Homework 9

CS 547

Due Friday, May 11, in class

The Towers of Hanoi

There's a good chance you've seen the famous Towers of Hanoi puzzle.

- The puzzle consists of three pegs and a set of disks, each of a different size
- All of the disks are stacked, from largest to smallest, on the first peg
- The goal is to move all of the disks from the first peg to the third peg without ever placing a larger disk on top of a smaller disk

Let f_n be a sequence denoting the number of moves required to complete the Tower of Hanoi puzzle when there are n disks. For any nonzero value of n, we can solve the puzzle in three basic steps.

- Move the $n-1$ smallest disks to the middle peg, taking f_{n-1} moves
- Move the largest disk to the destination peg, using one move
- Move the $n-1$ smallest disks from the middle peg to the destination peg, taking another $n-1$ moves

A recursive definition for the number of moves required for n disks is

$$f_n = 2f_{n-1} + 1$$

In this problem, you'll derive a closed-form solution for f_n using generating functions.

First, use the method described in class to show that the z-transform of the recurrence relation is

$$F(z) = \frac{z}{(1-z)(1-2z)}$$

Note that the recurrence relation contains a 1, which you'll have to simplify in your derivation using

$$\sum_{n=0}^{\infty} z^{n+1} = z\sum_{n=0}^{\infty} z^n = \frac{z}{1-z}$$

The transform is already in a rational form with a factored denominator. Use partial fractions to divide $F(z)$ into

$$F(z) = \frac{-1}{1-z} + \frac{1}{1-2z}$$

Invert the transform and write down the closed form solution for f_n.

Finally, use mathematical induction to prove that your solution is correct.

The Transform of a Transformed Sequence

Let the sequence $f_0,\ f_1,\ f_2,\ f_3, \ldots$ have a z-transform $F(z)$.

Consider the shifted and scaled sequence $g_0,\ g_1,\ g_2,\ g_3, \ldots$, where

$$g_n = af_n + b$$

for some constant values of a and b.

Show that the z-transform of the g sequence is

$$G(z) = aF(z) + \frac{b}{1-z}$$

A Convoluted Problem

Collecting urns and filling them with balls is a favorite hobby of probability theorists.

You have three urns. Suppose that

- each urn can hold at most four balls
- none of the urns can be empty

How many way can you put five balls into the three urns?

Hint: write down the generating function associated with a single urn, cube it, then find the coefficient associated with z^5.