]\!]^{\mathfrak{A},\varphi}$ = TRUE if for some $i \in \mathcal{D}$ $[\![\beta]\!]^{\mathfrak{A},\varphi[X \mapsto i]}$ = TRUE, and $[\![\alpha]\!]^{\mathfrak{A},\varphi}$ = FALSE otherwise.

Note: If α is a sentence then $[\![\alpha]\!]^{\mathfrak{A},\varphi}$ is the same for every valuation φ. Hence for any sentence α we may write $[\![\alpha]\!]^{\mathfrak{A}}$ without reference to a particular φ.

Let α be a sentence, A a set of sentences, and $\mathfrak{A}$ an interpretation.

- $\mathfrak{A}$ satisfies α iff $[\![\alpha]\!]^{\mathfrak{A}} = \text{TRUE}$, when we also say $\mathfrak{A}$ is a model of α.
- $\mathfrak{A}$ satisfies A iff $\mathfrak{A}$ satisfies all $a \in A$, when we also say $\mathfrak{A}$ is a model of A.
- A is satisfiable iff some interpretation satisfies A, i.e. A has a model.

$A \models \alpha$ (A entails α) iff every model of A is a model of α; α is a logical consequence of A.

Sets of sentences A and B are logically equivalent ($A \equiv B$) iff $A \models b$ for all $b \in B$ and $B \models a$ for all $a \in A$, i.e. A and B have the same models.

Example: Models

$\forall X(\forall Y((mother(X) \wedge child_of(Y,X)) \rightarrow loves(X,Y)))$

mother(mary)

child_of(tom,mary)

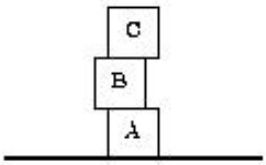

Figure 1.2: An alternative structure

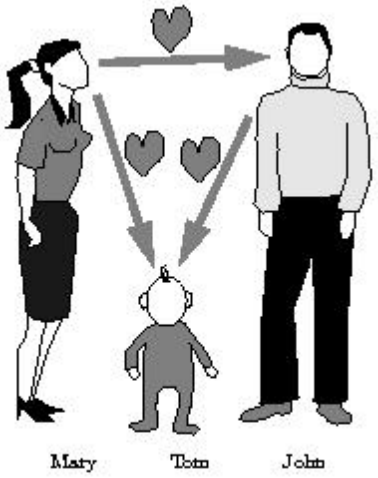

Figure 1.1: A family structure

Substitutions

A substitution is a function θ from expressions to expressions such that for every expression e:

(1) if e is a constant then $e\theta = e$

(2) if e is composed of $e_1, \ldots, e_n$ then $e\theta$ is composed of $e_1\theta, \ldots, e_n\theta$ in the same manner.

(3) if e is a variable then $e\theta$ is an expression.

We write a substitution θ as $\{X_1 \to t_1, \ldots, X_n \to t_n\}$ where $Dom(\theta) = \{X_1, \ldots, X_n\}$ and $X_i\theta = t_i$, $1 \le i \le n$.

The application of a substitution $\theta = \{X_1 \to t_1, \ldots, X_n \to t_n\}$ to an expression e is defined as follows: simultaneously replace t_i for every free occurrence of X_i in e.

Observation: $\theta_1 = \theta_2$ iff $V\theta_1 = V\theta_2$ for every variable V. So substitutions are entirely determined by their mappings for variables.

We limit our attention to "finite" substitutions — those for which $V\theta \neq V$ for only a finite set of variables. We call

$$\{V \mid V\theta \neq V\}$$

the domain of θ, denoted $Dom(\theta)$ (by a slight abuse of terminology).

Examples : Substitutions

$\theta = \{X \to f(Y,Z),\ Y \to Z\}$

$\delta = \{Y \to a\}$

$p(X,Z)\theta = p(f(Y,Z),Z)$

$p(X,Y)\theta = p(f(Y,Z),Z)$

$(p(X,Y)\theta)\delta = p(f(a,Z),Z)$

Note: θ is not idempotent but δ is. Definition of idempotent follows definition of composition.

Substitution Composition

Let $\theta = \{X_1 \to s_1, \ldots, X_m \to s_m\}$ and

$\delta = \{Y_1 \to t_1, \ldots, Y_n \to t_n\}$.

The composition $\theta\delta$ of θ and δ is obtained from the set

$$\{X_1 \to s_1\delta, \ldots, X_m \to s_m\delta, Y_1 \to t_1, \ldots, Y_n \to t_n\}$$

by removing all $X_i \to s_i\delta$ for which $X_i = s_i\delta$ and all $Y_j \to t_j$ for which $Y_j \in \{X_1, \ldots, X_m\}$.

For all expressions e and substitutions θ and δ, $e\theta\delta = (e\theta)\delta$ (Ex. 1.13).

Substitution Lemmas

(1) The identity function, $\mathcal{E}$, is a substitution. $\theta\mathcal{E} = \mathcal{E}\theta = \theta$ for any substitution θ.

(2) If θ and δ are substitutions then so is $\theta\delta$.

(3) The composition of substitutions is associative: $(\theta_1\theta_2)\theta_3 = \theta_1(\theta_2\theta_3)$ for any $\theta_1, \theta_2, \theta_3$.

Note that composition of substitutions is not commutative.

A substitution θ is idempotent iff $\theta = \theta\theta$.

A substitution θ is a renaming substitution iff: for any variables X and Y, $X\theta$ and $Y\theta$ are variables and $X\theta = Y\theta$ iff $X = Y$.

Although substitutions generally do not have inverses, renaming substitutions do.

Expression e_1 is a variant of e_2 iff $e_1 = e_2\theta$ for some renaming substitution (and hence $e_2 = e_1\theta^{-1}$).

Vars(e) denotes the set of variables in expression e. If Vars(e) is empty, we say that e is ground. A substitution is ground if it maps every variable in its domain to a ground expression.

e′ is an instance of e if $e' = e\theta$ for some substitution θ. If e′ is ground, we say e′ is a ground instance of e and θ is a grounding substitution for e. e_{gr} denotes the set of all ground instances of e. If E is a set of expressions E_{gr} denotes the set of all ground instances of expressions in E.