CS 547 Lecture 13: M/M/1 Residence Time Distribution

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In the previous lecture, we derived an expression for the average residence time in the M/M/1 queue. It's important to remember, however, that the actual residence time experienced by any particular customer is still a random value. In this lecture, we'll derive the distribution of the random residence time.

The Plan

Let $F_R(t) = P[R \leq t]$ denote the CDF of residence time in the M/M/1 queue. Our ultimate goal is to derive an expression for $F_R(t)$. To do so, we'll work through four steps.

1. Derive an expression for $F_{R_n}(t \mid n)$, the CDF of residence time conditioned on finding exactly n customers in the queue at an arrival instant
2. Differentiate $F_{R_n}(t \mid n)$ to obtain a conditional pdf, $f_{R_n}(t \mid n)$
3. Use total probability to remove the conditioning on finding n customers in the queue and obtain the unconditional density function $f_R(t)$
4. Integrate $f_R(t)$ to recover $F_R(t)$

The Conditional Residence Time Distribution

Assume that a new customer finds exactly n customers in the queue at the instant of its arrival. We'll now derive the CDF of the new customer's residence time under this condition.

The new customer's residence time is sum of $n + 1$ random service times.

- The residual life of the customer currently in service
- The service time of the $n - 1$ waiting customers
- The new customer's own service time

Each of these sources is a random value. Let Z denote the random residual service time, X_i denote the random service time of the i^{th} customer, and X_{n+1} denote the service time of the newly arriving customer. The residence time experienced by a new customer that finds n customers already in the queue has the form

$$R_n = Z + X_2 + X_3 + \ldots + X_n + X_{n+1}$$

All of the random variables Z and X_i are exponentially distributed with some service rate μ.

Therefore, if we regard a departure from the queue as an event, we can reason about a series of events with exponentially distributed interevent times. Recall that the probability of getting j events from such a process over a time period t is Poisson distributed with parameter μ.

$$P[j \text{ events in time } t] = \frac{e^{-\mu t}(\mu t)^j}{j!}$$

Now, suppose that the new customer finishes its waiting, receives service, and departs the queue by time t, implying that $R_n \leq t$. The new customer was in position $n+1$ of the queue, so this can only occur if $n+1$ or more departure events take place by time t. Conversely, if the new customer is still in the queue at time t, then *fewer* than $n+1$ departure events must have taken place. Therefore, the conditional residence time distribution has the form

$$P[R_n \leq t] = 1 - \sum_{j=0}^{n} \frac{e^{-\mu t}(\mu t)^j}{j!}$$

The summation, based on the Poisson distribution, represents the probability of *fewer* than $n+1$ departures taking place from the queue by time t.

This result is the *Erlang distribution*. It has two parameters: the *number of stages*, n, and the *rate*, μ. Note that the Erlang distribution is continuous, but it includes the discrete calculation based on the Poisson distribution as part of its formula.

The Conditional Density

The second step of our plan is to differentiate $F_{R_n}(t \mid n)$ to obtain the conditional pdf, $f_{R_n}(t \mid n)$.

First, break the summation into its individual terms.

$$P[R_n \leq t] = 1 - e^{-\mu t}\left(1 + \mu t + \frac{(\mu t)^2}{2!} + \ldots + \frac{(\mu t)^n}{n!}\right)$$

Differentiate, using the product rule.

$$\begin{aligned} &= \mu e^{-\mu t}\left(1 + \mu t + \frac{(\mu t)^2}{2!} + \ldots + \frac{(\mu t)^n}{n!}\right) \\ &\quad - e^{-\mu t}\left(\mu + \mu^2 t + \frac{\mu^3 t^2}{2!} + \ldots + \frac{\mu^n t^{n-1}}{(n-1)!}\right) \end{aligned}$$

Factoring μ out of the second term,

$$\begin{aligned} &= \mu e^{-\mu t}\left(1 + \mu t + \frac{(\mu t)^2}{2!} + \ldots + \frac{(\mu t)^{n-1}}{(n-1)!} + \frac{(\mu t)^n}{n!}\right) \\ &\quad - \mu e^{-\mu t}\left(1 + \mu t + \frac{\mu^2 t^2}{2!} + \ldots + \frac{\mu^{n-1} t^{n-1}}{(n-1)!}\right) \end{aligned}$$

Every term on the bottom is paired with one on the top, so almost everything cancels. The only term remaining is the final term on the top with $n!$ in its denominator.

$$f_{R_n}(t \mid n) = \mu e^{-\mu t}\frac{(\mu t)^n}{n!}$$

Removing the Conditioning

We'll now use the Theorem of Total Probability to remove the conditioning on finding exactly n customers in the queue at the arrival instant and recover the unconditional density $f_R(t)$.

$$\begin{aligned} f_R(t) &= \sum_{n=0}^{\infty} P[N=n]\, f_{R_n}(t \mid n) \\ &= \sum_{n=0}^{\infty} U^n(1-U)\mu e^{-\mu t}\frac{(\mu t)^n}{n!} \end{aligned}$$

Factor the terms that do not depend on n.

$$f_R(t) = (1-U)\mu e^{-\mu t} \sum_{n=0}^{\infty} \frac{(U\mu t)^n}{n!}$$

The summation simplifies to $e^{U\mu t}$ (this is one of the basic definitions of the exponential).

$$f_R(t) = (1-U)\mu e^{-\mu t} e^{U\mu t}$$

By the Utilization law, $U\mu = \lambda$. Simplifying yields the final result,

$$f_R(t) = (\mu - \lambda) e^{-(\mu-\lambda)t}$$

This is the pdf of an exponential distribution with parameter $\mu - \lambda$! Therefore, residence times in the M/M/1 queue are also exponentially distributed.

Integrating to obtain the final CDF is trivial.

$$F_R(t) = 1 - e^{-(\mu-\lambda)t}$$

The Waiting Time Distribution

It is also possible to derive the waiting time distribution, which has a very similar form.

$$F_W(t) = 1 - Ue^{-(\mu-\lambda)t}$$